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LIMITS OF NONLINEAR DIRICHLET PROBLEMS IN VARYING DOMAINS

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We study the general form of the limit, in the sense of Γ-convergence, of a sequence of nonlinear variational problems in varying domains with Dirichlet boundary conditions. The asymptotic problem is characterized in terms of the limit of suitable nonlinear capacities associated to the domains.

Introduction

The main purpose of this paper is the study of the asymptotic behavior, as $h \to +\infty$, of sequences of minimum problems in varying open sets with Dirichlet boundary conditions of the form

$$(0.1) \qquad \min_{u \in H_0^{1,p}(\Omega \setminus E_h)} \left\{ \int_{\Omega \setminus E_h} f(x,Du)\,dx + \int_{\Omega \setminus E_h} gu\,dx \right\} ,$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, and (E_h) is a sequence of closed subsets of Ω. We assume that $f(x,\xi)$ is measurable in x, convex and p-homogeneous in ξ, and that

$$c_1|\xi|^p \leq f(x,\xi) \leq c_2|\xi|^p$$

for suitable constants $0 < c_1 \leq c_2 < +\infty$, $1 < p \leq n$.

For every $g \in L^q(\Omega)$, $1/p + 1/q = 1$, we denote by $m_h(g)$ and $M_h(g)$ respectively the minimum value and the set of all minimum points of problem (0.1). We shall prove the following compactness theorem (Section 6): for every sequence (E_h) of closed subsets of Ω there exist a subsequence $(E_{\sigma(h)})$ and a non-negative Borel measure μ, vanishing on every subset of Ω with p-capacity zero, such that

$$(0.2) \qquad \lim_{h \to +\infty} m_{\sigma(h)}(g) = m(\mu,g)$$

for every $g \in L^q(\Omega)$, where

$$(0.3) \qquad m(\mu,g) = \min_{u \in H_0^{1,p}(\Omega)} \left\{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu + \int_\Omega gu\,dx \right\} .$$

Moreover, if $M(\mu,g)$ indicates the set of all minimum points of problem (0.3), then for every neighborhood U of $M(\mu,g)$ in $L^p(\Omega)$ there exists $k\in \mathbf{N}$ such that $M_{\sigma(h)}(g) \subseteq U$ for every $h \geq k$.

To achieve this result we introduce the class $\mathcal{M}_p(\Omega)$ of all non-negative Borel measures on Ω vanishing on all Borel sets with p-capacity zero. An important special case of such measures is given, for every Borel set $E \subseteq \Omega$, by the measure

$$(0.4) \qquad \infty_E(B) = \begin{cases} 0 & \text{if } C_p(E\cap B) = 0\,, \\ +\infty & \text{if } C_p(E\cap B) > 0\,. \end{cases}$$

Indeed, by taking this definition into account, the minimum problem (0.1) becomes equivalent to

$$(0.5) \qquad \min_{u\in H_0^{1,p}(\Omega)} \left\{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu_h + \int_\Omega gu\,dx \right\}$$

for $\mu_h = \infty_{E_h}$.

In the first part of this paper we analyze the dependence on $\mu\in \mathcal{M}_p(\Omega)$ of the minimum value $m(\mu,g)$ and of the set $M(\mu,g)$ of the minimum points of the problem

$$(0.6) \qquad \min_{u\in H_0^{1,p}(\Omega)} \left\{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu + \int_\Omega gu\,dx \right\} .$$

To reach this goal we introduce on $\mathcal{M}_p(\Omega)$ the notion of γ_f-convergence, which is a convergence of variational type related to the Γ-convergence (see [14], [13], [1]) of the corresponding functionals

$$\int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu .$$

We show that the γ_f-convergence is compact and metrizable on $\mathcal{M}_p(\Omega)$ (Theorems 3.3 and 3.5) by using some techniques developed in the study of limits of obstacle problems (see [7], [1], [2]). A well-known variational property of the Γ-convergence implies immediately the following result concerning the convergence of the minimum values and of the minimum points of problems of the form (0.6): if (μ_h) γ_f-converges to μ in $\mathcal{M}_p(\Omega)$, then $(m(\mu_h,g))$ converges to $m(\mu,g)$ and for every neighborhood U of $M(\mu,g)$ in $L^p(\Omega)$ there exists $k\in \mathbf{N}$ such that $M(\mu_h,g) \subseteq U$ for every $h \geq k$.

The results regarding the minimum problems (0.1), mentioned at the beginning, follow then rather easily from the compactness theorem applied to the sequence $\mu_h = \infty_{E_h}$.

Finally, the γ_f-convergence is characterized by means of the notion of μ-capacity (Section 5). This is a set function defined for every Borel set $B \subseteq \Omega$ by

$$C(f,\mu,B) = \min \left\{ \int_\Omega f(x,Du)\,dx + \int_B |u|^p d\mu \; : \; u-1 \in H_0^{1,p}(\Omega) \right\} .$$

We show the equivalence between the γ_f-convergence of a sequence of measures (μ_h)

in $\mathcal{M}_p(\Omega)$ and the weak convergence (in the sense of [15]) of the corresponding μ-capacities $C(f,\mu_h,\cdot)$. More precisely, by setting

$$\alpha'(K) = \liminf_{h\to+\infty} C(f,\mu_h,K) \quad , \qquad \alpha''(K) = \limsup_{h\to+\infty} C(f,\mu_h,K)$$

for every compact set $K \subseteq \Omega$, we show in Theorem 5.8 that (μ_h) γ_f-converges to a measure μ in $\mathcal{M}_p(\Omega)$ if and only if

$$\sup\{\alpha'(K) : K \text{ compact}, K \subseteq A\} = \sup\{\alpha''(K) : K \text{ compact}, K \subseteq A\} \tag{0.7}$$

for every open set $A \subseteq \Omega$. In this case both sides of (0.7) are equal to $C(f,\mu,A)$ and this allows us to obtain an explicit formula for μ in terms of the set functions α' and α'' by applying the main theorem of our previous paper [9].

These results take on an especially nice form in the case of the Dirichlet problems (0.1), as illustrated in Section 6.

In the case $p = 2$ and $f(x,\xi) = |\xi|^2$, the notion of γ_f-convergence has been extensively studied in [12], to which we refer for a wide bibliography on this subject. A probabilistic analysis of this notion of convergence is carried out in [4].

The first proof of the sufficiency of condition (0.7) in the case $f(x,\xi) = |\xi|^2$ was obtained in [5] by probabilisitic methods, under the hypothesis that μ has (locally) a bounded potential. A different proof, which holds for arbitrary μ, was given in [8] by Γ-convergence methods.

In the case $p \neq 2$, the results obtained in this paper are completely new. The only problems of this kind studied in the literature are two examples discussed in [2], Chapter 5, and [17], Chapter 4.2, under the assumption that the sets E_h have a periodic structure.

The results of the present paper were announced without proofs in [10].

1. Notation and Preliminaries

Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, and let p be a real constant with $1 < p \leq n$. We denote by $\mathcal{A}$ the class of all open subsets of Ω and we say that a subset $\mathcal{R}$ of $\mathcal{A}$ is *rich* in $\mathcal{A}$ if, for every family $(A_t)_{t\in\mathbf{R}}$ in $\mathcal{A}$, with $A_s \subset\subset A_t$ whenever $s, t \in \mathbf{R}$, $s < t$, the set $\{t\in\mathbf{R} : A_t \notin \mathcal{R}\}$ is at most countable. We indicate by $\mathcal{K}$ the class of all compact subsets of Ω and by $\mathcal{B}$ the σ-field of all Borel subsets of Ω.

For every $K\in\mathcal{K}$ we define the *p-capacity of* K *with respect to* Ω by

$$C_p(K) = \inf\{\int_\Omega |D\varphi|^p\,dx : \varphi\in C_0^\infty(\Omega)\,,\ \varphi \geq 1 \text{ on } K\}\ .$$

This definition is extended to $A\in\mathcal{A}$ by

$$C_p(A) = \sup \{C_p(K) : K \in \mathcal{K}, K \subseteq A\},$$

and to arbitrary sets $E \subseteq \Omega$ by

$$C_p(E) = \inf \{C_p(A) : A \in \mathcal{A}, E \subseteq A\}.$$

Let E be a subset of Ω. If a property $P(x)$ holds for all $x \in E$, except for a set $Z \subseteq E$ with $C_p(Z) = 0$, then we say that $P(x)$ holds *p-quasi everywhere* on E (p-q.e. on E) or *for p-quasi every* $x \in E$.

A set $U \subseteq \Omega$ is said to be *p-quasi open* (resp. *p-quasi closed*) in Ω if for every $\varepsilon > 0$ there exists an open (resp. closed) set $A \subseteq \Omega$ such that $C_p(U \Delta A) < \varepsilon$, where Δ denotes the symmetric difference and the topological notions are given in the relative topology of Ω. In a similar way we give the notion of a *p-quasi Borel* subset of Ω and denote by $\mathcal{B}_0$ the σ-field of all p-quasi Borel subsets of Ω.

By a Borel measure on Ω we mean a non-negative countably additive set function $\mu : \mathcal{B} \to [0,+\infty]$ such that $\mu(\emptyset) = 0$. We indicate by $\mathcal{M}_p(\Omega)$ the class of all Borel measures μ on Ω such that $\mu(B) = 0$ for every $B \in \mathcal{B}$ with $C_p(B) = 0$. Every measure μ of the class $\mathcal{M}_p(\Omega)$ can be extended to a unique measure, still denoted by μ, defined on the σ-field $\mathcal{B}_0$.

For every $u \in H^{1,p}(\Omega)$ and for every $x \in \Omega$ we assume that

$$\text{(1.1)} \qquad \liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \leq u(x) \leq \limsup_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy ,$$

where $B_r(x) = \{y \in \mathbf{R}^n : |x-y| < r\}$ and $|B_r(x)|$ is the Lebesgue measure of $B_r(x)$. With this convention, the pointwise value $u(x)$ is determined p-q.e. on Ω.

Let us finally recall the definition and the basic properties of Γ-convergence as formulated in abstract terms in an arbitrary metric space X (see [14]).

Definition 1.1. Let (F_h) be a sequence of functions from X into $\overline{\mathbf{R}}$, and let F be a function from X into $\overline{\mathbf{R}}$. We say that (F_h) *Γ-converges to* F in X if the following conditions are satisfied:

(a) for every $u \in X$ and for every sequence (u_h) converging to u in X

$$F(u) \leq \liminf_{h \to +\infty} F_h(u_h) ;$$

(b) for every $u \in X$ there exists a sequence (u_h) converging to u in X such that

$$F(u) \geq \limsup_{h \to +\infty} F_h(u_h) .$$

The main motivation of this convergence is given by the following variational property (see [14], Corollary 2.4).

Proposition 1.2. *Let* (F_h) *be a sequence of functions which* Γ*-converges in* X *to a function* F *and let* $G : X \to \mathbf{R}$ *be a continuous function. Suppose that for every* $\lambda \in \mathbf{R}$ *there exists a compact set* $K_\lambda \subseteq X$ *such that* $\{v \in X : F_h(v) + G(v) \leq \lambda\} \subseteq K_\lambda$ *for every* $h \in \mathbf{N}$. *Then* $F + G$ *attains its minimum in* X *and*

$$\lim_{h\to+\infty} \inf_{v\in X} [F_h(v) + G(v)] = \min_{v\in X} [F(v) + G(v)] .$$

Furthermore, if M_h *and* M *denote the set of all minimum points of* $F_h + G$ *and* $F + G$ *respectively in* X, *then for every neighborhood* U *of* M *there exists* $k \in \mathbf{N}$ *such that* $M_h \subseteq U$ *for every* $h \geq k$.

2. A Compactness Theorem

Let us fix a function $f : \Omega \times \mathbf{R}^n \to \mathbf{R}$ and two constants $0 < c_1 \leq c_2 < +\infty$ which satisfy the following conditions:

(2.1) $f(x,\xi)$ is Lebesgue measurable in x, convex and p-homogeneus in ξ ;

(2.2) $c_1|\xi|^p \leq f(x,\xi) \leq c_2|\xi|^p$ for every $(x,\xi) \in \Omega \times \mathbf{R}^n$.

For every $A \in \mathcal{A}$ and for every $u \in L^p(A)$ we define

$$(2.3) \qquad F(u,A) = \begin{cases} \displaystyle\int_A f(x,Du(x))dx & \text{if } u \in H^{1,p}(A) , \\ +\infty & \text{otherwise} . \end{cases}$$

Moreover, given $\mu \in \mathcal{M}_p(\Omega)$, we define for every $A \in \mathcal{A}$ and for every $u \in L^p(A)$

$$(2.4) \qquad G_\mu(u,A) = \begin{cases} \displaystyle\int_A |u|^p d\mu & \text{if } u \in H^{1,p}(A) , \\ +\infty & \text{otherwise}. \end{cases}$$

We can now state the main result of this section which is a compactness theorem, with respect to the Γ-convergence, for the family of all functionals of the form $F + G_\mu$ with $\mu \in \mathcal{M}_p(\Omega)$.

Theorem 2.1. *For every sequence* (μ_h) *in* $\mathcal{M}_p(\Omega)$ *there exist a subsequence* $(\mu_{\sigma(h)})$ *of* (μ_h), *a measure* μ *in* $\mathcal{M}_p(\Omega)$, *and a family* $\mathcal{R}$, *rich in* $\mathcal{A}$, *such that*

$$[F(\cdot,A) + G_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad in\ L^p(A)$$

for every $A \in \mathcal{R}$.

To prove this theorem we establish first an analogous result for functionals of the form $F + G^1_\mu$, $\mu \in \mathcal{M}_p(\Omega)$, where G^1_μ is defined as follows: for every $A \in \mathcal{A}$ and for every $u \in L^p(A)$ we set

$$G^1_\mu(u,A) = \begin{cases} \int_A (u^+)^p d\mu & \text{if } u \in H^{1,p}(A), \\ +\infty & \text{otherwise}, \end{cases}$$

where $u^+ = \max\{u,0\}$. Then the following lemma holds.

Lemma 2.2. *For every sequence* (μ_h) *in* $\mathcal{M}_p(\Omega)$ *there exist a subsequence* $(\mu_{\sigma(h)})$ *of* (μ_h), *a measure* $\mu \in \mathcal{M}_p(\Omega)$, *and a family* $\mathcal{R}$, *rich in* $\mathcal{A}$, *such that*

$$[F(\cdot,A) + G^1_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G^1_\mu(\cdot,A)] \ \ in\ L^p(A)$$

for every $A \in \mathcal{R}$.

Before starting with the proof of this lemma let us introduce the notion of local functional. Let $X(\Omega)$ be a space of functions defined (a.e.) on Ω. By a *local functional* on $X(\Omega)$ we mean a functional $G : X(\Omega) \times \mathcal{A} \to \overline{\mathbf{R}}$ such that $G(u,A) = G(v,A)$ for every $A \in \mathcal{A}$ and for every pair of functions $u, v \in X(\Omega)$ which agree almost everywhere in A.

Let then G be a local functional on $L^p(\Omega)$ and let $A \in \mathcal{A}$. The function $G(\cdot,A)$, defined on $L^p(\Omega)$, can be extended in a natural way to $L^p(A)$: for every $u \in L^p(A)$ we define $G(u,A) = G(v,A)$, where v is an arbitrary function of $L^p(\Omega)$ which extends u. Since G is local, the definition of $G(u,A)$ does not depend on the extension v.

Proof of Lemma 2.2. Let $(\mu_h) \in \mathcal{M}_p(\Omega)$. By a general compactness theorem with respect to the Γ-convergence (see [11], Theorem 4.18 and Proposition 4.11) there exist a subsequence $(\mu_{\sigma(h)})$ of (μ_h), a family $\mathcal{R}$, rich in $\mathcal{A}$, and a local functional $H : L^p(\Omega) \times \mathcal{A} \to [0,+\infty]$ such that

(2.5) for every $A \in \mathcal{R}$, the functionals $[F(\cdot,A) + G^1_{\mu_{\sigma(h)}}(\cdot,A)]$ Γ-converge to $H(\cdot,A)$ in $L^p(\Omega)$ (hence in $L^p(A)$);

(2.6) for every $A \in \mathcal{A}$, the function $H(\cdot,A)$ is lower semicontinuous on $L^p(\Omega)$ (hence on $L^p(A)$);

(2.7) for every $u \in L^p(\Omega)$, the set function $H(u,\cdot)$ is a measure, i.e. $H(u,\cdot)$ is the trace on $\mathcal{A}$ of a Borel measure defined on $\mathcal{B}$.

For every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$ we define

(2.8) $$G(u,A) = H(u,A) - F(u,A).$$

Then G is a non-negative local functional on $H^{1,p}(\Omega)$. By definition it follows also immediately that the set function $G(u,\cdot)$ is a measure for every $u\in H^{1,p}(\Omega)$ and that $G(\cdot,A)$ is lower semicontinuous on $H^{1,p}(\Omega)$ for every $A\in\mathcal{A}$. As in Lemma 3.3 (3) of [2] we get finally that for every $A\in\mathcal{A}$ the function $G(\cdot,A)$ is increasing. Thus, the integral representation Theorem 5.7 of [6] yields the existence of a Borel function $g:\Omega\times\mathbf{R}\to[0,+\infty]$ and of two non-negative Radon measures λ and ν such that

(i) for every $u\in H^{1,p}(\Omega)$ and for every $A\in\mathcal{A}$

$$G(u,A) = \int_A g(x,u(x))\,d\lambda(x) + \nu(A) \; ; \tag{2.9}$$

(ii) λ belongs to $H^{-1,q}(\Omega)$, $1/p + 1/q = 1$, hence to $\mathcal{M}_p(\Omega)$;

(iii) for every $x\in\Omega$ the function $g(x,\cdot)$ is increasing and lower semicontinuous on $\mathbf{R}$.

Let $A\in\mathcal{A}$ with a Lipschitz boundary. Since G is local and every $u\in H^{1,p}(A)$ can be extended to a function of $H^{1,p}(\Omega)$, the function $G(\cdot,A)$ is well defined on $H^{1,p}(A)$ and the integral representation for G in (2.9) is still valid on $H^{1,p}(A)$. Since G is a measure and every open set A can be approximated by means of open sets with a Lipschitz boundary, it is easy to show that (2.9) holds for every $A\in\mathcal{A}$ and for every $u\in H^{1,p}(\Omega)$. Since $F(0,A)=0$ and $G^1_{\mu_{\sigma(h)}}(0,A)=0$ for every $A\in\mathcal{A}$, by (2.5) and (2.8) we get $G(0,A)=0$ for every $A\in\mathcal{R}$. By (2.9) this implies $\nu\equiv 0$ on $\mathcal{A}$ and $g(x,0)=0$ λ-q.e. on Ω. To accomplish the proof of the theorem it remains only to show that there exists $\mu\in\mathcal{M}_p(\Omega)$ such that

$$\int_A g(x,u(x))\,d\lambda = \int_A (u^+)^p\,d\mu \tag{2.10}$$

for every $A\in\mathcal{A}$ and for every $u\in H^{1,p}(A)$.

To this aim let us observe that, since $(F+G^1_{\mu_h})(\cdot,A)$ is positively p-homogeneous, the functional $(F+G)(\cdot,A)$ is positively p-homogeneous on $H^{1,p}(A)$ for every $A\in\mathcal{R}$. Furthermore, our assumptions on F imply that $G(\cdot,A)$ is positively p-homogeneous on $H^{1,p}(A)$ for every $A\in\mathcal{R}$, and hence for every $A\in\mathcal{A}$. Therefore we can apply the next lemma which proves (2.10), and concludes the proof of Lemma 2.2. ∎

Lemma 2.3. *Let* $\lambda\in\mathcal{M}_p(\Omega)$ *and let* $g:\Omega\times\mathbf{R}\to[0,+\infty]$ *be a Borel function such that*

(i) *for every* $x\in\Omega$, *the function* $g(x,\cdot)$ *is increasing and lower semicontinuous on* $\mathbf{R}$;

(ii) *for every* $A\in\mathcal{A}$, *the function* $u\to\int_A g(x,u)\,d\lambda$ *is positively p-homogeneous on* $H^{1,p}(A)$;

(iii) $g(x,0)=0$ λ*-q.e. on* Ω.

Then, setting $a(x) = g(x,1)$ *for* $x \in \Omega$, *we get*

$$\int_A g(x,u)\, d\lambda = \int_A a(x)(u^+)^p\, d\lambda \tag{2.11}$$

for every $A \in \mathcal{A}$ *and for every* $u \in H^{1,p}(A)$.

Proof of Lemma 2.3. The proof of the lemma is standard when $\int_\Omega g(x,u(x))d\lambda < +\infty$ for every $u \in H^{1,p}(\Omega)$. To prove the lemma in the general case, we consider the set $K = \{u \in H^{1,p}(\Omega) : u \geq 0 \text{ on } \Omega, \int_\Omega g(x,u(x))d\lambda < +\infty\}$. Since $H^{1,p}(\Omega)$ is a separable metric space there exists a sequence (u_h) in K which is dense in K in the strong topology of $H^{1,p}(\Omega)$. According to the convention (1.1) we define the pointwise values of u_h by

$$u_h(x) = \liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u_h(y)\, dy ,$$

and we set $E = \bigcup_1^\infty \{u_h > 0\}$. By the density of (u_h) we obtain that $\{u > 0\} \subseteq E$ p-q.e. for every $u \in K$.

Let us prove that the function $g(x,\cdot)$ is positively p-homogeneous on $\mathbf{R}$ for λ-q.e. $x \in E$. For every $A \in \mathcal{A}$, for every $\tau > 0$, and for every $h \in \mathbf{N}$ we have

$$\int_A g(x,\tau u_h(x))\, d\lambda = \tau^p \int_A g(x,u_h(x))\, d\lambda < +\infty .$$

Therefore, there exists $N \in \mathcal{B}$ such that $\lambda(N) = 0$ and

$$g(x,\tau u_h(x)) = \tau^p\, g(x,u_h(x)) \tag{2.12}$$

for every $x \in \Omega \setminus N$, $h \in \mathbf{N}$, and $\tau \in \mathbf{Q}$, $\tau > 0$. By assumption (i) the function $g(x,\cdot)$ is continuous from the left for every $x \in \Omega$ and therefore (2.12) holds also for every $\tau \in \mathbf{R}$, $\tau > 0$. By (iii) there exists a Borel subset N' of Ω such that $\lambda(N') = 0$ and $g(x,\eta) = 0$ for every $\eta \leq 0$ and for every $x \in \Omega \setminus N'$. Let $x \in E \setminus (N \cup N')$ and $t > 0$. By the definition of E there exists $h \in \mathbf{N}$ such that $u_h(x) > 0$; so we can choose $\tau = t/u_h(x)$. By (2.12) we get

$$g(x,t) = t^p \frac{g(x,u_h(x))}{(u_h(x))^p}$$

and therefore, for $t = 1$, we have

$$g(x,1) = \frac{g(x,u_h(x))}{(u_h(x))^p} ,$$

hence

$$g(x,t) = g(x,1)\, t^p .$$

Since $g(x,\eta) = 0$ for every $\eta \le 0$, we conclude that

$$g(x,t) = g(x,1)\,(t^+)^p \tag{2.13}$$

for every $t \in \mathbf{R}$ and for every $x \in E \setminus (N \cup N')$, which proves our assertion.

Let us prove now that

$$\int_\Omega g(x,u)\,d\lambda = \int_\Omega a(x)(u^+)^p\,d\lambda \tag{2.14}$$

for every $u \in H^{1,p}(\Omega)$, where $a(x) = g(x,1)$ for $x \in \Omega$.

Let $u \in H^{1,p}(\Omega)$, $u \ge 0$. If $\int_\Omega g(x,u)\,d\lambda < +\infty$, by the density property of (u_h) there exists a subsequence $(u_{\sigma(h)})$ which converges to u p-q.e. on Ω, which yields that $\{u > 0\} \subseteq E$ p-q.e. By (2.13) we have

$$g(x,u(x)) = a(x)(u(x))^p \qquad \lambda\text{-q.e. on } \{u > 0\}.$$

Since $g(x,0) = 0 = a(x)\cdot 0$ λ-q.e. on Ω, we get

$$g(x,u(x)) = a(x)(u(x))^p \qquad \lambda\text{-q.e. on } \Omega,$$

which implies (2.14) under the additional assumption that $\int_\Omega g(x,u)d\lambda < +\infty$.

If $\int_\Omega g(x,u(x))d\lambda = +\infty$, let us suppose by contradiction that $\int_\Omega g(x,1)(u)^p d\lambda < +\infty$. Then $\int_E g(x,1)(u)^p d\lambda < +\infty$. Since $\int_E g(x,1)(u)^p d\lambda = \int_E g(x,u)d\lambda$, it follows that $\int_{\Omega\setminus E} g(x,u)d\lambda = +\infty$. This yields that $\lambda(\{u > 0\} \cap (\Omega \setminus E)) > 0$. By the continuity of λ along increasing sequences there exists $\varepsilon > 0$ such that $\lambda(\{u > \varepsilon\} \cap (\Omega \setminus E)) > 0$, which implies

$$\int_\Omega g(x,1_{\{u>\varepsilon\}})\,d\lambda = \int_{\{u>\varepsilon\}} g(x,1)\,d\lambda < \frac{1}{\varepsilon^p}\int_\Omega g(x,1)(u)^p\,d\lambda < +\infty. \tag{2.15}$$

Since $\{u > \varepsilon\}$ is p-quasi open, by Lemma 1.5 of [6] there exists an increasing sequence (v_h) in $H^{1,p}(\Omega)$ converging to $1_{\{u>\varepsilon\}}$ p-q.e. on Ω. By (2.15) it follows that

$$\int_\Omega g(x,v_h(x))\,d\lambda < +\infty,$$

so $v_h \in K$ for every $h \in \mathbf{N}$. Hence $\{v_h > 0\} \subseteq E$ p-q.e. and therefore $\{u > \varepsilon\} \subseteq E$ p-q.e., which contradicts $\lambda(\{u > \varepsilon\} \cap (\Omega \setminus E)) > 0$. So we conclude that $\int_\Omega g(x,1)(u)^p d\lambda = +\infty$, proving (2.14).

To accomplish the proof of the lemma it is clearly enough to show that (2.11) holds for every $u \in H^{1,p}(A)$, $u \ge 0$. Let $A' \in \mathcal{A}$, $A' \subset\subset A$, and let v be a function of $H_0^{1,p}(\Omega)$ such that $\operatorname{spt} v \subset\subset A$, $v = u$ on A', and $0 \le v \le u$ on A. Then (2.14) implies that

$$\int_{A'} g(x,u)\, d\lambda = \int_{A'} g(x,v)\, d\lambda \le \int_{\Omega} g(x,v)\, d\lambda = \int_{\Omega} a(x)(v)^p\, d\lambda = \int_{A} a(x)(v)^p\, d\lambda \le \int_{A} a(x)(u)^p\, d\lambda \,.$$

By taking the supremum for $A' \subset\subset A$ we get

$$\int_{A} g(x,u)\, d\lambda \le \int_{A} a(x)(u)^p\, d\lambda \,.$$

In a similar way we obtain the opposite inequality and conclude so the proof of Lemma 2.3. ∎

Proof of Theorem 2.1. Let (μ_h) be a sequence of measures of $\mathcal{M}_p(\Omega)$. By Lemma 2.2 there exist a subsequence $(\mu_{\sigma(h)})$, a measure $\mu \in \mathcal{M}_p(\Omega)$, and a family $\mathcal{R}$, rich in $\mathcal{A}$, such that

$$(2.16) \qquad [F(\cdot,A) + G^1_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G^1_{\mu}(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A \in \mathcal{R}$.

For every $\nu \in \mathcal{M}_p(\Omega)$, $A \in \mathcal{A}$, and $u \in L^p(A)$ we define

$$G^2_\nu(u,A) = \begin{cases} \displaystyle\int_A (u^-)^p\, d\mu & \text{if } u \in H^{1,p}(A), \\ +\infty & \text{otherwise}, \end{cases}$$

where $u^- = \max\{-u,0\}$. Since $G^2_\nu(u,A) = G^1_\nu(-u,A)$ and $F(u,A) = F(-u,A)$, from (2.16) we obtain that

$$[F(\cdot,A) + G^2_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G^2_{\mu}(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A \in \mathcal{R}$. Since $F(0,A) = G^1_{\mu_{\sigma(h)}}(0,A) = G^2_{\mu_{\sigma(h)}}(0,A) = 0$, we can apply Theorem 3.12 of [2], which yields that

$$[F(\cdot,A) + G^1_{\mu_{\sigma(h)}}(\cdot,A) + G^2_{\mu_{\sigma(h)}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G^1_{\mu}(\cdot,A) + G^2_{\mu}(\cdot,A)]$$

in $L^p(A)$ for every $A \in \mathcal{R}$. The conclusion follows now from the fact that $G_\nu = G^1_\nu + G^2_\nu$ for every $\nu \in \mathcal{M}_p(\Omega)$. ∎

3. γ_f-convergence

In this section we introduce the notion of γ_f-convergence for sequences of measures in $\mathcal{M}_p(\Omega)$ and study the main properties of this convergence. In particular, we show that the γ_f-convergence is compact and metrizable on $\mathcal{M}_p(\Omega)$.

To define the γ_f-convergence, we introduce the functional F_0 defined for every for every $A \in \mathcal{A}$ and for every $u \in L^p(A)$ by

$$(3.1) \qquad F_0(u,A) = \begin{cases} F(u,A) & \text{if } u \in H_0^{1,p}(A) \ , \\ +\infty & \text{otherwise} \ . \end{cases}$$

Let us point out that the effective domain of this functional takes into account the boundary condition $u = 0$ on ∂A.

Definition 3.1. Let (μ_h) be a sequence in $\mathcal{M}_p(\Omega)$ and let $\mu \in \mathcal{M}_p(\Omega)$. We say that (μ_h) *γ_f-converges to* μ if

$$[F_0(\cdot,\Omega) + G_{\mu_h}(\cdot,\Omega)] \quad \Gamma\text{-converges to} \quad [F_0(\cdot,\Omega) + G_{\mu}(\cdot,\Omega)] \quad \text{in } L^p(\Omega)$$

according to Definition 1.1.

In the case $p = 2$ and $f(x,\xi) = |\xi|^2$, the γ_f-convergence coincides with the γ-convergence introduced in [12] and studied in [4] and [8].

Our main goal in this section is to prove the following theorem.

Theorem 3.2. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *and let* $\mu \in \mathcal{M}_p(\Omega)$. *Then the following conditions are equivalent:*

(i) (μ_h) *γ_f-converges to* μ ;

(ii) for every $A \in \mathcal{A}$

$$[F_0(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_0(\cdot,A) + G_{\mu}(\cdot,A)] \quad \text{in } L^p(A) \ ;$$

(iii) there exists a family $\mathcal{R}$, *rich in* $\mathcal{A}$, *such that for every* $A \in \mathcal{R}$

$$[F(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_{\mu}(\cdot,A)] \quad \text{in } L^p(A) \ .$$

Proof of Theorem 3.2.

(iii) $\Rightarrow$ (ii) : Assume (iii) and define for every $u \in L^p(A)$

(3.2) $$H'(u,A) = \inf\{\liminf_{h\to+\infty} [F_0(u_h,A) + G_{\mu_h}(u_h,A)] : u_h \to u \ \text{ in } L^p(A)\} ,$$

(3.3) $$H''(u,A) = \inf\{\limsup_{h\to+\infty} [F_0(u_h,A) + G_{\mu_h}(u_h,A)] : u_h \to u \ \text{ in } L^p(A)\} .$$

It is easy to see (by a diagonal argument) that the infima in (3.2) and (3.3) are achieved by suitable sequences and that $H'(\cdot,A)$ and $H''(\cdot,A)$ are lower semicontinuous on $L^p(A)$ (see [14], Proposition 1.8).

To prove (ii) we have to show that

$$H''(u,A) \le F_0(u,A) + G_\mu(u,A) \le H'(u,A)$$

for every $A \in \mathcal{A}$ and for every $u \in L^p(A)$.

Let us prove that

(3.4) $$F_0(u,A) + G_\mu(u,A) \le H'(u,A) .$$

Fix $A \in \mathcal{A}$ and $u \in L^p(A)$ such that $H'(u,A) < +\infty$. Let (u_h) be a sequence converging to u in $L^p(A)$ such that

$$H'(u,A) = \liminf_{h\to+\infty} [F_0(u_h,A) + G_{\mu_h}(u_h,A)] .$$

Since $H'(u,A) < +\infty$, there exist a constant $c \in \mathbf{R}$ and a subsequence $(u_{\sigma(h)})$ of (u_h) such that $F_0(u_{\sigma(h)},A) < c$ for every $h \in \mathbf{N}$. Hence $u_{\sigma(h)} \in H_0^{1,p}(A)$ and, by the coerciveness of F_0, we may assume that $(u_{\sigma(h)})$ converges weakly to u in $H_0^{1,p}(A)$. This implies that $u \in H^{1,p}(A)$, hence

(3.5) $$F_0(u,A) = F(u,A) .$$

By (iii) there exists a family $\mathcal{R}$, rich in $\mathcal{A}$, such that

$$F(u,A') + G_\mu(u,A') \le \liminf_{h\to+\infty} [F(u_h,A') + G_{\mu_h}(u_h,A')]$$

for every $A' \in \mathcal{R}$ with $A' \subseteq A$. By taking the supremum over all such A' we get

$$F(u,A) + G_\mu(u,A) \le \liminf_{h\to+\infty} [F(u_h,A) + G_{\mu_h}(u_h,A)] \le H'(u,A) .$$

This inequality together with (3.5) yields (3.4).

Let us prove that

(3.6) $$H''(u,A) \le F_0(u,A) + G_\mu(u,A) .$$

Fix $A \in \mathcal{A}$ and $u \in L^p(A)$ such that $F_0(u,A) + G_\mu(u,A) < +\infty$, so that $u \in H_0^{1,p}(A)$

and $F_0(u,A) = F(u,A)$. To prove (3.6) it is enough to show that for every $\eta > 0$ there exists a sequence (u_h) in $H_0^{1,p}(A)$ converging to u in $L^p(A)$ such that

$$(3.7) \qquad F(u,A) + G_\mu(u,A) + \eta \geq \limsup_{h\to+\infty} [F(u_h,A) + G_{\mu_h}(u_h,A)] .$$

We first consider the special case $\operatorname{spt} u \subseteq A$. By (iii) for every $A' \in \mathcal{R}$, with $A' \subset\subset A$, there exists a sequence (w_h) in $H^{1,p}(A')$ which converges to u in $L^p(A')$ and satisfies

$$F(u,A') + G_\mu(u,A') \geq \limsup_{h\to\infty} [F(w_h,A') + G_{\mu_h}(w_h,A')] .$$

To construct the sequence (u_h) which fulfils (3.7) we use the J-property introduced in [11], Definition 2.2, which holds uniformly for the sequence $F + G_{\mu_h}$ (see Theorem 6.1 and Proposition 2.6 of [11]).

Let us fix $\varepsilon > 0$ and $A' \in \mathcal{R}$ with $A' \subset\subset A$ and choose a compact set K such that $\operatorname{spt} u \subseteq K \subseteq A' \subset\subset A$. By applying the J-property of $F + G_{\mu_h}$ to connect the functions w_h (on A') and 0 (on $A\setminus K$), we obtain a constant $M > 0$ and a sequence (u_h) in $H_0^{1,p}(A)$ converging to u in $L^p(A)$ such that

$$F(u_h,A) + G_{\mu_h}(u_h,A) \leq (1+\varepsilon)[F(w_h,A') + G_{\mu_h}(w_h,A')] + \varepsilon[\|w_h\|^p_{L^p(A')} + 1] + \\ + M\|w_h\|^p_{L^p(A'\setminus K)}$$

for every $h \in \mathbf{N}$. It follows that

$$\limsup_{h\to+\infty} [F(u_h,A) + G_{\mu_h}(u_h,A)] \leq (1+\varepsilon)[F(u,A') + G_\mu(u,A')] + \varepsilon[\|u\|^p_{L^p(A')} + 1] \\ \leq (1+\varepsilon)[F(u,A) + G_\mu(u,A)] + \varepsilon[\|u\|^p_{L^p(A)} + 1] .$$

Since ε can be choosen arbitrarily small, we obtain (3.7), and hence (3.6), under the additional assumption that $\operatorname{spt} u \subseteq A$.

To prove (3.6) in the general case $u \in H_0^{1,p}(A)$ we observe that there exists a sequence (v_h) in $H_0^{1,p}(A)$ with $\operatorname{spt} v_h \subseteq A$ such that (v_h) converges to u in $H_0^{1,p}(A)$ and $|v_h|^p \uparrow |u|^p$ p-q.e. on A. By applying the previous result to v_h we get

$$H''(v_h,A) \leq F(v_h,A) + G_\mu(v_h,A)$$

for every $h \in \mathbf{N}$. By the lower semicontinuity of $H''(\cdot,A)$ on $L^p(A)$ it follows that

$$H''(u,A) \leq \liminf_{h\to+\infty} H''(v_h,A) \leq \lim_{h\to+\infty} [F(v_h,A) + G_{\mu_h}(v_h,A)] .$$

Since the functional $F(\cdot,A)$ is continuous in the strong topology of $H^{1,p}(A)$ and

$G_\mu(v_h,A)$ converges to $G_\mu(u,A)$ as $h\to+\infty$ by Beppo Levi's theorem, we conclude that

$$H''(u,A) \leq F(u,A) + G_\mu(u,A) \ ,$$

which implies (3.6). The proof of (iii) $\Rightarrow$ (ii) is so accomplished.

(ii) $\Rightarrow$ (i) : By taking $A = \Omega$ in (ii) we get immediately (i).

(i) $\Rightarrow$ (iii) : By Theorem 2.1 for every subsequence $(\mu_{\sigma(h)})$ of (μ_h) there exist a subsequence $(\mu_{\sigma(\tau(h))})$ of $(\mu_{\sigma(h)})$, a measure $\nu\in\mathcal{M}_p(\Omega)$ and a family $\mathcal{R}$, rich in $\mathcal{A}$, such that for every $A\in\mathcal{R}$

$$\text{(3.8)} \qquad [F(\cdot,A) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\nu(\cdot,A)] \quad \text{in } L^p(A) \ .$$

Since (iii) implies (i) it follows that

$$[F_0(\cdot,\Omega) + G_{\mu_{\sigma(\tau(h))}}(\cdot,\Omega)] \quad \Gamma\text{-converges to} \quad [F_0(\cdot,\Omega) + G_\nu(\cdot,\Omega)] \quad \text{in } L^p(\Omega) \ .$$

By assumption (i) we get then $G_\nu(u,\Omega) = G_\mu(u,\Omega)$ for every $u\in H_0^{1,p}(\Omega)$ which implies that $G_\nu(u,A) = G_\mu(u,A)$ for every $A\in\mathcal{A}$ and for every $u\in H^{1,p}(A)$. By taking this into account in (3.8) we obtain that

$$[F(\cdot,A) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A\in\mathcal{R}$. Since the limit functional does not depend on the subsequence, property (iii) follows immediately from Proposition 4.14 of [11].

The proof of Theorem 3.2 is so accomplished. ∎

An immediate consequence of Theorems 3.2 and 2.1 is the following result which asserts that the class of measures $\mathcal{M}_p(\Omega)$ is sequentially compact under the γ_f-convergence.

Theorem 3.3. *For every sequence (μ_h) in $\mathcal{M}_p(\Omega)$ there exists a subsequence $(\mu_{\sigma(h)})$ which γ_f-converges to a measure μ of the class $\mathcal{M}_p(\Omega)$.*

The notion of γ_f-convergence is defined by means of the functionals $F_0 + G_\mu$. Clearly two measures μ and ν may give rise to the same functional (see [12], Example 4.5). This leads to the following defintion.

Definition 3.4. We say that two measures $\mu, \nu \in\mathcal{M}_p(\Omega)$ are *equivalent* if $G_\mu(u,\Omega) = G_\nu(u,\Omega)$ for every $u\in H_0^{1,p}(\Omega)$.

It is easy to see that μ and ν are equivalent if and only if $G_\mu(u,A) = G_\nu(u,A)$ for every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$. Moreover, by adapting the proof of Theorem 2.6 of [8], we can show that μ and ν are equivalent if and only if $\mu(U) = \nu(U)$ for every p-quasi open set $U \subseteq \Omega$.

In the next theorem we still denote by $\mathcal{M}_p(\Omega)$ the quotient space with respect to the equivalence relation introduced in Defintion 3.4, and we identify each measure with its equivalence class. Note that the defintion of γ_f-convergence is clearly independent of the choice of μ in its equivalence class in $\mathcal{M}_p(\Omega)$.

Theorem 3.5. *The γ_f-convergence is metrizable on $\mathcal{M}_p(\Omega)$.*

Proof. We shall use the following general result for the Γ-convergence (see [1], Section 2.8.3). Let X be a separable metric space and let $\mathcal{S}(X,\psi)$ be the family of all lower semicontinuous functions $F : X \to \overline{\mathbf{R}}$ such that $F(v) \geq \psi(v)$ for every $v \in X$, where $\psi : X \to \overline{\mathbf{R}}$ is a given lower semicontinuous coercive function. Then the Γ-convergence in $\mathcal{S}(X,\psi)$ is metrizable, that is, there exists a metric d in $\mathcal{S}(X,\psi)$ such that (F_h) Γ-converges to F if and only if $d(F_h,F) \to 0$.

The metrizability of $\mathcal{M}_p(\Omega)$ can now be obtained by identifying each element μ of $\mathcal{M}_p(\Omega)$ with the corresponding functional $F_0(\cdot,\Omega) + G_\mu(\cdot,\Omega)$ defined on $L^p(\Omega)$. ∎

4. Localization and Boundary Conditions

In the first part of this section we aim to prove a localization property for measures on $\mathcal{M}_p(\Omega)$ with respect to the γ_f-convergence. More precisely, we shall establish the following theorem.

Theorem 4.1. *Let (μ_h) be a sequence in $\mathcal{M}_p(\Omega)$ which γ_f-converges to $\mu \in \mathcal{M}_p(\Omega)$. Then there exists a family $\mathcal{R}'$, rich in $\mathcal{A}$, such that*

$$[F(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,\Omega) + G_\mu(\cdot,A)] \quad \text{in } L^p(\Omega)$$

for every $A \in \mathcal{R}'$.

To prove this result we introduce the functionals H' and H'' defined for every $A,B \in \mathcal{A}$ with $A \subseteq B$ and for every $u \in L^p(B)$ by

$$(4.1) \qquad H'(u,B,A) = \inf\{ \liminf_{h\to+\infty} [F(u_h,B) + G_{\mu_h}(u_h,A)] : u_h \to u \text{ in } L^p(B) \} ,$$

$$(4.2) \qquad H''(u,B,A) = \inf\{ \limsup_{h\to+\infty} [F(u_h,B) + G_{\mu_h}(u_h,A)] : u_h \to u \text{ in } L^p(B) \} .$$

Moreover, for every $u \in H^{1,p}(B)$ we set

(4.3) $$G'(u,B,A) = H'(u,B,A) - F(u,B) ,$$

(4.4) $$G''(u,B,A) = H''(u,B,A) - F(u,B) .$$

Note that the infima in (4.1) and (4.2) are actually achieved, as one can easily see by a diagonal argument.

In the next lemma we collect some properties of the functionals G' and G", which imply immediately Theorem 4.1.

Lemma 4.2. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *which* γ_f*-converges to* $\mu \in \mathcal{M}_p(\Omega)$*. Let* $A_1, A_2 \in \mathcal{A}$ *with* $A_1 \subset\subset A_2$ *and* $u \in H^{1,p}(\Omega)$*. Then*

(4.5) $$G_\mu(u,A_1) \le G'(u,\Omega,A_2) \le G''(u,\Omega,A_2) ,$$

(4.6) $$G'(u,\Omega,A_1) \le G''(u,\Omega,A_1) \le G_\mu(u,A_2) .$$

Proof. Let us prove (4.5). Let A_1, A_2 and u be as required in the lemma. The inequality $G'(u,\Omega,A_2) \le G''(u,\Omega,A_2)$ is trivial. Let us prove that $G_\mu(u,A_1) \le G'(u,\Omega,A_2)$. By (4.3) and (4.1) there exists (u_h) in $H^{1,p}(\Omega)$ converging to u in $L^p(\Omega)$ such that

(4.7) $$F(u,\Omega) + G'(u,\Omega,A_2) = \liminf_{h\to+\infty} [F(u_h,\Omega) + G_{\mu_h}(u_h,A_2)] .$$

We may assume that the right hand side of the equality is finite and that the lower limit is a limit, so that the sequence (u_h) converges to u weakly in $H^{1,p}(\Omega)$ by the coerciveness of F. Since the function

$$u \to \int_B f(x,Du)\,dx$$

is lower semicontinuous in the weak topology of $H^{1,p}(\Omega)$ for every $B \in \mathcal{B}$, we have

(4.8) $$\int_{\Omega\setminus A'} f(x,Du)\,dx \le \liminf_{h\to+\infty} \int_{\Omega\setminus A'} f(x,Du_h)\,dx$$

for every $A' \in \mathcal{A}$. On the other hand, by Theorem 3.2 there exists a family $\mathcal{R}$, rich in $\mathcal{A}$, such that for every $A' \in \mathcal{R}$

(4.9) $$F(u,A') + G_\mu(u,A') \le \liminf_{h\to+\infty} [F(u_h,A') + G_{\mu_h}(u_h,A')] .$$

By adding (4.8) and (4.9) we get immediately

$$F(u,\Omega) + G_\mu(u,A') \le \liminf_{h\to+\infty} [F(u_h,\Omega) + G_{\mu_h}(u_h,A_2)] = F(u,\Omega) + G'(u,\Omega,A_2)$$

for every $A' \in \mathcal{R}$, $A' \subseteq A_1$. Since $G_\mu(u,\cdot)$ is a measure, by taking the supremum in

A' we obtain finally

$$G_\mu(u,A_1) \leq G'(u,\Omega,A_2) \quad ,$$

which concludes the proof of (4.5).

Let us prove (4.6). The inequality $G'(u,\Omega,A_1) \leq G''(u,\Omega,A_1)$ is trivial. It remains to prove that $G''(u,\Omega,A_1) \leq G_\mu(u,A_2)$.

Let $\mathcal{R}$ be the family, rich in $\mathcal{A}$, given by Theorem 3.2. Thus, for every $A'\in\mathcal{R}$ with $A_1 \subset\subset A' \subset\subset A_2$ we have

$$G''(u,A',A') = G_\mu(u,A') \leq G_\mu(u,A_2) \ .$$

By the monotonicity of the function $G''(u,A',\cdot)$ the proof of (4.6) will be accomplished if we show that

$$G''(u,\Omega,A_1) \leq G''(u,A',A_1) \ . \tag{4.10}$$

Let (w_h) be a sequence in $H^{1,p}(\Omega)$ converging to u in $L^p(A')$ such that

$$F(u,A') + G''(u,A',A_1) = \limsup_{h\to+\infty} [F(w_h,A') + G_{\mu_h}(w_h,A_1)] \ . \tag{4.11}$$

Fix $\varepsilon > 0$ and let K be a compact set with $A_1 \subseteq K \subseteq A'$ and $F(u,A'\setminus K) < \varepsilon$. Again by the J-property of F (see [11], Theorem 6.1) there exist a constant $M > 0$ and a sequence (u_h) of functions in $H^{1,p}_0(\Omega)$ converging to u in $L^p(\Omega)$ such that $u_h = w_h$ on a neighborhood of K, $u_h = u$ on $\Omega\setminus A'$ and

$$F(u_h,\Omega) \leq (1+\varepsilon)[F(w_h,A') + F(u,\Omega\setminus K)] + \varepsilon(\|w_h\|^p_{L^p(A')} + \|u\|^p_{L^p(\Omega\setminus K)} + 1) + \\ + M\|w_h - u\|^p_{L^p(A'\setminus K)} \tag{4.12}$$

for every $h\in\mathbf{N}$. By the Γ-convergence and by (4.12) we get

$$\begin{aligned} F(u,\Omega) + G''(u,\Omega,A_1) &\leq \liminf_{h\to+\infty} [F(u_h,\Omega) + G_{\mu_h}(u_h,A_1)] \\ &\leq (1+\varepsilon) \limsup_{h\to+\infty} [F(w_h,A') + G_{\mu_h}(w_h,A_1)] + (1+\varepsilon)F(u,\Omega\setminus K) + \\ &\quad + \varepsilon(2\|u\|^p_{L^p(\Omega)} + 1) \\ &\leq (1+\varepsilon) \limsup_{h\to+\infty} [F(w_h,A') + G_{\mu_h}(w_h,A_1)] + (1+\varepsilon)\varepsilon + \\ &\quad + (1+\varepsilon)F(u,\Omega\setminus A') + \varepsilon(2\|u\|^p_{L^p(\Omega)} + 1) \ . \end{aligned}$$

By (4.11) it follows that

$$F(u,\Omega) + G''(u,\Omega,A_1) \leq (1+\varepsilon)[F(u,\Omega) + G''(u,A',A_1)] + (1+\varepsilon)\varepsilon + \varepsilon(2\|u\|^p_{L^p(\Omega)} + 1) \ .$$

Since ε is arbitrarily small, it follows $G''(u,\Omega,A_1) \le G''(u,A',A_1)$ which concludes the proof of (4.10) and therefore of the lemma. ∎

Proof of Theorem 4.1. By Lemma 4.2 we obtain that

$$G_\mu(u,A) = \sup\{G'(u,\Omega,A') : A' \in \mathcal{A}\,,\ A' \subset\subset A\} = $$
$$= \sup\{G''(u,\Omega,A') : A' \in \mathcal{A}\,,\ A' \subset\subset A\}$$

for every $A \in \mathcal{A}$ and $u \in H^{1,p}(\Omega)$. The functionals $G_\mu(u,A)$, $G'(u,\Omega,A)$, $G''(u,\Omega,A)$ are increasing with respect to A and lower semicontinuous with respect to u on $H^{1,p}(\Omega)$. Therefore, by Proposition 1.14 of [11] there exists a family $\mathcal{R}'$, rich in $\mathcal{A}$, such that

$$G_\mu(u,A) = G'(u,\Omega,A) = G''(u,\Omega,A)$$

for every $A \in \mathcal{R}'$ and for every $u \in H^{1,p}(\Omega)$. By the definitions of G' and G'', these equalities are equivalent to the assertion of the theorem. ∎

We now take into account non-homogeneous boundary conditions on $\partial\Omega$.

Let $\varphi \in H^{1,p}(\Omega)$. For every $A \in \mathcal{A}$ and for every $u \in L^p(A)$ we define

$$(4.13) \qquad F_\varphi(u,A) = \begin{cases} F(u,A) & \text{if } u - \varphi \in H_0^{1,p}(A)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Then the following theorem holds.

Theorem 4.3. *Let* $\varphi \in H^{1,p}(\Omega)$. *Fix* $A \subset\subset \Omega$ *and let* (μ_h) *and* μ *be measures on* $\mathcal{M}_p(\Omega)$ *such that*

$$[F(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,\Omega) + G_\mu(\cdot,A)] \quad \text{in } L^p(\Omega)\,.$$

Then

$$[F_\varphi(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_\varphi(\cdot,\Omega) + G_\mu(\cdot,A)] \quad \text{in } L^p(\Omega)\,.$$

Proof. We shall prove first that, given $u \in L^p(\Omega)$, there exists a sequence (u_h) converging to u in $L^p(\Omega)$ such that

$$(4.14) \qquad F_\varphi(u,\Omega) + G_\mu(u,A) \ge \limsup_{h\to+\infty} [F_\varphi(u_h,\Omega) + G_{\mu_h}(u_h,A)]\,.$$

We may assume that the left hand side of (4.14) is finite, which implies by (4.13) that

$u-\varphi \in H_0^{1,p}(\Omega)$, and therefore $u \in H^{1,p}(\Omega)$. Now, by assumption there exists a sequence (v_h) converging to u in $L^p(\Omega)$ such that

$$F_\varphi(u,\Omega) + G_\mu(u,A) = \lim_{h\to+\infty} [F(v_h,\Omega) + G_{\mu_h}(v_h,A)] \ . \tag{4.15}$$

Fix $\varepsilon > 0$ and let K be a compact set with $A \subseteq K \subseteq \Omega$ such that $F(u,\Omega\setminus K) < \varepsilon$. Moreover, let A' be an open set with $K \subseteq A' \subset\subset \Omega$. By the J-property of F (see [11], Theorem 6.1) there exist a constant $M > 0$ and a sequence (u_h) of functions in $H_0^{1,p}(\Omega)$ converging to u in $L^p(\Omega)$ such that $u_h = v_h$ on a neighborhood of K, $u_h = u$ on $\Omega\setminus A'$ (and therefore $u_h-\varphi \in H_0^{1,p}(\Omega)$), and

$$F(u_h,\Omega) \le (1+\varepsilon)[F(v_h,\Omega) + F(u,\Omega\setminus K)] + \varepsilon(\|v_h\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} + 1) + \\ + M\|v_h - u\|^p_{L^p(\Omega)} \ .$$

This inequality together with (4.15) yields

$$\begin{aligned} \limsup_{h\to+\infty} [F_\varphi(u_h,\Omega) + G_{\mu_h}(u_h,A)] &\le (1+\varepsilon) \limsup_{h\to+\infty} [F(v_h,\Omega) + G_{\mu_h}(v_h,A)] + \\ &\quad + (1+\varepsilon)F(u,\Omega\setminus K) + \varepsilon[2\|u\|^p_{L^p(\Omega)} + 1] \\ &\le (1+\varepsilon)[F_\varphi(u,\Omega) + G_\mu(u,A)] + (1+\varepsilon)F(u,\Omega\setminus K) + \\ &\quad + \varepsilon[2\|u\|^p_{L^p(\Omega)} + 1] \ . \end{aligned}$$

Since $F(u,\Omega\setminus K) < \varepsilon$ and $\varepsilon > 0$ is arbitrary, we get immediately (4.14).

It remains to prove that for every $u \in L^p(\Omega)$ and for every sequence (u_h) converging to u in $L^p(\Omega)$ we have

$$F_\varphi(u,\Omega) + G_\mu(u,A) \le \liminf_{h\to+\infty} [F_\varphi(u_h,\Omega) + G_{\mu_h}(u_h,A)] \ . \tag{4.16}$$

Let $u \in L^p(\Omega)$ and (u_h) be a sequence in $L^p(\Omega)$ converging to u in $L^p(\Omega)$. We may assume that the right hand side of (4.16) is finite and that the lower limit is a limit. By passing, if necessary, to a subsequence, we may assume that (u_h) converges to u weakly in $H^{1,p}(\Omega)$ by the coerciveness of F. Since $u_h-\varphi \in H_0^{1,p}(\Omega)$ we get $u-\varphi \in H_0^{1,p}(\Omega)$, so (4.16) follows easily from the definition of F_φ and from our assumption concerning the Γ-convergence of $F(\cdot,\Omega) + G_{\mu_h}(\cdot,A)$. ∎

The next theorem collects some conditions which are equivalent to the γ_f-convergence in $\mathcal{M}_p(\Omega)$.

Theorem 4.4. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$, *let* $\mu\in\mathcal{M}_p(\Omega)$, *and let* $\varphi\in H^{1,p}(\Omega)$. *Then the following conditions are equivalent:*

(i) (μ_h) γ_f*-converges to* μ ;

(ii) *for every* $A\in\mathcal{A}$

$$[F_0(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_0(\cdot,A) + G_\mu(\cdot,A)] \quad in\ L^p(A)\ ;$$

(iii) *there exists a family* $\mathcal{R}$, *rich in* $\mathcal{A}$, *such that for every* $A\in\mathcal{R}$

$$[F(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad in\ L^p(A)\ ;$$

(iv) *there exists a family* $\mathcal{R}'$, *rich in* $\mathcal{A}$, *such that for every* $A\in\mathcal{R}'$

$$[F(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,\Omega) + G_\mu(\cdot,A)] \quad in\ L^p(\Omega)\ ;$$

(v) *there exists a family* $\mathcal{R}'$, *rich in* $\mathcal{A}$, *such that for every* $A\in\mathcal{R}'$, $A\subset\subset\Omega$,

$$[F_\varphi(\cdot,\Omega) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_\varphi(\cdot,\Omega) + G_\mu(\cdot,A)] \quad in\ L^p(\Omega)\ .$$

Proof. By Theorem 3.2 follows that the conditions (i), (ii), and (iii) are equivalent. Theorem 4.1 guarantees that (i) implies (iv), while (v) follows from (iv) by Theorem 4.3. To conclude the proof of the theorem we shall show that (v) implies (iii). By Theorem 2.1 for every subsequence $(\mu_{\sigma(h)})$ of (μ_h) there exist a subsequence $(\mu_{\sigma(\tau(h))})$ of $(\mu_{\sigma(h)})$, a measure $\nu\in\mathcal{M}_p(\Omega)$, and a family $\tilde{\mathcal{R}}$, rich in $\mathcal{A}$, such that

$$(4.17) \qquad [F(\cdot,A) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\nu(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A\in\tilde{\mathcal{R}}$. Since (iii) implies (v), there exists a family $\mathcal{R}''$, rich in $\mathcal{A}$, such that

$$(4.18) \qquad [F_\varphi(\cdot,\Omega) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_\varphi(\cdot,\Omega) + G_\nu(\cdot,A)] \quad \text{in } L^p(\Omega)$$

for every $A\in\mathcal{R}''$, $A\subset\subset\Omega$. On the other hand, by assumption (v) there exists a family $\mathcal{R}'$, rich in $\mathcal{A}$, such that

$$(4.19) \qquad [F_\varphi(\cdot,\Omega) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F_\varphi(\cdot,\Omega) + G_\mu(\cdot,A)] \text{ in } L^p(\Omega)\ .$$

for every $A\in\mathcal{R}'$, with $A\subset\subset\Omega$. By (4.18) and (4.19) we have

$$(4.20) \qquad G_\mu(u,A) = G_\nu(u,A)$$

for every $A \in \mathcal{R}' \cap \mathcal{R}''$, $A \subset\subset \Omega$, and for every $u \in \varphi + H_0^{1,p}(\Omega)$.

We prove now that (4.20) holds for every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$. Let us fix A and u as required. For every $A \in \mathcal{R}' \cap \mathcal{R}''$, with $A' \subset\subset A$, there exists $u' \in \varphi + H_0^{1,p}(\Omega)$ such that $u' = u$ on A'. Since the functionals G_μ and G_ν are local, by (4.20) we get $G_\mu(u,A') = G_\mu(u',A') = G_\nu(u',A') = G_\nu(u,A')$. By taking the limit as $A' \uparrow A$ we obtain

$$G_\mu(u,A) = G_\nu(u,A) \tag{4.21}$$

for every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$.

By (4.17) and (4.21)

$$[F(\cdot,A) + G_{\mu_{\sigma(\tau(h))}}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A \in \tilde{\mathcal{R}}$. Since the limit does not depend on the subsequence, by Proposition 4.14 of [11] we conclude that there exists a family $\mathcal{R}$, rich in $\mathcal{A}$, such that

$$[F(\cdot,A) + G_{\mu_h}(\cdot,A)] \quad \Gamma\text{-converges to} \quad [F(\cdot,A) + G_\mu(\cdot,A)] \quad \text{in } L^p(A)$$

for every $A \in \mathcal{R}$.

The proof of Theorem 4.4 is so accomplished. ∎

Finally, from the properties of the Γ-convergence we derive some variational properties of the γ_f-convergence.

For every $\mu \in \mathcal{M}_p(\Omega)$, $A \in \mathcal{A}$, and $g \in L^q(A)$, $1/p + 1/q = 1$, we denote by $m(\mu,g,A)$ and $M(\mu,g,A)$ respectively the minimum value and the set of mimimum points of the problem

$$\min_{u \in H_0^{1,p}(A)} \{ \int_A f(x,Du)\,dx + \int_A |u|^p\,d\mu + \int_A gu\,dx \} .$$

By applying Theorem 4.4 (the equivalence between (i) and (ii)) and Proposition 1.2 we get to our next result.

Theorem 4.5. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *which* γ_f*-converges to* $\mu \in \mathcal{M}_p(\Omega)$. *Then for every* $A \in \mathcal{A}$ *and for every* $g \in L^q(A)$, $1/p + 1/q = 1$, *the following properties hold:*

(i) $\lim_{h\to+\infty} m(\mu_h,g,A) = m(\mu,g,A)$;

(ii) for every neighborhood U *of* $M(\mu,g,A)$ *in* $L^p(A)$ *there exists* $k \in \mathbf{N}$ *such that* $M(\mu_h,g,A) \subseteq U$ *for every* $h \geq k$.

Remark 4.6. It is clear that results analogous to those of Theorem 4.5 can also be achieved for the minimum problems associated to the other functionals considered in Theorem 4.4. For example, condition (v) of Theorem 4.4 implies that

$$\lim_{h\to+\infty} \{ \min_{u-\varphi\in H_0^{1,p}(\Omega)} [F(u,\Omega) + G_{\mu_h}(u,A)] \} = \min_{u-\varphi\in H_0^{1,p}(\Omega)} [F(u,\Omega) + G_\mu(u,A)]$$

for every $A\in\mathcal{R}'$ with $A \subset\subset \Omega$.

5. γ_f-convergence and μ-capacity

In this section we establish the equivalence between the γ_f-convergence of a sequence of measures (μ_h) of $\mathcal{M}_p(\Omega)$ and the weak convergence (in the sense of [15]) of the corresponding capacities $C(f, \mu_h,\cdot)$.

According to [9], Section 3, for every $\mu\in\mathcal{M}_p(\Omega)$ and for every $B\in\mathcal{B}_0$ the μ-capacity of B, relative to f, is defined by

$$(5.1) \qquad C(f,\mu,B) = \min \{ \int_\Omega f(x,Du)\, dx + \int_B |u|^p\, d\mu : u - 1 \in H_0^{1,p}(\Omega) \} .$$

For every $\mu\in\mathcal{M}_p(\Omega)$ the set function $C(f,\mu,\cdot)$ is non-negative, increasing, and countably subadditive on $\mathcal{B}_0$. Moreover, it is strongly subadditive and continuous along increasing sequences in $\mathcal{B}_0$ (for a review on the properties of the μ-capacity we refer to [9], Theorem 3.2).

The measure μ is uniquely determined by $C(f,\mu,\cdot)$. In fact, as proved in [9], Theorem 4.2, μ is the least measure greater than or equal to $C(f,\mu,\cdot)$ on $\mathcal{B}$; therefore for every $B\in\mathcal{B}$

$$(5.2) \qquad \mu(B) = \sup \sum_{i\in I} C(f,\mu,B_i) \quad ,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B.

By Remark 4.6 it follows immediately that the γ_f-convergence of a sequence (μ_h) implies the convergence of the sequence of the corresponding μ-capacities $C(f, \mu_h,\cdot)$ on a family which is rich in $\mathcal{A}$. This allows us to obtain the following result.

Theorem 5.1. *Let (μ_h) be a sequence in $\mathcal{M}_p(\Omega)$ which γ_f-converges to μ in $\mathcal{M}_p(\Omega)$. Then*

$$(5.3) \qquad C(f,\mu,A) \le \liminf_{h\to+\infty} C(f,\mu_h,A)$$

and

(5.4) $$C(f,\mu,A) \geq \limsup_{h\to+\infty} C(f,\mu_h,K)$$

for every $A\in\mathcal{A}$ *and for every* $K\in\mathcal{K}$ *with* $K\subseteq A$.

Proof. Let A and K be as required in the theorem. By Remark 4.6 there exists a family $\mathcal{R}'$, rich in $\mathcal{A}$, such that for every $A'\in\mathcal{R}'$

$$C(f,\mu,A') = \lim_{h\to+\infty} C(f,\mu_h,A') .$$

For every $A'\in\mathcal{R}'$ with $A'\subset\subset A$ we have

$$C(f,\mu,A') \leq \liminf_{h\to+\infty} C(f,\mu_h,A) ,$$

which implies immediately (5.3) by the continuity properties of the μ-capacity.

On the other hand, for $A'\in\mathcal{R}'$ with $K\subseteq A'\subset\subset A$ it follows that

$$C(f,\mu,A) \geq C(f,\mu,A') = \lim_{h\to+\infty} C(f,\mu_h,A') \geq \limsup_{h\to+\infty} C(f,\mu_h,K) ,$$

which proves (5.4). ∎

The next corollary follows easily from Theorem 5.1 and from the continuity properties of the μ-capacity mentioned at the beginning of this section.

Corollary 5.2. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *which* γ_f*-converges to* μ *in* $\mathcal{M}_p(\Omega)$. *Then*

(5.5) $$\sup_{\substack{K\subseteq A\\ K\in\mathcal{K}}} \liminf_{h\to+\infty} C(f,\mu_h,K) = \sup_{\substack{K\subseteq A\\ K\in\mathcal{K}}} \limsup_{h\to+\infty} C(f,\mu_h,K) = C(f,\mu,A)$$

for every $A\in\mathcal{A}$.

To identify the set function $C(f,\mu,\cdot)$ on $\mathcal{B}_0$ we introduce a class of measures contained in $\mathcal{M}_p(\Omega)$.

Definition 5.3. We denote by $\overset{*}{\mathcal{M}}_p(\Omega)$ the class of all measures $\mu\in\mathcal{M}_p(\Omega)$ such that

$$\mu(B) = \inf\{\mu(U) : U \text{ p-quasi open}, B\subseteq U\}$$

for every $B\in\mathcal{B}$.

In the case $p = 2$ the properties of the class $\overset{*}{\mathcal{M}}_p(\Omega)$ have been studied in [8], Section 3. Analogous properties can be obtained without any difficulty for $1 < p \leq n$ and shall be summarized in Propositions 5.5 and 5.7.

Definition 5.4. Let $\mu \in \mathcal{M}_p(\Omega)$. We denote by μ^* the set function defined by

$$\mu^*(B) = \inf\{\mu(U) : U \text{ p-quasi open}, B \subseteq U\}$$

for every $B \in \mathcal{B}$.

As in Theorems 3.9 and 3.10 of [8] we obtain the following proposition.

Proposition 5.5. *Let $\mu \in \mathcal{M}_p(\Omega)$. Then the set function μ^* is a Borel measure which belongs to $\mathcal{M}^*_p(\Omega)$ and μ^* is equivalent to μ, i.e.*

$$\int_A |u|^p \, d\mu = \int_A |u|^p \, d\mu^* \tag{5.6}$$

for every $A \in \mathcal{A}$ and for every $u \in H^{1,p}(A)$.

Remark 5.6. By (5.6) and (5.1) we have $C(f,\mu,A) = C(f,\mu^*,A)$ for every $A \in \mathcal{A}$. Furthermore, (5.6) implies that a sequence (μ_h) in $\mathcal{M}_p(\Omega)$ γ_f-converges to $\mu \in \mathcal{M}_p(\Omega)$ if and only if (μ_h) γ_f-converges to μ^*.

Let us finally analyze the relationship between $C(f,\mu,\cdot)$ and $C(f,\mu^*,\cdot)$ on $\mathcal{B}_0$. As in Proposition 3.11 of [8] we obtain the following result.

Proposition 5.7. *Let $\mu \in \mathcal{M}_p(\Omega)$. Then*

$$C(f,\mu^*,B) = \inf\{C(f,\mu^*,A) : A \in \mathcal{A}, B \subseteq A\} = \inf\{C(f,\mu,A) : A \in \mathcal{A}, B \subseteq A\} \tag{5.7}$$

for every $B \in \mathcal{B}_0$.

We now come to the main result of this section.
Let (μ_h) be a sequence of measures in $\mathcal{M}_p(\Omega)$. For every $K \in \mathcal{K}$ we define

$$\alpha'(K) = \liminf_{h \to +\infty} C(f,\mu_h,K) ,$$

$$\alpha''(K) = \limsup_{h \to +\infty} C(f,\mu_h,K) .$$

For every $A \in \mathcal{A}$ we set

$$\beta'(A) = \sup\{\alpha'(K) : K \in \mathcal{K}, K \subseteq A\} ,$$

$$\beta''(A) = \sup\{\alpha''(K) : K \in \mathcal{K}, K \subseteq A\} ,$$

and for every $B \in \mathcal{B}$ we define

$$\beta'(B) = \inf\{\beta'(A) : A \in \mathcal{A}, B \subseteq A\} , \tag{5.8}$$

$$\beta''(B) = \inf\{\beta''(A) : A \in \mathcal{A}, B \subseteq A\} . \tag{5.9}$$

Then the following characterization of the γ_f-convergence in $\mathcal{M}_p(\Omega)$ holds.

Theorem 5.8. *Let* (μ_h) *be a sequence in* $\mathcal{M}_p(\Omega)$ *and let* β' *and* β'' *be the set functions defined by* (5.8) *and* (5.9). *Then* (μ_h) γ_f*-converges to a measure* μ *in* $\mathcal{M}_p(\Omega)$ *if and only if* $\beta' = \beta''$ *on* $\mathcal{B}$. *In this case, for every* $B \in \mathcal{B}$ *we have* $\beta'(B) = \beta''(B) = C(f,\mu^*,B)$ *and*

$$(5.10) \qquad \mu^*(B) = \sup \sum_{i\in I} \beta'(B_i) \quad ,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ *of* B.

Proof. Let (μ_h) be a sequence in $\mathcal{M}_p(\Omega)$ which γ_f-converges to $\mu \in \mathcal{M}_p(\Omega)$. By Corollary 5.2 we obtain

$$(5.11) \qquad C(f,\mu,A) = \beta'(A) = \beta''(A)$$

for every $A \in \mathcal{A}$. By taking (5.11) into account, from Proposition 5.7 together with (5.8) and (5.9) we get

$$C(f,\mu^*,B) = \inf\{\beta'(A) : A \in \mathcal{A}, B \subseteq A\} = \inf\{\beta''(A) : A \in \mathcal{A}, B \subseteq A\}$$
$$= \beta'(B) = \beta''(B)$$

for every $B \in \mathcal{B}$. Finally, (5.10) follows from (5.2) applied to μ^*.

Let now (μ_h) be a sequence in $\mathcal{M}_p(\Omega)$ and suppose that $\beta' = \beta''$ on $\mathcal{B}$. Let us define the measure μ by the formula

$$\mu(B) = \sup \sum_{i\in I} \beta'(B_i) \quad ,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B. Furthermore, since the γ_f-convergence on $\mathcal{M}_p(\Omega)$ is compact and metrizable (Theorems 3.3 and 3.5), we may assume that (μ_h) γ_f-converges to a measure $\nu \in \mathcal{M}_p(\Omega)$. Since β' and β'' do not change if we pass to a subsequence of (μ_h), by Corollary 5.2 we have $\beta'(A) = \beta''(A) = C(f,\nu,A)$ for every $A \in \mathcal{A}$; hence $\beta' = \beta'' = C(f,\nu^*,\cdot)$ on $\mathcal{B}$ by (5.7), (5.8), and (5.9). By applying (5.2) to ν^*, we obtain that ν^* is the least measure greater than or equal to β' on $\mathcal{B}$. By definition of μ we have to conclude that $\nu^* = \mu$. Therefore $\mu^* = \mu$ and Remark 5.6 implies that (μ_h) γ_f-converges to μ in $\mathcal{M}_p(\Omega)$. The proof of Theorem 5.8 is so accomplished. ∎

6. Nonlinear Dirichlet Problems on Varying Open Sets

We may now apply the results obtained in the previous sections to analyze the asymptotic behavior of sequences of nonlinear variational problems in varying open sets with Dirichlet boundary conditions of the form

$$(6.1) \qquad \min_{u\in H_0^{1,p}(\Omega\setminus E_h)} \{ \int_{\Omega\setminus E_h} f(x,Du)\,dx + \int_{\Omega\setminus E_h} gu\,dx \} ,$$

where (E_h) is a sequence of closed subsets of Ω and $g\in L^q(\Omega)$ with $1/p + 1/q = 1$.

We indicate by $m_h(g)$ and $M_h(g)$ respectively the minimum value and the set of all minimum points of problem (6.1) and we identify each $u\in H_0^{1,p}(\Omega\setminus E_h)$ with the function of $H_0^{1,p}(\Omega)$ obtained by the usual extension $u = 0$ on E_h.

To put this study in the general setting, for every $E\in \mathcal{B}$ we consider the Borel measure ∞_E defined by

$$(6.2) \qquad \infty_E(B) = \begin{cases} 0 & \text{if } C_p(E\cap B) = 0 , \\ +\infty & \text{if } C_p(E\cap B) > 0 . \end{cases}$$

Note that the measure ∞_E belongs to $\mathcal{M}_p(\Omega)$.

For every $h\in \mathbf{N}$ the minimum problem (6.1) is equivalent to the minimum problem

$$(6.3) \qquad \min_{u\in H_0^{1,p}(\Omega)} \{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\infty_{E_h} + \int_\Omega gu\,dx \} .$$

in the sense that both problems have the same minimum values and the same minimum points. In fact, for a function $u\in H_0^{1,p}(\Omega)$ the condition $u = 0$ p-q.e. on E is equivalent to $u\in H_0^{1,p}(\Omega\setminus E)$ for arbitrary closed sets $E\subseteq\Omega$ (see [3], Theorem 4, and [16], Lemma 4).

The equivalence between (6.1) and (6.3) enables us to state the convergence properties of the sequences $(m_h(g))$ and $(M_h(g))$ by relying on the properties of the γ_f-convergence proved in the previous sections. According to Theorem 3.3, there exist a subsequence $(E_{\sigma(h)})$ of (E_h) and a measure $\mu\in \mathcal{M}_p(\Omega)$ such that $(\infty_{E_{\sigma(h)}})$ γ_f-converges to μ. The convergence of the corresponding minimum values $m_{\sigma(h)}(g)$ to the minimum value $m(\mu,g)$ of

$$(6.4) \qquad \min_{u\in H_0^{1,p}(\Omega)} \{ \int_\Omega f(x,Du)\,dx + \int_\Omega |u|^p d\mu + \int_\Omega gu\,dx \}$$

follows then immediately from Theorem 4.5. Moreover, if $M(\mu,g)$ denotes the set of all minimum points of (6.4), then Theorem 4.5 implies also that for every neighborhood U of $M(\mu,g)$ in $L^p(\Omega)$ there exists $k\in \mathbf{N}$ such that $M_{\sigma(h)}(g) \subseteq U$ for every $h \geq k$.

Finally, we point out that the main result of Section 5, concerning a characterization of the variational convergence by means of the convergence of μ-capacities, is particularly meaningful in the case $\mu_h = \infty_{E_h}$. It can be stated by using the capacity associated to f and defined for every $K \in \mathcal{K}$ by

$$C(f,K) = \inf \{ \int_\Omega f(x,Du)\, dx : u \in C_0^\infty(\Omega) , u \geq 1 \text{ on } K \} . \tag{6.5}$$

In fact, since $C(f,\infty_{E_h}, K) = C(f,K \cap E_h)$, for every $K \in \mathcal{K}$ the set functions α' and α'', introduced in Section 5, become

$$\alpha'(K) = \liminf_{h\to\infty} C(f,K\cap E_h) \quad , \quad \alpha''(K) = \limsup_{h\to\infty} C(f,K\cap E_h) \ .$$

Hence, the sequence (∞_{E_h}) γ_f-converges to a measure μ in $\mathcal{M}_p(\Omega)$ if and only if

$$\sup \{\alpha'(K) : K \in \mathcal{K} , K \subseteq A\} = \sup \{\alpha''(K) : K \in \mathcal{K} , K \subseteq A\}$$

for every $A \in \mathcal{A}$. Furthermore, formula (5.10) allows us to reconstruct the measure μ from the knowledge of the function f and of the sequence (E_h) by means of the set function $C(f,\cdot)$ defined in (6.5).

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