The Laplacian in polar and spherical coordinates.

Polar coordinates.

The Laplacian is defined with respect the canonical base of $\mathbb{R}^N$. Let us consider, for instance, the following problem

$$-\triangle u = 0\,, \qquad \text{in } B_{\bar{r}}(0)\,,$$

where $B_r(0) := \{x \in \mathbb{R}^2 \,:\, |x| < \bar{r}\}$ is the ball of radius $\bar{r}$ centered at the origin. Since the set $B_r(0)$ has a spherical symmetry, it is more convenient to describe it with spherical coordinates (radius r and angle θ). That is, we describe points $(x, y) \in \mathbb{R}^2$ as

$$\begin{cases} x = r\cos\theta\,, \\ y = r\sin\theta\,. \end{cases}$$

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Let us consider the change of variable $P : (0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \setminus \{(0, 0)\}$ given by

$$P(r, \theta) := (r\cos\theta, r\sin\theta)\,.$$

Notice that, for technical reasons of the change of variable, we are not considering the origin in the target space. This will not affect our computations. Now, define the function

$$v(r, \theta) := u\left(P(r, \theta)\right) = u\left(r\cos\theta, r\sin\theta\right)\,.$$

We want to understand what equation v has to solve in the rectangle $[0, \bar{r}) \times [0, 2\pi)$ (that is, the set that describes the ball $B_{\bar{r}}(0)$), in order for the function u to solve the Laplace equation. In other terms, we want to understand how to write the Laplacian in polar coordinates. Namely, we would like to write

$$-\triangle u = -\partial^2_{xx} u - \partial^2_{yy} u\,,$$

as something involving only derivatives with respect to r and θ. For, let's reason as follows: we first write the first derivatives $\partial_x u$ and $\partial_y u$ in terms of $\partial_r v$ and $\partial_\theta v$. By applying the chain rule, we have that

$$\begin{cases} \partial_r v = \partial_x u\, \partial_r x + \partial_y u\, \partial_r y\,, \\ \partial_\theta v = \partial_x u\, \partial_\theta x + \partial_y u\, \partial_\theta y\,. \end{cases}$$

That is

$$\begin{cases} \partial_r v = \partial_x u \cos\theta + \partial_y u\, \sin\theta\,, \\ \partial_\theta v = -r\partial_x u\, \sin\theta + r\partial_y u\, \cos\theta\,. \end{cases}$$

That is, in a matrix form,

$$\begin{pmatrix} \partial_r v \\ \partial_\theta v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = DP(r, \theta) \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}\,,$$

where $DP(r, \theta)$ denotes the differential of the map P at the point (r, θ). Since the above equation is true for every u (and the correspondent v), we can simply write it as an *equality between differential operators* as

$$\begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = DP(r, \theta) \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}\,.$$

What we want is ∂_x and $\partial_y u$ in terms of $\partial_r v$ and $\partial_\theta v$. so, we have to invert the above equality, that is

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \left(DP(r, \theta)\right)^{-1} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix}\,,$$

where $\left(DP(r, \theta)\right)^{-1}$ is the inverse of the matrix $DP(r, \theta)$, that is

$$\left(DP(r, \theta)\right)^{-1} = \begin{pmatrix} \cos\theta & -\frac{\sin}{r}\theta \\ \sin\theta & \frac{\cos}{r}\theta \end{pmatrix}\,.$$

Thus, we get

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin}{r}\theta \\ \sin\theta & \frac{\cos}{r}\theta \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix},$$

that is

$$\begin{cases} \partial_x = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\,, \\ \\ \partial_y = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\,. \end{cases}$$

We now want to compute ∂^2_{xx}. We have that

$$\begin{aligned}
\partial^2_{xx} = \partial_x\partial_x &= \left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right) \\
&= \cos\theta\partial_r\left(\cos\theta\partial_r\right) + \cos\theta\partial_r\left(-\frac{\sin\theta}{r}\partial_\theta\right) - \frac{\sin\theta}{r}\partial_\theta\left(\cos\theta\partial_r\right) - \frac{\sin\theta}{r}\partial_\theta\left(-\frac{\sin\theta}{r}\partial_\theta\right) \\
&= \cos\theta(\partial_r\cos\theta)\partial_r + \cos^2\theta\partial_r\partial_r + \cos\theta\partial_r\left(-\frac{\sin\theta}{r}\right)\partial_\theta - \frac{\cos\theta\sin\theta}{r}\partial_r\partial_\theta \\
&\quad - \frac{\sin\theta}{r}\partial_\theta\left(\cos\theta\right)\partial_r - \frac{\sin\theta\cos\theta}{r}\partial_\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\left(-\frac{\sin\theta}{r}\right)\partial_\theta + \frac{\sin^2\theta}{r^2}\partial_\theta\partial_\theta \\
&= 0 + \cos^2\theta\partial^2_{rr} + \frac{1}{r^2}\cos\theta\sin\theta\partial_\theta - \frac{\cos\theta\sin\theta}{r}\partial^2_{r\theta} \\
&\quad + \frac{\sin^2\theta}{r}\partial_r - \frac{\cos\theta\sin\theta}{r}\partial^2_{r\theta} + \frac{\sin\theta\cos\theta}{r^2}\partial_\theta + \frac{\sin^2\theta}{r^2}\partial^2_{\theta\theta}\,.
\end{aligned}$$

With similar computations, we get

$$\begin{aligned}
\partial^2_{yy} = \partial_y\partial_y &= \left(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\left(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\right) \\
&= \sin^2\theta\partial^2_{rr} - \frac{1}{r^2}\cos\theta\sin\theta\partial_\theta + \frac{\cos\theta\sin\theta}{r}\partial^2_{r\theta} \\
&\quad + \frac{\cos^2\theta}{r}\partial_r + \frac{\cos\theta\sin\theta}{r}\partial^2_{r\theta} - \frac{\sin\theta\cos\theta}{r^2}\partial_\theta + \frac{\cos^2\theta}{r^2}\partial^2_{\theta\theta}\,.
\end{aligned}$$

Thus, we can write

$$\boxed{\triangle = \partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}\,.}$$

The above is the expression of the Laplacian in polar coordinates. Notice that it is made by a *radial* component

$$\partial^2_{rr} + \frac{1}{r}\partial_r\,,$$

and by an *angular* one

$$\partial_{\theta\theta}\,.$$

In our example, this means that, u solves the Laplace equation in the ball $B_r(0)$ if and only if v solves the equation

$$\partial^2_{rr}v + \frac{1}{r}\partial_r v + \frac{1}{r^2}\partial_{\theta\theta}v = 0\,,$$

in the rectangle $[0, r) \times [0, 2\pi)$. Even if at a first glance this does not seem like a good simplification of the problem we will see that it is possible to solve the equation for v.

Spherical coordinates.

We would like to perform the same computation in dimension $N = 3$ with the spherical coordinates, that is, when we describe a point $(x, y, z) \in \mathbb{R}^3$ as

$$\begin{cases} x = r\sin\varphi\cos\theta\,, \\ y = r\sin\varphi\sin\theta\,, \\ z = r\cos\varphi\,. \end{cases}$$

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In order to do so, we will take advantage of the previous computations and we will add an additional variable (that we will get rid in the end). We call

$$s := \sqrt{x^2+y^2} = r\sin\varphi\,,$$

and we write the above system as

$$\begin{cases} x = s\cos\theta\,, \\ y = s\sin\theta\,, \\ z = r\cos\varphi\,. \end{cases}$$

By considering the planes xy and, for θ fixed, the plane sz, by the previous computations, we have that

$$\partial^2_{xx} + \partial^2_{yy} = \partial^2_{ss} + \frac{1}{s}\partial_s + \frac{1}{s^2}\partial_{\theta\theta}\,,$$

$$\partial^2_{ss} + \partial^2_{zz} = \partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\varphi\varphi}\,.$$

Then

$$\triangle = \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz} = \partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\varphi\varphi} + \frac{1}{s}\partial_s + \frac{1}{s^2}\partial_{\theta\theta}\,. \tag{1}$$

We now just have to rewrite the last two terms with respect to derivatives in r, θ, φ. For, notice that, by the definition of s,

$$\frac{1}{s^2}\partial_{\theta\theta} = \frac{r^2\sin^2\varphi}{\partial^2_{\theta\theta}}\,.$$

Moreover, by the chain rule, we have that

$$\partial_s = \partial_r\frac{\partial r}{\partial s} + \partial_\varphi\frac{\partial \varphi}{\partial s} + \partial_\theta\frac{\partial \theta}{\partial s}\,.$$

We have that

$$\frac{\partial\theta}{\partial s} = 0\,,$$

since the variable θ does not depend on s, while

$$\frac{\partial r}{\partial s} = \frac{\partial}{\partial s}\sqrt{s^2+z^2} = \frac{s}{r}\,.$$

Finally, in order to compute $\frac{\partial\varphi}{\partial s}$, we reason as follows: by definition

$$s = r\sin\varphi\,.$$

By differentiating both terms with respect to s, we get

$$1 = \frac{r}{s}\sin\varphi + r\cos\theta\frac{\partial\varphi}{\partial s}\,.$$

Thus,

$$\frac{\partial\varphi}{\partial s} = \frac{r - s\sin\varphi}{r^2\cos\varphi} = \frac{\cos\varphi}{r}\,.$$

By plugging in these expression in (1), we finally get

$$\boxed{\triangle = \partial^2_{rr} + \frac{2}{r}\partial_r + \frac{1}{r^2}\left[\partial^2_{\varphi\varphi} + \frac{\cos\varphi}{\sin\varphi}\partial_\varphi + \frac{1}{\sin^2\varphi}\partial^2_{\theta\theta}\right]\,.}$$

Again, also here we notice that the radial and the angular part are separated. This last one, is called the **Laplace-Beltrami operator**, and functions w defined on the sphere (the boundary of the ball!) for which

$$\partial^2_{\varphi\varphi} w + \frac{\cos\varphi}{\sin\varphi}\partial_\varphi w + \frac{1}{\sin^2\varphi}\partial^2_{\theta\theta} w = 0\,,$$

are called **spherical harmonics**.