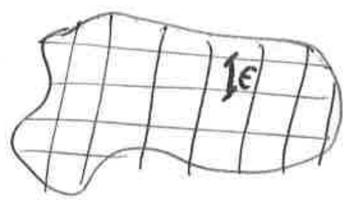
- Adrian – Homogenization, asymptotic expansion & 2-scale convergence – **Part 1**

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(x)\right) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \rightsquigarrow$$

if good bounds on A: $u_\varepsilon \to u$; goal: find A_{hom} s.t. $-\mathrm{div}(A_{hom} \nabla u) = f(x)$

where $A : Y \to \mathbb{R}^{d\times d}$, $Y := [0,1]^d$, periodic.

- Idea:



medium with a periodic cell structure

$\to$ Multiscale expansion:

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right)$$

large scale behaviour

microscopic behaviour

$\to$ homogenization kills fine details by averaging them

[see Bensousson – Lions – Papanicolaou]

We have that:

$$\nabla_x \left[u_0\left(x, \frac{x}{\varepsilon}\right) \right] = \nabla_x u_0\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} \nabla_y u_0\left(x, \frac{x}{\varepsilon}\right).$$

$\vdots$

we have that: (by plugging therein into the eq.) $\left[\frac{x}{\varepsilon} = y \right]$

- ε^{-2}: $-\operatorname{div}_y \left(A(y) \nabla_y u_0(x,y) \right) = 0$
- ε^{-1}: $-\operatorname{div}_y \left(A(y) \left[\nabla_x u_0(x,y) + \nabla_y u_1(x,y) \right] \right)$

  $-\operatorname{div}_x \left(A(y) \nabla_y u_0 \right) = 0$
- ε^{0}: $-\operatorname{div}_x \left(A(y) \left[\nabla_x u_0 + \nabla_y u_1 \right] \right)$

  $-\operatorname{div}_y \left(A(y) \left[\nabla_x u_1 + \nabla_y u_2 \right] \right) = f$

Let us consider the eq. for ε^{-2}: [integrate by parts]

$$\int_Y A(y) \nabla_y u_0(x,y) \cdot \nabla_y u_0(x,y)\, dy = 0$$

$$\Downarrow \text{ A uniformly elliptic}$$

$$\| \nabla_y u_0 \|_{L^2} = 0$$

$$\Downarrow$$

$$u_0(x,y) = u_0(x)$$

- Using the one with ε^{-1}:

$$-\operatorname{div}_y \left(A(y) \left(\nabla_x u_0 + \nabla_y u_1 \right) \right) = 0$$

Define: $w_i \in H^1_{per}(Y)$ as the solution of

$$-\operatorname{div}_y \left(A(y) \nabla w_i \right) = \operatorname{div}(A(y) e_i)$$

Then:

<u>cell problem</u>

$$-\operatorname{div}_y \left(A(y) \nabla_y u_1 \right) = \sum_i \operatorname{div}\left(A(y) e_i \right) \partial_i u$$

$$\Downarrow$$

$$\boxed{u_1 = \sum_{i=1}^{d} \underbrace{\partial_i u_0(x)}_{\substack{\text{large} \\ \text{scale}}} \underbrace{w_i(y)}_{\substack{\text{micro} \\ \text{scale}}}}$$

- Using the one with ε^0: [by averaging it]

$$-\operatorname{div}_x \left[\int_Y A(y) \left(\nabla_x u_0 + \partial_i u_0 \, w_i \right) dy \right] = f(x)$$

$$\Downarrow$$

$$\boxed{(A_{eff})_{JK} = \int_Y \left[A_{JK}(y) + A_{Je} \partial_e w_K \right] dy}$$

$$\Rightarrow \left[\begin{aligned} u_\varepsilon(x, \tfrac{x}{\varepsilon}) &= u_0(x) + \varepsilon\, u_1(x, \tfrac{x}{\varepsilon}) \\ &= u_0(x) + \varepsilon\, \partial_i u_0(x) \underbrace{w_i(\tfrac{x}{\varepsilon})} \end{aligned} \right.$$

principle directions where u_ε oscillates.

- So far, everything is just formal.
A more rigorous idea is the following:

- __2-scale convergence__ [by Nguetseng '89, Allaire '92]

given $(u_\varepsilon)_\varepsilon \subset L^2(\Omega)$, we say that $[u_\varepsilon = u_\varepsilon(x)]$

$$u_\varepsilon \xrightarrow{2-s} u(x,y) \in L^2(\Omega \times Y)$$

if $\forall \varphi \in C_c^\infty(\Omega; C_{per}^\infty(Y))$ it holds:

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\, \varphi(x, \tfrac{x}{\varepsilon})\, dx = \int_{\Omega \times Y} u(x,y)\, \varphi(x,y)\, dx\, dy$$

__oscillating test functions!__

what happens in the __domain__ [__Remark__: Young measures take care of oscillations on the __target__ space]

We have: $u_\varepsilon \xrightarrow{2-s} u \Rightarrow u_\varepsilon \longrightarrow \int_Y u(\cdot, y)\,dy$

- $(u_\varepsilon)_\varepsilon \subseteq L^2$ bounded $\Rightarrow u_\varepsilon \xrightarrow{2-s} u$ (up to a subsequence)

[see Allaire & Briane for multi-scale convergence]

- $(u_\varepsilon)_\varepsilon$ bdd in $H^1(\Omega) \Rightarrow \exists\, u_0 \in H^1(\Omega),\ u_1 \in L^2(\Omega, H^1_{per}(Y))$ s.t.
  - $u_\varepsilon \longrightarrow u_0$ in H^1
  - $u_\varepsilon \longrightarrow u_0$ in L^2
  - $u_\varepsilon \xrightarrow{2-s} u_0$
  - $\nabla u_\varepsilon \xrightarrow{2-s} \nabla_x u_0 + \nabla_y u_1$