The wave equation in bounded domains.
Separation of variables

Let us consider the wave equation in an interval $[0, L]$:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } [0,L] \times (0,\infty)\,, \\ u(x,0) = g(x) & \text{in } [0,L]\,, \\ u_t(x,0) = h(x) & \text{in } [0,L]\,, \\ u(0,t) = 0 & \text{for } t > 0\,, \\ u(L,t) = 0 & \text{for } t > 0\,, \end{cases} \tag{1}$$

Notice that, we have **two** initial conditions, $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$, and some boundary conditions, telling us what is going on at the boundary points $x = 0$ and $x = L$. The particular conditions we are considering here are the **homogeneous** Dirichlet conditions

$$u(0,t) = u(L,t) = 0,\,,$$

for $t > 0$, saying that our string is fixed at the ending points. We would like to apply the strategy of separation of variables, developed for the case of the heat equation in bounded domains, to solve the above problem. We recall that the basic idea is the following: since we don't know what the solution can be, we look for a particular kind of solution, namely one of the form:

$$u(x,t) = T(t)X(x)\,,$$

for some (one variable) functions T and X. In order for such a function u to solve the equation

$$u_{tt} - c^2 u_{xx} = 0\,,$$

we need the functions T and X to satisfy

$$T''(t)X(x) - c^2 T(t)X''(x) = 0\,.$$

By dividing by $DT(t)X(x)$ (here we are assuming that it is possible to divide by $T(t)X(x)$. This is just a formal operation. It is possible, and we will do it later on during the course, to perform this step in a more rigorous way), we get

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}\,.$$

since the left-hand side is a function just of the variable t, while the right-hand side is a function just of the variable x, in order to have the above equality in force for every $x \in [0, L]$ and every $t > 0$, we must impose both sides to be constant. So, we are lead to the equations

$$\begin{cases} T''(t) = \lambda c^2 T(t)\,, \\ X''(x) = \lambda X(x)\,, \end{cases}$$

for some *arbitrary* $\lambda \in \mathbb{R}$. Let us first consider the equation for X. The BCs we have to impose for u are

$$0 = u(0,t) = T(t)X(0)\,, \qquad 0 = u(L,t) = T(t)X(L)\,,$$

for every $t > 0$. In order to satisfy them, we can have $T \equiv 0$ (but this would implies that $u \equiv 0$, and this is a solution if and only if $g \equiv h \equiv 0$), or we need to impose

$$X(0) = 0\,, \qquad X(L) = 0\,.$$

So, we have to solve the problem

$$\begin{cases} X''(x) = \lambda X(x) & \text{in } [0, L]\,, \\ X(0) = 0\,, \\ X(L) = 0\,. \end{cases} \tag{2}$$

As we did for the heat equation, we have to consider three cases (recall that λ is an arbitrary number!):

- $\lambda > 0$: in this case, the general solution of the above equation is
$$X(x) = a \sinh(\sqrt{\lambda} x) + b \cosh(\sqrt{\lambda} x)\,.$$
We now have to impose the boundary conditions. So,
$$0 = X(0) = b\,,$$
and hence $X(x) = a \sinh(\sqrt{\lambda} x)$. Moreover, we have to impose impose
$$0 = X(L) = a \sinh(\sqrt{\lambda} L)\,.$$
Since $\sinh y \neq 0$ if $y \neq 0$ (and this is the case, since $\sqrt{\lambda} L \neq 0$ - recall that we are in the case $\lambda > 0$), we get that the above equation can be satisfied only if $a = 0$. So, we obtain the trivial solution $X \equiv 0$, leading to the function
$$u(x,t) = T(t)X(x) \equiv 0\,.$$
So, u can be a solution of problem (1) if and only if $g \equiv h \equiv 0$ (it is the only way for the null function to match the initial condition). In the case $g, h \not\equiv 0$, this cannot be a solution, and thus we have to exclude it.

- $\lambda = 0$, in this case, the general solution of the above equation is
$$X(x) = ax + b\,.$$
By imposing the boundary conditions, we get $a = b = 0$. Thus, $X \equiv 0$. By arguing as before, for a nontrivial initial data, this function cannot lead to a solution of our problem.

- $\lambda < 0$: in this case, the general solution of the equation is given by
$$X(x) = a \cos(\sqrt{-\lambda} x) + b \sin(\sqrt{-\lambda} x)\,.$$
By imposing the boundary conditions at $x = 0$, we get
$$0 = X(0) = a\,,$$
and hence $X(x) = a \cos x$. Then, by imposing the boundary conditions at $x = L$, we get
$$0 = X(L) = b \sin(\sqrt{-\lambda} L)\,.$$
Since for $b = 0$ we obtain the trivial solution, we want to impose $\sin(\sqrt{-\lambda} L) = 0$. By recalling that
$$\sin x = 0 \quad \Leftrightarrow \quad x = n\pi\,,$$
for some $n \in \mathbb{N}$, we get
$$\sqrt{-\lambda} L = n\pi \quad \Rightarrow \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2\,.$$
So, for every $n = 1, 2, 3, \ldots$ (because, for $n = 0$, we obtain $\lambda = 0$, and we already discussed that case), we get that the function
$$x \mapsto b_n \sin\left(\frac{n\pi}{L} x\right)\,,$$
where $b_n \in \mathbb{R}$ is *arbitrary*, is a solution of problem (2) with $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$.

Let us now consider, for every $n = 1, 2, 3, \dots$, the corresponding equation for T:

$$T''(t) = \lambda_n c^2 T(t)\,,$$

whose general solution is of the form

$$T_n(t) := a_n \cos\left(\frac{n\pi}{L}ct\right) + b_n \sin\left(\frac{n\pi}{L}ct\right)\,.$$

Thus, for every $n = 1, 2, 3, \dots$, we obtain that the function

$$u_n(x,t) = \left[a_n \cos\left(\frac{n\pi}{L}ct\right) + b_n \sin\left(\frac{n\pi}{L}ct\right)\right] \sin\left(\frac{n\pi}{L}x\right)$$

satisfies the wave equation and matches the boundary conditions. In order to solve the problem we also have to satisfy the initial condition. Notice that

$$u_n(x,0) = a_n \sin\left(\frac{n\pi}{L}x\right)\,,$$

and

$$\partial_t u_n(x,0) = b_n \frac{n\pi}{L} \sin\left(\frac{n\pi}{L}x\right)\,.$$

That is, the function u_n solves the problem (1) if and only if the initial data g is of the form

$$g(x) = a \sin\left(\frac{n\pi}{L}x\right)\,,$$

and h is of the form

$$h(x) = b \sin\left(\frac{n\pi}{L}x\right)\,,$$

for some $a, b \in \mathbb{R}$. This is too restrictive! We would like to use the above functions u_n's to build a solution for a generic initial data g and h. To this purpose, let us consider

$$v(x,t) := \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi}{L}ct\right) + b_n \sin\left(\frac{n\pi}{L}ct\right)\right] \sin\left(\frac{n\pi}{L}x\right)\,, \tag{3}$$

for some $N \in \mathbb{N}$. Since the wave equation is **linear**, any *finite* sum of the above functions still satisfies it. So, v still satisfies the equation. Moreover, since the boundary conditions are **homogeneous**, v will match them. Again, we see that finite linear combinations of the u_n's require too restrictive assumptions of the form of the initial data. So, as we did for the heat equation, we set $N = +\infty$ in (3) (making the sum a series), forgetting about asking ourselves whether it makes sense or not. We just do it!, obtaining the function

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}ct\right) + b_n \sin\left(\frac{n\pi}{L}ct\right)\right] \sin\left(\frac{n\pi}{L}x\right)\,.$$

So, we can say that **formally** (that means, if we believe it!), the above function is still a solution of the heat equation and it matches the boundary condition (since every *finite* sum of the u_n's does). Then, in order for u to satisfy the initial problem, it just need to match the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = g(x)\,,$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi}{L}x\right) = h(x)\,,$$

for $x \in [0, L]$. In order to have the above equality satisfies, we will make a further assumption: we will **assume** that g is of the form

$$g(x) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right)\,, \tag{4}$$

and that

$$h(x) = \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{L}x\right) ,$$

for some $g_n, h_n \in \mathbb{R}$. Is this too restrictive? We will see, thanks to the theory of Fourier series, that it is not! So, in order for u to match the initial condition, we need to have

$$\sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) ,$$

and

$$\sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi}{L}x\right)$$

for every $x \in [0, L]$. But two series of functions are equal if and only if all the terms in the series are the same. That is, we need to impose

$$g_n \sin\left(\frac{n\pi}{L}x\right) = a_n \sin\left(\frac{n\pi}{L}x\right) ,$$

and

$$h_n \sin\left(\frac{n\pi}{L}x\right) = b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi}{L}x\right) .$$

for every $x \in [0, L]$ and every $n = 1, 2, 3, \ldots$. But these conditions boil down to impose

$$a_n = g_n , \qquad b_n = \frac{L}{n\pi c} h_n ,$$

for every $n = 1, 2, 3, \ldots$. Thus, we get that the function u defined as

$$u(x,t) := \sum_{n=1}^{\infty} \left[g_n \cos\left(\frac{n\pi}{L}ct\right) + \frac{L}{n\pi c} h_n \sin\left(\frac{n\pi}{L}ct\right) \right] \sin\left(\frac{n\pi}{L}x\right) ,$$

is a solution of the problem (1).