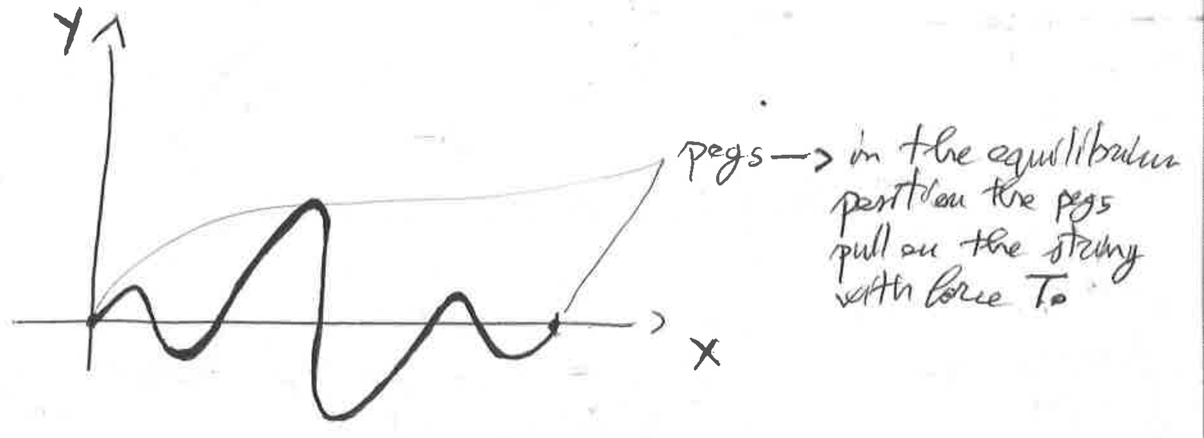
# 5) The wave equation:

## 5.1) Derivation of the wave eq.:

We want to derive the equation of motion of a vibrating string whose ends are fixed.
For simplicity, we'll suppose the string to always stay on a fixed plane $xy$.
Moreover, we are not taking into consideration the resistance of the air.
The derivation of this model will also help us in understanding the reasons behind the Hadamard's definition of well posedness for a PDE.
Let us start by setting some notation:

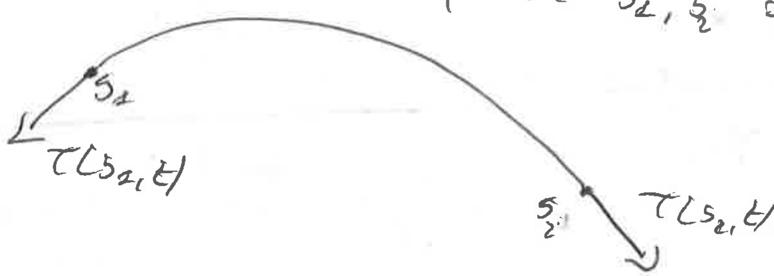



- Reference configuration: $[0, L]$
- Equilibrium configuration: $x(s,0) = s$
- string is so thin that its cross section moves as a single pt
- coordinates: $\begin{cases} x = x(s,t) \\ y = y(s,t) \end{cases}$

- flexibility: no effort to bend it $\Rightarrow$ the force that one piece of the string exerts on the other is tangential to the string $\to$ tension $T(s,t)$
- we neglect long-range forces
- continuous density: $s \mapsto \rho(s)$
- initial conditions:

$$\begin{cases} y(s,0) = f(s) \\ \dfrac{\partial y}{\partial t}(s,0) = g(s) \end{cases} \qquad \begin{cases} x(s,0) = s \\ \dfrac{\partial x}{\partial t}(s,0) = 0 \end{cases}$$

- external force $F(x,t)$ per unit mass acting on the $y$-direction
- let us consider a portion $s_1, s_2$ of the string:




- Newton's law in the $x$-direction:

$$\int_{s_1}^{s_2} \rho(s)\, \frac{\partial^2 x}{\partial t^2}\, ds = \frac{T(s_2,t)\, \dfrac{\partial x}{\partial s}(s_2,t)}{\sqrt{\left(\dfrac{\partial x}{\partial s}\right)^2 + \left(\dfrac{\partial y}{\partial s}\right)^2}} - \frac{T(s_1,t)\, \dfrac{\partial x}{\partial s}(s_1,t)}{\sqrt{\left(\dfrac{\partial x}{\partial s}\right)^2 + \left(\dfrac{\partial y}{\partial s}\right)^2}}$$

by differentiating w.r.t. $s$

$$(1) \qquad \frac{\partial}{\partial s}\left[ \frac{T\, \dfrac{\partial x}{\partial s}}{\sqrt{\left(\dfrac{\partial x}{\partial s}\right)^2 + \left(\dfrac{\partial y}{\partial s}\right)^2}} \right] = \rho\, \frac{\partial^2 x}{\partial t^2}$$

• Same for the y-direction:

$$(2) \quad \frac{\partial}{\partial s}\left[ T \frac{\frac{\partial y}{\partial s}}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}} \right] = \rho \frac{\partial^2 y}{\partial t^2} - \rho F(x,t)$$

• Assuming perfect elasticity, the tension at any pt $s$ is determined by the local stretching per unit length:

$$e(s,t) := \sqrt{\left(\frac{\partial x}{\partial s}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2} - 1, \longrightarrow \text{recall the area factor}$$

that is:

$$T(s,t) = \Upsilon(e(s,t),\ s)$$

[function describing the elastic properties of the string]

• equilibrium position $\begin{cases} x(s,t) = s \\ y(s,t) = 0 \end{cases} \underset{(1) \text{ with } F=0}{\Longrightarrow}$ $T \equiv$ constant

$\Downarrow$ $e \equiv 0$ in this case

$\longrightarrow \boxed{\Upsilon(0,s) = T_0}$ $\hookrightarrow$ tension at the end pts

$\longrightarrow$ <u>The above pb is very difficult to solve</u>!

Further simplifications are needed!

• small vibrations around the equilibrium position

• $\frac{\partial x}{\partial s} \neq 0$ [the string is never vertical]

$\Downarrow$

$$s = s(x,t) \Rightarrow y = v(x,t) \text{ s.t.}$$

$$(\circ) \quad y(s,t) = v(x(s,t), t)$$

From (1) we deduce that;

- $$\frac{\partial y}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s}$$

- $$\frac{\partial y}{\partial t} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial t}$$

- $$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 v}{\partial x \partial t}\frac{\partial x}{\partial t} + \frac{\partial^2 v}{\partial t^2}\frac{\partial v}{\partial x}$$

Then, (2) becomes;

- $$\frac{\partial}{\partial s}\left[T\frac{\frac{\partial x}{\partial s}\frac{\partial v}{\partial x}}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}}\right] = \rho[\quad] - \rho F$$

by (1)

- $$\frac{\partial}{\partial s}\left[\frac{\partial v}{\partial x}\right] T\frac{\frac{\partial x}{\partial s}}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}} + \frac{\partial v}{\partial x}\boxed{\rho\frac{\partial^2 x}{\partial t^2}} = \rho[-] - \rho F$$

- $$\frac{\partial^2 v}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2\frac{\gamma(e,s)}{e+1} + \frac{\partial v}{\partial x}\rho\frac{\partial^2 x}{\partial t^2} = \rho[-] - \rho F$$

(3) $$\left[\frac{\gamma(e,s)}{e+1}\left(\frac{\partial x}{\partial s}\right)^2 - \rho\left(\frac{\partial x}{\partial t}\right)^2\right]\boxed{\frac{\partial^2 v}{\partial x^2}} - 2\rho\frac{\partial x}{\partial t}\boxed{\frac{\partial^2 v}{\partial x \partial t}} - \rho\boxed{\frac{\partial^2 v}{\partial t^2}} = -\rho F$$

→ **Pb**: eq. in $v$, but with coefficients depending on $x(s,t)$! and **e**!

→ To eliminate this dependence, we assume

- $\dfrac{T(e(s))}{(e+1)T_0}\left(\dfrac{\partial x}{\partial s}\right)^2 - 1$ ⇝ $x \sim s$

(*)
- $\dfrac{\rho}{T_0}\left(\dfrac{\partial x}{\partial t}\right)^2$ ⇝ $x$ is almost constant

- $\dfrac{\rho(x)}{\rho(s)} - 1$ ⇝ $x \sim s$

are negligible relative to 1.

⇒ Then, the PDE:

(4) $$T_0 \frac{\partial^2 u}{\partial x^2} - \rho(x)\frac{\partial^2 u}{\partial t^2} = -\rho(x)F(x,t)$$

has coefficients that are closed to the ones in (3).

Setting $c := \sqrt{\dfrac{T_0}{\rho}}$, we get

the vibrating string pb

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = F(x,t) & x \in (0,L),\ t>0 \\ u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

- Some considerations:
  - the idealization of the string is a <u>mathematical model</u>, and we made the <u>physical assumption</u> that an actual string acts like the mathematical model
  - For $f = g = F = 0$, the solution of (3) is $v = 0$, $x = s$, thus, we can expect that, for $f, g, F$ small, $\frac{\partial v}{\partial x}$, $\frac{\partial x}{\partial t}$, $\frac{\partial x}{\partial s} - 1$ are small $\rightarrow$ <u>this can be proved</u> !
  - the fact that $u \sim v$ because the coefficients of (3) & (4) are closed <u>can be proved</u>!
  - since the vibrating string pb is removed from the physical pb, we cannot expect existence & uniqueness to follow by empirical observations $\rightarrow$ they have to be proved!
  - since $f, g, F$ cannot be measured exactly, we expect that small modification of them lead to small variations of the solution

$\rightarrow$ The above observations lead to the so called <u>well-position</u> of a pb <u>in the sense of Hadamard</u> [1923]

- $\exists$ <u>existence</u>
- ! <u>uniqueness</u>
- <u>continuous dependence</u> on the (initial) data