Nonlinear stochastic homogenization and ergodic theory

By *Gianni Dal Maso* at Udine and *Luciano Modica* at Pisa

Introduction

In a recent paper [2] we have studied the stochastic homogenization in the class $\mathfrak{F}$ of all integral functionals

$$F(u, A) = \int_A f(x, Du(x))\, dx$$

whose integrand $f(x, p)$ is Lebesgue measurable in x, convex in p and fulfils the inequalities

$$c_1 |p|^\alpha \leqq f(x, p) \leqq c_2 (1 + |p|^\alpha),$$

where $\alpha > 1$, $c_2 \geqq c_1 > 0$ are fixed constants.

More precisely, first we constructed a metric d on $\mathfrak{F}$ so that $(\mathfrak{F}, d)$ is a compact metric space and the minimum value of the Dirichlet problem

$$m(F, u_0, A) = \min_u \{F(u, A) : u - u_0 \in W_0^{1,\alpha}(A)\}$$

is a continuous function of F in $(\mathfrak{F}, d)$ for any bounded open subset A of $\mathbb{R}^n$ and for any fixed boundary value $u_0 \in W^{1,\alpha}(A)$.

Second, we defined a stochastic homogenization process in $\mathfrak{F}$ as a family $(F_\varepsilon)_{\varepsilon > 0}$ of random integral functionals (i.e. measurable maps on a fixed probability space Ω with values in $(\mathfrak{F}, d)$) such that F_ε has the same law of $\varrho_\varepsilon F$, where F is a given random integral functional with integrand $f(\omega, x, p)$ and $\varrho_\varepsilon F$ is defined by

$$[(\varrho_\varepsilon F)(\omega)](u, A) = \int_A f\left(\omega, \frac{x}{\varepsilon}, Du(x)\right) dx.$$

Then we proved that, under the assumption that F is periodic in law and satisfies a particular condition of independence at large distances, there exists a random integral functional F_0 such that (F_ε) converges in probability to F_0 as $\varepsilon \to 0^+$. In particular, we obtained by continuity that the minima of the Dirichlet problems for F_ε converge in probability as $\varepsilon \to 0^+$ to the corresponding minimum of F_0 and so we solved a large class of physical nonlinear stochastic homogenization problems. For instance, the homogenization of materials with random chessboard structure sketched in figure 1 or of the structure with randomly positioned impurities sketched in figure 2.

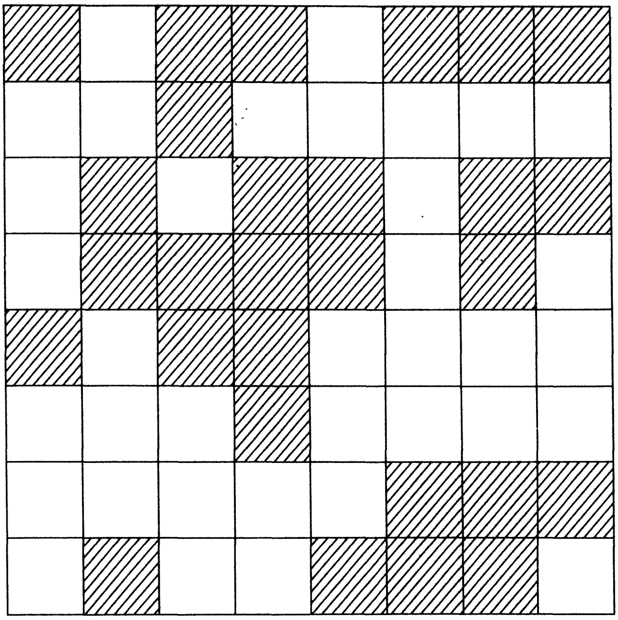

Fig. 1: The random chessboard

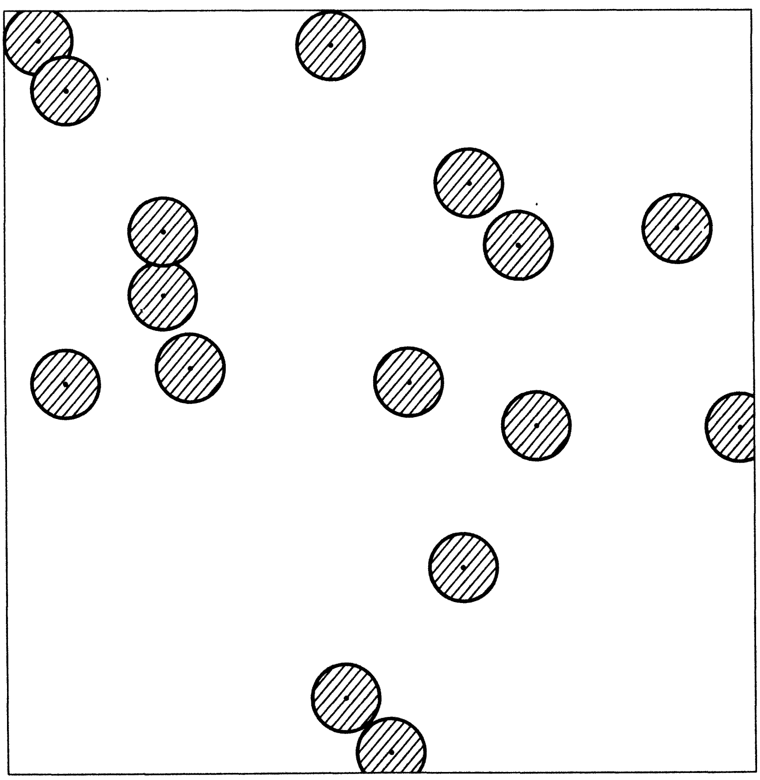

Fig. 2: Balls in random position

Nevertheless, our result was not completely satisfactory for two reasons. The former is that the independence at large distances is not verified in some other interesting cases of stochastic homogenization, as for instance the chessboard structure with cells of random size sketched in figure 3.

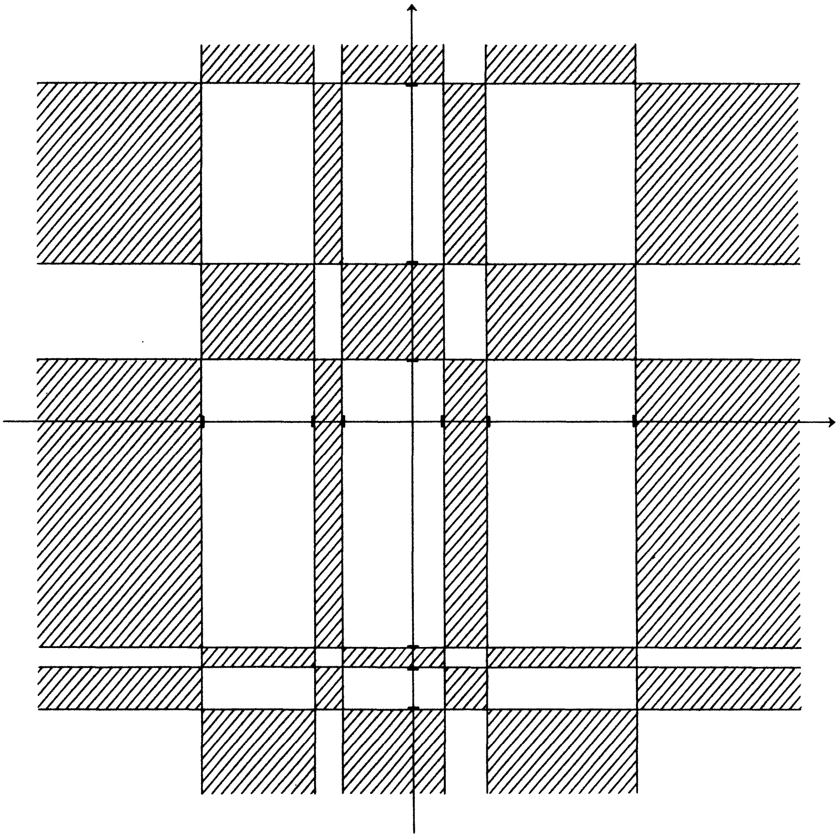

Fig. 3: Infinite chessboard with cells of random size

The latter is that, while convergence in probability is the best possible if we give as above the hypotheses in terms of laws, the problem arises whether there is almost everywhere convergence in the case $F_\varepsilon = \varrho_\varepsilon F$, as already proved in linear stochastic homogenization by S. M. Kozlov [7], V. V. Yurinskij [10], G. C. Papanicolaou and S. R. S. Varadhan [8].

Both these difficulties can be overcome by a more general proof of our theorem (see section 1) which relies on recent results in Ergodic Theory and on a characterization of convergence in $\mathfrak{F}$ (G. Dal Maso-L. Modica [3]). We are indebted to L. Russo who, recognizing in the independence at large distances a "mixing" hypothesis, signaled us the nice Subadditive Ergodic Theorem due to M. A. Akcoglu and U. Krengel [1].

Section 2 of this paper is devoted to the non-trivial problem of passing from hypotheses, like ergodicity, on the integrands to the same ones on the integral functionals. Finally, section 3 contains some examples.

1. The main theorem

In this section we extensively use notations and results of our paper [2]. We repeat here only those definitions which are necessary in the statement of the main theorem.

We denote by $\mathfrak{A}_0$ the family of all bounded open subsets of $\mathbb{R}^n$ and for every $A \in \mathfrak{A}_0$ we denote by $W^{1,\alpha}(A)$ the Sobolev space of the functions of $L^\alpha(A)$ whose first weak derivatives belong to $L^\alpha(A)$.

Let us fix $\alpha > 1$, $c_2 \geqq c_1 > 0$. We denote by $\mathfrak{F} = \mathfrak{F}(c_1, c_2, \alpha)$ the class of all functionals $F : L^\alpha_{\mathrm{loc}}(\mathbb{R}^n) \times \mathfrak{A}_0 \to \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$F(u, A) = \begin{cases} \int_A f(x, Du(x))\, dx & \text{if} \quad u|_A \in W^{1,\alpha}(A), \\ +\infty & \text{otherwise}, \end{cases}$$

where $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is any function satisfying the following conditions:

(i) $f(x, p)$ is Lebesgue measurable in x and convex in p;

(ii) $c_1 |p|^\alpha \leqq f(x, p) \leqq c_2(1 + |p|^\alpha) \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n$.

We consider $\mathfrak{F}$ as equipped with the metric d introduced in [2], prop. 1.12, so that $\mathfrak{F}$ is a compact metric space ([2], cor. 1.22). For each $F \in \mathfrak{F}$, $A \in \mathfrak{A}_0$ and $u_0 \in W^{1,\alpha}(A)$ we may consider the Dirichlet problem for F on A with boundary value u_0: letting

$$m(F, u_0, A) = \min_u \{F(u, A) : u - u_0 \in W_0^{1,\alpha}(A)\},$$

we have that $m(F, u_0, A)$ is continuous in F with respect to the metric d ([2], cor. 1.23).

We denote by $\mathscr{T}_F$ the Borel σ-algebra of $\mathfrak{F}$. Let $(\Omega, \mathscr{T}, P)$ be a fixed probability space; we call random integral functional any $(\mathscr{T}, \mathscr{T}_F)$-measurable map $F : \Omega \to \mathfrak{F}$.

The additive group $\mathbb{Z}^n$ and the multiplicative group $\mathbb{R}_+$ act on $\mathfrak{F}$ by the translation operator τ_z $(z \in \mathbb{Z}^n)$ defined by

$$(\tau_z F)(u, A) = F(\tau_z u, \tau_z A),$$

where $(\tau_z u)(x) = u(x - z)$, $\tau_z A = \{x \in \mathbb{R}^n : x - z \in A\}$, and by the homothety operator ϱ_ε $(\varepsilon > 0)$ defined by

$$(\varrho_\varepsilon F)(u, A) = \varepsilon^n F(\varrho_\varepsilon u, \varrho_\varepsilon A)$$

where $(\varrho_\varepsilon u)(x) = \frac{1}{\varepsilon} u(\varepsilon x)$, $\varrho_\varepsilon A = \{x \in \mathbb{R}^n : \varepsilon x \in A\}$.

Let us recall that, if F is a random integral functional, then $\tau_z F$ and $\varrho_\varepsilon F$ are also random integral functionals (see [2], cor. 2.4) for every $z \in \mathbb{Z}^n$ and $\varepsilon > 0$.

We shall consider ergodicity in $\mathfrak{F}$ with respect to $\mathbb{Z}^n$. Namely, we say that a random integral functional F is ergodic if $P[F \in S] = 0$ or 1 for every $\mathscr{T}_F$-measurable subset S of $\mathfrak{F}$ such that $\tau_z(S) = S$ for every $z \in \mathbb{Z}^n$.

Now, we state our main theorem. Let Q_t be the cube

$$\{x \in \mathbb{R}^n : |x_i| < t,\ i=1,\dots,n\},$$

let $|Q_t|$ be its Lebesgue measure and let l_p be the linear function on $\mathbb{R}^n$ with gradient p, i.e. $l_p(x) = p \cdot x$.

Theorem I. *Let F be a random integral functional and define $F_\varepsilon = \varrho_\varepsilon F$. If F is periodic in law (i.e. F and $\tau_z F$ have the same law for every $z \in \mathbb{Z}^n$), then F_ε converges P-almost everywhere as $\varepsilon \to 0^+$ to a random integral functional F_0. Moreover, there exists $\Omega' \subseteq \Omega$ of full measure such that the limit*

$$\lim_{t \to +\infty} \frac{m(F(\omega), l_p, Q_t)}{|Q_t|} = f_0(\omega, p)$$

exists for every $\omega \in \Omega'$, $p \in \mathbb{R}^n$ and

$$F_0(\omega)(u, A) = \int_A f_0(\omega, Du(x))\, dx$$

for every $\omega \in \Omega'$, $A \in \mathfrak{A}_0$, $u \in L^\alpha_{\text{loc}}(\mathbb{R}^n)$ with $u|_A \in W^{1,\alpha}(A)$. If, in addition, F is ergodic, then F_0 is constant or equivalently $f_0(\omega, p)$ does not depend on ω and

$$f_0(p) = \lim_{t \to +\infty} \int_\Omega \frac{m(F(\omega), l_p, Q_t)}{|Q_t|}\, dP(\omega)$$

for every $p \in \mathbb{R}^n$.

In order to prove theorem I, we need a few definitions and a lemma. A set function $\mu : \mathfrak{A}_0 \to \mathbb{R}$ is said to be subadditive if

$$\mu(A) \leqq \sum_{k \in K} \mu(A_k)$$

for every $A \in \mathfrak{A}_0$ and for every finite family $(A_k)_{k \in K}$ in $\mathfrak{A}_0$ such that

$$A_k \subseteqq A \ \ \forall k \in K, \qquad A_h \cap A_k = \emptyset \ \ \forall h, k \in K,\ h \neq k, \qquad |A - \bigcup_{k \in K} A_k| = 0.$$

Let $\mathfrak{M} = \mathfrak{M}(c)$ be the family of the subadditive functions $\mu : \mathfrak{A}_0 \to \mathbb{R}$ such that

$$0 \leqq \mu(A) \leqq c|A| \quad \forall A \in \mathfrak{A}_0,$$

where c is a fixed real constant. We denote by $\mathcal{T}_M$ the trace on $\mathfrak{M}$ of the product σ-algebra of $\mathbb{R}^{\mathfrak{A}_0}$.

Let $(\Omega, \mathcal{T}, P)$ be a given probability space. A $(\mathcal{T}, \mathcal{T}_M)$-measurable map $\mu : \Omega \to \mathfrak{M}$ is called a subadditive process.

The group $\mathbb{Z}^n$ acts on $\mathfrak{M}$ by the formula

$$(\tau_z \mu)(A) = \mu(\tau_z A)$$

so that we say that a subadditive process is ergodic if $P[\mu \in S] = 0$ or 1 for every $\mathcal{T}_M$ measurable subset S of $\mathfrak{M}$ such that $\tau_z S = S$ for every $z \in \mathbb{Z}^n$.

The main tool in the proof of theorem I will be the following proposition, which is substantially the subadditive ergodic theorem of M. A. Akcoglu and U. Krengel [1].

Proposition 1. *Let $\mu : \Omega \longrightarrow \mathfrak{M}$ be a subadditive process. If μ is periodic in law, that is μ and $\tau_z \mu$ have the same law for every $z \in \mathbb{Z}^n$, then there exist a $\mathcal{T}$-measurable function $\varphi : \Omega \longrightarrow \mathbb{R}$ and a subset $\Omega' \subseteqq \Omega$ of full measure such that*

$$\lim_{t \to +\infty} \frac{\mu(\omega)(tQ)}{|tQ|} = \varphi(\omega),$$

for every $\omega \in \Omega'$ and for every cube Q in $\mathbb{R}^n$. If, in addition, μ is ergodic, then φ is constant.

Proof. An immediate consequence of theorem 2.7 of [1] is that there exist a $\mathcal{T}$-measurable function $\varphi : \Omega \longrightarrow \mathbb{R}$ and a subset $\Omega' \subseteqq \Omega$ of full measure such that

$$\lim_{\substack{t \to +\infty \\ t \in \mathbb{N}}} \frac{\mu(\omega)(tQ)}{|tQ|} = \varphi(\omega)$$

for every $\omega \in \Omega'$ and for every cube Q in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$. Now, by the inequality

$$\mu(\omega)(B) \leqq \mu(\omega)(A) + c\,|B - A| \quad \forall \omega \in \Omega,\ A, B \in \mathfrak{A}_0,\ A \subseteqq B,\ |\partial A| = 0,$$

we first obtain that

$$\lim_{\substack{t \to +\infty \\ t \in \mathbb{R}}} \frac{\mu(\omega)(tQ)}{|tQ|} = \varphi(\omega)$$

for every $\omega \in \Omega'$ and for every cube Q in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$, and then we pass to general cubes in $\mathbb{R}^n$ by an easy approximation argument.

The fact that φ is constant when μ is ergodic follows from a remark of [1] (p. 59), so the proposition is proved.

Let us return to the proof of theorem I.

Proof of theorem I. Let us fix $p \in \mathbb{R}^n$ and define

$$\mu_p(\omega)(A) = m(F(\omega), l_p, A)$$

for $\omega \in \Omega$ and $A \in \mathfrak{A}_0$. It is easy to check that $\mu_p(\omega) \in \mathfrak{M}(c)$ with $c = c_2(1 + |p|^\alpha)$ for every $\omega \in \Omega$ and that $\mu_p : \Omega \longrightarrow \mathfrak{M}$ is $(\mathcal{T}, \mathcal{T}_M)$-measurable because $m(\cdot, l_p, A)$ is continuous on $\mathfrak{F}$. Note that, for every $z \in \mathbb{Z}^n$, $\omega \in \Omega$, $A \in \mathfrak{A}_0$

$$\begin{aligned}(\tau_z \mu_p)(\omega)(A) &= \mu_p(\omega)(\tau_z A) \\ &= \min_u \{(\tau_z F)(\omega)(\tau_{-z} u, A) : \tau_{-z} u - \tau_{-z} l_p \in W_0^{1,\alpha}(A)\} \\ &= \min_v \{(\tau_z F)(\omega)(v + l_p(z), A) : v - l_p \in W_0^{1,\alpha}(A)\}.\end{aligned}$$

As the integrand of F depends only on x and Du, we have that

$$(\tau_z F)(\omega)(v + l_p(z), A) = (\tau_z F)(\omega)(v, A),$$

hence

$$(\tau_z \mu_p)(\omega)(A) = m((\tau_z F)(\omega), l_p, A)$$

for every $z \in \mathbb{Z}^n$, $\omega \in \Omega$, $A \in \mathfrak{A}_0$ so that μ_p is periodic in law because $\tau_z F$ and F have the same law and $m(\cdot, l_p, A)$ is continuous on $\mathfrak{F}$.

Then proposition 1 gives that there exist a subset $\Omega'_p \subseteq \Omega$ of full measure and a $\mathcal{T}$-measurable function $\varphi_p : \Omega \to \mathbb{R}$ such that

$$\lim_{t \to +\infty} \frac{\mu_p(\omega)(tQ)}{|tQ|} = \varphi_p(\omega)$$

for every $\omega \in \Omega'_p$ and for every cube Q in $\mathbb{R}^n$. Let Q_t be the cube $\{x \in \mathbb{R}^n : |x_i| < t,\ i = 1, \dots, n\}$ and let $f_0 : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$f_0(\omega, p) = \limsup_{t \to +\infty} \frac{\mu_p(\omega)(Q_t)}{|Q_t|} \quad \forall (\omega, p) \in \Omega \times \mathbb{R}^n.$$

Now, let us remark that the functions

$$p \mapsto \frac{\mu_p(\omega)(A)}{|A|} \qquad (\omega \in \Omega,\ A \in \mathfrak{A}_0)$$

are convex (this is an easy consequence of convexity in u of $F(\omega)(u, A)$) and equibounded between 0 and $c_2(1 + |p|^\alpha)$, hence locally equicontinuous. It follows that $f_0(\omega, p)$ is convex in p and, denoting

$$\Omega' = \bigcap_{p \in \mathbb{Q}^n} \Omega'_p$$

($\mathbb{Q}$ is the set of rational numbers), we have $P(\Omega') = 1$ and

$$\lim_{t \to +\infty} \frac{\mu_p(\omega)(tQ)}{|tQ|} = f_0(\omega, p)$$

for every $\omega \in \Omega'$, $p \in \mathbb{R}^n$ and for every cube Q in $\mathbb{R}^n$. Finally, observe that

$$\mu_p(\omega)(tQ) = t^n m((\varrho_{\frac{1}{t}} F)(\omega), l_p, Q),$$

hence, as $\varrho_\varepsilon F = F_\varepsilon$, we have that

$$\lim_{\varepsilon \to 0^+} \frac{m(F_\varepsilon(\omega), l_p, Q)}{|Q|} = f_0(\omega, p)$$

for every $\omega \in \Omega'$, $p \in \mathbb{R}^n$ and for every cube Q in $\mathbb{R}^n$. Recalling theorem IV of [3], it follows that for every $\omega \in \Omega'$ there exists an integral functional $F_0(\omega) \in \mathfrak{F}$ such that $F_\varepsilon(\omega)$ converges to $F_0(\omega)$ as $\varepsilon \to 0^+$. Let us calculate the integrand $g_0(\omega, x, p)$ of $F_0(\omega)$. Fix $\omega \in \Omega'$. If we denote

$$Q_\varrho(x) = \{y \in \mathbb{R}^n : |y_i - x_i| < \varrho,\ i = 1, \dots, n\},$$

theorem I of [3] and the continuity of $m(\cdot, l_p, A)$ give that there exists a subset N of $\mathbb{R}^n$ with $|N| = 0$ such that

$$g_0(\omega, x, p) = \lim_{\varrho \to 0^+} \frac{m(F_0(\omega), l_p, Q_\varrho(x))}{|Q_\varrho(x)|} = \lim_{\varrho \to 0^+} \lim_{\varepsilon \to 0^+} \frac{m(F_\varepsilon(\omega), l_p, Q_p(x))}{|Q_\varrho(x)|}$$

$$= \lim_{\varrho \to 0^+} \lim_{\varepsilon \to 0^+} \frac{\mu_p(\omega)\left(\frac{1}{\varepsilon} Q_\varrho(x)\right)}{\left|\frac{1}{\varepsilon} Q_\varrho(x)\right|} = f_0(\omega, p)$$

for every $x \in \mathbb{R}^n \setminus N$, $p \in \mathbb{R}^n$, hence

$$F_0(\omega)(u, A) = \int_A f_0(\omega, Du(x))\, dx$$

for every $\omega \in \Omega'$, $A \in \mathfrak{A}_0$, $u \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u|_A \in W^{1,\alpha}(A)$. Finally, if F is ergodic, then μ_p is ergodic and φ_p does not depend on ω (see proposition 1), so theorem I is completely proved.

Remarks. (a) Note that the almost everywhere convergence of $F_\varepsilon(\omega)$ is obtained in theorem I under the assumption $F_\varepsilon = \varrho_\varepsilon F$. If we suppose only that F_ε and $\varrho_\varepsilon F$ have the same law, we can not obtain almost everywhere convergence but we may deduce that F_ε converges in law to F_0 from compactness of $\mathfrak{F}$ and Lebesgue's dominated convergence theorem. If, in addition, F is ergodic, then F_ε converges in probability to F_0, being F_0 a constant random integral functional.

(b) Theorem I contains the results by S. M. Kozlov [7], V. V. Yurinskij [10], G. C. Papanicolaou and S. R. S. Varadhan [8] about the homogenization of Dirichlet boundary value problems for second order elliptic partial differential equations in divergence form:

$$\begin{cases} \sum_{i,j=1}^{n} D_i\left(a_{ij}\left(\omega, \dfrac{x}{\varepsilon}\right) D_j u\right) = \varphi & \text{on } A, \\ u = u_0 & \text{on } \partial A. \end{cases}$$

Indeed, as we will see in the remark of section 2, the functional F associated to the integrand

$$f(\omega, x, p) = \sum_{i,j=1}^{n} a_{ij}(\omega, x)\, p_i p_j$$

is periodic in law (resp. ergodic) if the matrix-valued random field (a_{ij}) is homogeneous (resp. ergodic), so we may apply theorem I for obtaining almost everywhere convergence of $F_\varepsilon = \varrho_\varepsilon F$ in $\mathfrak{F}$. Now, in the symmetric case $a_{ij} = a_{ji}$, the P-almost everywhere convergence in $L^2(A)$ of the solutions of the Dirichlet problems is an easy consequence of corollary 1.23 of [2].

(c) The convergence of $F_\varepsilon(\omega)$ to $F_0(\omega)$ for P-almost all $\omega \in \Omega$ gives also the P-almost everywhere convergence of minimum values and minimizers of general boundary value problems for F_ε: we refer again to corollary 1.23 of [2].

(d) The easiest way for proving ergodicity of F is to verify a mixing condition (or independence at large distances), as for example the following one: there exists $M > 0$ such that the two real-extended vector random variables

$$(F(\cdot)(u_i, A_j))_{i=1,\dots,I;\, j=1,\dots,J} \qquad (F(\cdot)(v_k, B_l))_{k=1,\dots,K;\, l=1,\dots,L}$$

are independent whenever $u_1, \dots, u_I, v_1, \dots, v_K \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$, $A_1, \dots, A_J, B_1, \dots, B_L \in \mathfrak{A}_0$ and $\mathrm{dist}(A_j, B_l) \geq M$ for $j = 1, \dots, J$ and $l = 1, \dots, L$.

The proof that this assumption implies ergodicity is not completely trivial: it requires the standard technique for passing from mixing hypothesis to ergodicity (see for example [9]) and also the characterization of $\mathcal{T}_F$ contained in theorem 1.26 of [2].

Finally, note that the above condition of independence at large distance is slightly stronger than the corresponding one in theorem 3.2 of [2], but it is verified in all meaningful examples. Therefore theorem I is substantially — but not strictly speaking — a generalization of our previous result.

2. Integrands and integral functionals

In this section we are concerned with the following problem. What are the hypotheses on the integrand $f(\omega, x, p)$ of a random integral functional $F(\omega) \in \mathfrak{F}$ in order that theorem I holds? In particular, in order that F is ergodic?

Let us begin by giving some notations. By $\mathfrak{I} = \mathfrak{I}(c_1, c_2, \alpha)$ we denote the class of the integrands, that is of the functions $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $f(x, p)$ is Lebesgue measurable in x, convex in p and

$$c_1 |p|^\alpha \leqq f(x, p) \leqq c_2(1 + |p|^\alpha) \qquad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n$$

where $\alpha > 1$, $c_2 \geqq c_1 > 0$ are fixed real constants.

Let $(\Omega, \mathscr{T}, P)$ be a probability space. A function $f: \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be a random integrand if f is $(\mathscr{T} \times \mathscr{B}_n \times \mathscr{B}_n, \mathscr{B})$-measurable ($\mathscr{B}_n$ denotes the Borel σ-algebra on $\mathbb{R}^n$, $\mathscr{B}$ that one on $\mathbb{R}$) and $f(\omega, \cdot, \cdot)$ is an integrand of the class $\mathfrak{I}$ for every $\omega \in \Omega$.

If $f: \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a random integrand, we denote by $\tilde{f}: \Omega \to \mathfrak{I}$ the map defined by

$$\tilde{f}(\omega)(x, p) = f(\omega, x, p).$$

Let $\mathscr{T}_I$ be the trace on $\mathfrak{I}$ of the product σ-algebra of $\mathbb{R}^{\mathbb{R}^n \times \mathbb{R}^n}$, that is the smallest σ-algebra $\mathscr{S}$ on $\mathfrak{I}$ such that all the evaluation maps

$$f \mapsto f(x, p) \qquad (x \in \mathbb{R}^n,\ p \in \mathbb{R}^n)$$

are $(\mathscr{S}, \mathscr{B})$-measurable.

A first trivial remark is that, if $f: \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a random integrand, then $\tilde{f}$ is $(\mathscr{T}, \mathscr{T}_I)$ measurable. The converse is not true, because the joint $\mathscr{T} \times \mathscr{B}_n \times \mathscr{B}_n$ measurability of f may fail.

Now, we may consider the map $J: \mathfrak{I} \to \mathfrak{F}$ which transforms an integrand of the class $\mathfrak{I}$ in the corresponding integral functional of the class $\mathfrak{F}$, that is

$$J(f)(u, A) = \begin{cases} \int_A f(x, Du(x))\, dx & \text{if } u|_A \in W^{1,\alpha}(A), \\ +\infty & \text{otherwise} \end{cases}$$

for every $u \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$ and $A \in \mathfrak{A}_0$.

The difficulty which arises here is that J is not $(\mathscr{T}_I, \mathscr{T}_F)$-measurable ($\mathscr{T}_F$ is the Borel σ-algebra on $\mathfrak{F}$). Let $\hat{\mathscr{T}}_I$ be the smallest σ-algebra on $\mathfrak{I}$ which contains $\mathscr{T}_I$ and such that J is $(\hat{\mathscr{T}}_I, \mathscr{T}_F)$-measurable. Then the main result proved in this section will be the following theorem.

Theorem II. *Let $f, g : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be two random integrands. Then the functions $\tilde{f}, \tilde{g} : \Omega \to \mathfrak{I}$ are $(\mathcal{T}, \hat{\mathcal{T}}_I)$-measurable. If, in addition, the two measures $\mu = \tilde{f}_\# P$, $\nu = \tilde{g}_\# P$ (defined on $\hat{\mathcal{T}}_I$) agree on $\mathcal{T}_I$, then $\mu = \nu$ on $\hat{\mathcal{T}}_I$.*

Proof. As $\tilde{f}$ is obviously $(\mathcal{T}, \mathcal{T}_I)$-measurable, the $(\mathcal{T}, \hat{\mathcal{T}}_I)$-measurability of $\tilde{f}$ can be proved by verifying that $J \circ \tilde{f}$ is $(\mathcal{T}, \mathcal{T}_F)$-measurable or equivalently, recalling theorem 1.26 of [2], that the map $\omega \mapsto J(\tilde{f}(\omega))(u, A)$ is $(\mathcal{T}, \mathcal{B})$-measurable for any fixed $u \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$, $A \in \mathfrak{A}_0$. But this is obvious by the definition of random integrand and by the theorem of measurability of integrals depending on a parameter.

Let us pass to prove that, if $\mu = \tilde{f}_\# P$ and $\nu = \tilde{g}_\# P$ agree on $\mathcal{T}_I$, then $\mu = \nu$ on $\hat{\mathcal{T}}_I$. This is an immediate consequence of the following lemma.

Lemma 1. *Let $(f_m)_{m \in M}$ be a countable family of random integrands and let $\mu_m = (\tilde{f}_m)_\# P$ for every $m \in M$. Then, for every $H \in \hat{\mathcal{T}}_I$, there exists $E \in \mathcal{T}_I$ such that $\mu_m(E \triangle H) = 0$ for every $m \in M$.*

Proof of the lemma. Let $\mathcal{T}_0$ be the subfamily of the elements H of $\hat{\mathcal{T}}_I$ for which the thesis holds. Then $\mathcal{T}_0$ is a σ-algebra which contains $\mathcal{T}_I$. If we prove that J is $(\mathcal{T}_0, \mathcal{T}_F)$-measurable, then $\mathcal{T}_0 \supseteq \hat{\mathcal{T}}_I$ and the lemma is proved. Let us fix $u \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$, $A \in \mathfrak{A}_0$ such that $u|_A \in W^{1,\alpha}(A)$, and $t_0 \in \mathbb{R}$: recalling theorem 1.26 of [2], it is enough to prove that

$$H = \left\{\varphi \in \mathfrak{I} : (J\varphi)(u, A) = \int_A \varphi(x, Du(x))\, dx > t_0\right\} \in \mathcal{T}_0 .$$

Let $S = A^{\mathbb{N}}$ be the set of all sequences in A, endowed with the σ-algebra $\mathcal{S}$, which is the infinite product of the Borel σ-algebras on A, and with the probability measure λ given by the infinite product of the normalized Lebesgue measures on A. Then the Birkhoff's ergodic theorem (see [9]) gives that, for every $\varphi \in \mathfrak{I}$, there exists $S_\varphi \in \mathcal{S}$ such that $\lambda(S_\varphi) = 1$ and

$$\frac{1}{|A|} \int_A \varphi(x, Du(x))\, dx = \lim_{k \to +\infty} \frac{1}{k} \sum_{h=1}^{k} \varphi(s_h, Du(s_h))$$

for every $s = (s_h) \in S_\varphi$.

Let us consider the set U of the pairs $(\omega, s) \in \Omega \times S$ such that

$$\frac{1}{|A|} \int_A f_m(\omega, x, Du(x))\, dx = \limsup_{k \to +\infty} \frac{1}{k} \sum_{h=1}^{k} f_m(\omega, s_h, Du(s_h))$$

for every $m \in M$. Of course, $U \in \mathcal{T} \times \mathcal{S}$ and

$$\lambda(\{s \in S : (\omega, s) \in U\}) = 1 \qquad \forall \omega \in \Omega,$$

hence, by Fubini's theorem, there exists $\bar{s} \in S$ such that

$$P(\{\omega \in \Omega : (\omega, \bar{s}) \in U\}) = 1 .$$

Now, define

$$E = \left\{\varphi \in \mathfrak{I} : \limsup_{k \to +\infty} \frac{1}{k} \sum_{h=1}^{k} \varphi(\bar{s}_h, Du(\bar{s}_h)) > \frac{t_0}{|A|}\right\} .$$

It is obvious that $E \in \mathscr{T}_I$, so it remains only to prove that $\mu_m(E \triangle H) = 0$ for every $m \in M$. But

$$\mu_m(E \triangle H) = P(\tilde{f}_m^{-1}(E \triangle H))$$
$$\leqq P\left(\left\{\omega \in \Omega : \frac{1}{|A|} \int_A f_m(\omega, x, Du(x))\,dx \neq \limsup_{k \to +\infty} \frac{1}{k} \sum_{h=1}^{k} f_m(\omega, \bar{s}_h, Du(\bar{s}_h))\right\}\right)$$
$$\leqq P(\{\omega \in \Omega : (\omega, \bar{s}) \notin U\}) = 0$$

for every $m \in M$, so the lemma and theorem II are completely proved.

The following corollaries are straightforward applications of theorem II.

Corollary 1. *Let f, g be two random integrands. Then $\tilde{f}_\# P = \tilde{g}_\# P$ on $\hat{\mathscr{T}}_I$ if and only if the laws of the two vector random variables*

$$(f(\cdot, x_k, p_k))_{k \in K}, \quad (g(\cdot, x_k, p_k))_{k \in K}$$

are equal for every finite family $\{(x_k, p_k)\}_{k \in K}$ in $\mathbb{R}^n \times \mathbb{R}^n$.

The group $\mathbb{Z}^n$ acts on $\mathfrak{J}$ by the translation operator τ_z defined by

$$(\tau_z f)(x, p) = f(x + z, p).$$

A random integrand f is said to be periodic in law if $\tilde{f}_\# P = (\tau_z \tilde{f})_\# P$ (on $\hat{\mathscr{T}}_I$) for every $z \in \mathbb{Z}^n$.

Corollary 2. *A random integrand f is periodic in law if and only if the laws of the two vector random variables*

$$(f(\cdot, x_k, p_k))_{k \in K} \quad (f(\cdot, x_k + z, p_k))_{k \in K}$$

are equal for every $z \in \mathbb{Z}^n$ and for every finite family $\{(x_k, p_k)\}_{k \in K}$ in $\mathbb{R}^n \times \mathbb{R}^n$.

A random integrand is said to be ergodic if $(\tilde{f}_\# P)(H) = 0$ or 1 for every subset $H \in \hat{\mathscr{T}}_I$ such that $\tau_z(H) = H$ for every $z \in \mathbb{Z}^n$.

Corollary 3. *A random integrand f is ergodic if and only if $(\tilde{f}_\# P)(E) = 0$ or 1 for every subset $E \in \mathscr{T}_I$ such that $\tau_z(E) = E$ for every $z \in \mathbb{Z}^n$.*

Proof. Assume that $(\tilde{f}_\# P)(E) = 0$ or 1 for every subset $E \in \mathscr{T}_I$ such that $\tau_z(E) = E$ for every $z \in \mathbb{Z}^n$, and let $H \in \hat{\mathscr{T}}_I$ be a set for which $\tau_z(H) = H$ for every $z \in \mathbb{Z}^n$. We have to prove that $(\tilde{f}_\# P)(H) = 0$ or 1. If we apply lemma 1 to the family $(\tau_z f)_{z \in \mathbb{Z}^n}$ of random integrands, we obtain that there exists $E \in \mathscr{T}_I$ such that $(\tilde{f}_\# P)(H \triangle \tau_z(E)) = 0$ for every $z \in \mathbb{Z}^n$. This implies that $(\tilde{f}_\# P)(E \triangle \tau_z(E)) = 0$ for every $z \in \mathbb{Z}^n$, therefore $(\tilde{f}_\# P)(E) = 0$ or 1 by hypothesis. Since $(\tilde{f}_\# P)(H) = (\tilde{f}_\# P)(E)$, we obtain $(\tilde{f}_\# P)(H) = 0$ or 1, and the corollary is proved.

Remark 1. The previous corollaries allow to construct random integrands periodic in law and ergodic in a standard way. Let $g : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ be any Borel function such that $g(y, p)$ is convex in p and

$$c_1 |p|^\alpha \leqq g(y, p) \leqq c_2(1 + |p|^\alpha) \qquad \forall (y, p) \in \mathbb{R}^k \times \mathbb{R}^n.$$

Let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^k$ be any random field (i.e. any $(\mathscr{T} \times \mathscr{B}_n, \mathscr{B}_k)$-measurable map) and let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$f(\omega, x, p) = g(a(\omega, x), p).$$

Then f is a random integrand and f is periodic in law (resp. ergodic) if a is periodic in law (resp. ergodic). For example, the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(\omega, x)\, p_i p_j$$

is periodic in law (resp. ergodic) if the matrix-valued random field (a_{ij}) is periodic in law (resp. ergodic).

Remark 2. A random integrand f is ergodic if it satisfies the following mixing condition:

$$\lim_{\substack{|z| \to +\infty \\ z \in \mathbb{Z}^n}} P(\{\omega \in \Omega : f(\omega, x_i, p_i) > s_i \ \forall i \in I,\ f(\omega, y_j + z, q_j) > t_j \ \forall j \in J\})$$
$$= P(\{\omega \in \Omega : f(\omega, x_i, p_i) > s_i \ \forall i \in I\}) \cdot P(\{\omega \in \Omega : f(\omega, y_j, q_j) > t_j \ \forall j \in J\})$$

for every pair of finite families $\{(x_i, p_i, s_i)\}_{i \in I}$ and $\{(y_j, q_j, t_j)\}_{j \in J}$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Indeed this condition implies that

$$\lim_{\substack{|z| \to +\infty \\ z \in \mathbb{Z}^n}} (\tilde{f}_{\#} P)(E \cap \tau_z(E)) = [(\tilde{f}_{\#} P)(E)]^2$$

for every $E \in \mathcal{T}_I$. Therefore, if $E \in \mathcal{T}_I$ and $\tau_z(E) = E$ for every $z \in \mathbb{Z}^n$ we have

$$(\tilde{f}_{\#} P)(E) = [(\tilde{f}_{\#} P)(E)]^2$$

so $(\tilde{f}_{\#} P)(E) = 0$ or 1, and f is ergodic by corollary 3.

Finally, we obtain the result which was the goal of this section.

Theorem III. *Let f, g be two random integrands and $F = J \circ \tilde{f}$, $G = J \circ \tilde{g}$ be the corresponding random integral functionals.*

(i) *If $\tilde{f}_{\#} P = \tilde{g}_{\#} P$ on $\hat{\mathcal{T}}_I$, then F and G have the same law.*

(ii) *If f is periodic in law, then F is periodic in law.*

(iii) *If f is ergodic, then F is ergodic.*

Proof. It follows immediately from the $(\hat{\mathcal{T}}_I, \mathcal{T}_F)$-measurability of J and from $J \circ \tau_z = \tau_z \circ J$ for every $z \in \mathbb{Z}^n$.

Let us remark that the converse of the statements (i), (ii), (iii) in theorem III is not true because an integral functional F does not univocally determine its integrand $f(x, p)$ for every $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Nevertheless, the formula

$$\mathcal{J}(F)(x, p) = \limsup_{h \to +\infty} \frac{m(F, l_p, B_{\frac{1}{h}}(x))}{|B_{\frac{1}{h}}(x)|}$$

where $B_{\frac{1}{h}}(x) = \left\{ y \in \mathbb{R}^n : |y - x| < \frac{1}{h} \right\}$, selects a particular integrand $\mathcal{J}(F) \in \mathfrak{I}$ of $F \in \mathfrak{F}$ (see [3], theorem I) and it is possible to prove that F ergodic implies $\mathcal{J}(F)$ ergodic. The same holds for the converse of (i) and (ii).

3. Examples

(3.1) The two examples contained in [2], § 4 — homogenization with regular cells occupied by two materials randomly chosen and homogenization with cells of bounded random size alternatively occupied by two materials — may be also treated by theorem I of the present paper. In particular, these two examples verify the hypothesis of independence at large distances in the strong form (see remark (d) in section 1).

(3.2) As we said in remark (b) of section 1, the linear stochastic homogenization of second order elliptic equation is a particular case of theorem I.

(3.3) For the other examples we need the definition of multi-dimensional Poisson process (see, for example, [6]). Let $\mathscr{B}_n^0$ be the family of all bounded Borel subsets of $\mathbb{R}^n$. A Poisson process with parameter λ is a family $(X_A)_{A \in \mathscr{B}_n^0}$ of random variables on a fixed probability space $(\Omega, \mathscr{T}, P)$ with non-negative integer values such that

(a) $X_{A \cup B}(\omega) = X_A(\omega) + X_B(\omega)$ for P-almost all $\omega \in \Omega$ whenever $A, B \in \mathscr{B}_n^0$ and $A \cap B = \emptyset$;

(b) for every finite and disjoint family $(A_k)_{k \in K}$ in $\mathscr{B}_n^0$, the random variables X_{A_k} $(k \in K)$ are independent;

(c) $P[X_A = m] = e^{-\lambda |A|} \dfrac{\lambda^m |A|^m}{m!}$ for every $A \in \mathscr{B}_n^0$ (that is the law of X_A is the Poisson law with parameter $\lambda |A|$).

Roughly speaking, a Poisson process is a counting process for a uniform random distribution of points in $\mathbb{R}^n$.

The first example models a material with random spherical inclusions (for instance, a concrete). Let (X_A) be a Poisson process and define

$$a(\omega, x) = c_1 + (c_2 - c_1) \min\{1, X_{B(x,r)}(\omega)\}.$$

In other words, $a(\omega, x)$ holds c_2 on the union of the balls of radius r centered in the random points associated to (X_A) and c_1 elsewhere. Now, let F be the random integral functional given by

$$F(\omega)(u, A) = \int_A a(\omega, x) |Du|^\alpha \, dx$$

and $F_\varepsilon = \varrho_\varepsilon F$ (F is a random integral functional by remark 1 of section 2). As F is periodic in law (for any real period) and independent at large distances, hence ergodic, theorem I applies, so there exists a functional F_0 of the form

$$F_0(u, A) = \int_A f_0(Du(x)) \, dx$$

such that $F_\varepsilon(\omega)$ converges to F_0 as $\varepsilon \to 0^+$ for P-almost all $\omega \in \Omega$. It can be proved that F_0 is homogeneous of degree α because the family of functionals which are homogeneous of degree α is closed in $\mathfrak{F}$. Moreover the invariance under rotations of F implies that $f_0(p) = c|p|^\alpha$ with $c_1 \leqq c \leqq c_2$.

The same physical situation may be modelled in another way. Let (X_A^ε) be, for $\varepsilon>0$, Poisson processes with parameter $\lambda_\varepsilon=\lambda\varepsilon^n$: this corresponds to reducing up to a factor ε the mean distance of the random points associated to the Poisson process. Define

$$a_\varepsilon(\omega, x)=c_1+(c_2-c_1)\min\{1, X^\varepsilon_{B(x,r_\varepsilon)}(\omega)\},$$

where $r_\varepsilon=r\varepsilon$, and

$$F_\varepsilon(\omega)(u, A)=\int_A a_\varepsilon(\omega, x)\,|Du|^\alpha\,dx.$$

In this case F_ε is not equal to $\varrho_\varepsilon F$ but F_ε and $\varrho_\varepsilon F$ have the same law, so remark (a) of section 1 applies, hence F_ε converges in probability to the constant random functional F_0.

(3. 4) The random chessboard structures (with cells of arbitrary size, not bounded a priori) may be treated by means of n one-dimensional Poisson processes which give random partitions of the axes. For example, in two dimensions, let (X_A), (Y_A) be two independent one-dimensional Poisson processes, let V be a random variable with equiprobable values c_1, c_2, independent of the processes (X_A), (Y_A).

For every $t\in\mathbb{R}$, we set $A(t)=[0,t[$ if $t>0$, $A(t)=[t,0[$ if $t<0$, $A(t)=\emptyset$ if $t=0$.

Define for every $\omega\in\Omega$ and $x=(x_1, x_2)\in\mathbb{R}^2$

$$a(\omega, x)=\begin{cases} V(\omega) & \text{if } X_{A(x_1)}(\omega)+Y_{A(x_2)}(\omega) \text{ is even,}\\ c_1+c_2-V(\omega) & \text{if } X_{A(x_1)}(\omega)+Y_{A(x_2)}(\omega) \text{ is odd.}\end{cases}$$

Intuitively, for every $\omega\in\Omega$ the function $a(\omega, x)$ is constant on each cell of the random chessboard structure, with alternated values c_1 and c_2, and the value at the origin is given by the random variable V.

Then, taking $\alpha=2$, the random integral functional F, whose integrand is $a|Du|^2$, is periodic in law (for any real period) and ergodic (but not independent at large distances) so the random functionals $F_\varepsilon=\varrho_\varepsilon F$ converge P-almost everywhere as $\varepsilon\to 0^+$ toward

$$F_0(u, A)=c\int_A |Du|^2\,dx.$$

It can be proved that $c=\sqrt{c_1c_2}$ as in the deterministic regular chessboard structure (see [7] and [4]).

A similar structure in the one-dimensional case has been studied by G. Facchinetti and L. Russo [5].

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Instituto di Matematica, via Mantica, 3, 33100 Udine, Italy

Dipartimento di Matematica, via Buonarroti, 2, 56100 Pisa, Italy

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