Source in the heat equation in bounded domains

We want to solve the problem

$$\begin{cases} u_t - Du_{xx} = f(x,t) & \text{in } (0,L)\times(0,\infty)\,, \\ u(x,0) = g(x) & \text{in } (0,L)\,, \\ u(0,t) = 0 & \text{for } t>0\,, \\ u(L,t) = 0 & \text{for } t>0\,, \end{cases} \tag{1}$$

where $f:(0,L)\times(0,\infty)\to\mathbb{R}$ is a given function. The above problem models, for instance, the temperature of a thin object $(0,L)$, where f represents the (density of the) source of heat.

The idea in order to solve the above problem is similar to the one of the variation of coefficients for ODEs: we know that the solution of the above problem in the case $f\equiv 0$ is of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi}{L}x\right)\,,$$

for some functions $b_n:[0,\infty)\to\mathbb{R}$. The idea is that the solution of (1) is of the same form. We just have to find the functions b_n's. Formally, we derive the function u above in order to find

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} b_n(t)\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi}{L}x\right)\,,$$

$$u_t(x,t) = \sum_{n=1}^{\infty} b_n'(t)\sin\left(\frac{n\pi}{L}x\right)\,.$$

We assume that, for every fixed $t>0$ it is possible to expand the function $x\mapsto f(x,t)$ as

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\left(\frac{n\pi}{L}x\right)\,,$$

for some $f_n:[0,\infty)\to\mathbb{R}$. So, the equation

$$u_t - Du_{xx} = f(x,t)\,,$$

writes as

$$\sum_{n=1}^{\infty}\left[b_n'(t) + D\left(\frac{n\pi}{L}\right)^2 b_n(t)\right]\sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} f_n(t)\sin\left(\frac{n\pi}{L}x\right)\,.$$

In order to have the above equation in force for every $x\in(0,L)$ and every $t>0$, we need to impose the terms of the two series to be equal, that is, we have to impose

$$b_n'(t) + D\left(\frac{n\pi}{L}\right)^2 b_n(t) = f_n(t)\,,$$

for $t>0$, for every $n=1,2,3,\ldots$. Thus, the solution of a PDE boils down to solve a countably many ODEs (in a similar way, in the case $f\equiv 0$, the solution of the PDE is down to compute the Fourier coefficients of the initial data, that is, to solve countably many integrals!). Each of the above equation is couple with an initial condition that we can derive from the initial condition that u has to satisfy. So, if we assume

$$g(x) = \sum_{n=1}^{\infty} g_n\sin\left(\frac{n\pi}{L}x\right)\,,$$

the initial condition is $b_n(0) = g_n$.

So, let us fix $n = 1, 2, 3, \ldots$, and let us consider the problem

$$\begin{cases} b_n'(t) + D\left(\frac{n\pi}{L}\right)^2 b_n(t) = f_n(t)\,, \\ b_n(0) = g_n\,. \end{cases}$$

The solution of this problem is

$$b_n(t) = e^{-D\left(\frac{n\pi}{L}\right)^2 t}\left[g_n + \int_0^t f_n(s) e^{D\left(\frac{n\pi}{L}\right)^2 s}\,\mathrm{d}s\right].$$

Notice that these coefficients are the coefficients you would obtain in the case $f \equiv 0$ with an additional term due to the source term f. Hence, the solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{L}\right)^2 t}\left[g_n + \int_0^t f_n(s) e^{D\left(\frac{n\pi}{L}\right)^2 s}\,\mathrm{d}s\right] \sin\left(\frac{n\pi}{L}x\right).$$

Bottom line. In order to solve a problem of the form

$$\begin{cases} u_t - Du_{xx} = f(x,t) & \text{in } (0,L)\times(0,\infty)\,, \\ u(x,0) = g(x) & \text{in } (0,L)\,, \\ \text{boundary conditions at} & \\ x = 0 \text{ and } x = L & \text{for } t > 0\,, \end{cases} \tag{2}$$

we proceed as follows: let

$$u(x,t) = \sum_n T_n(t) X_n(x)\,,$$

be the general form of the solution in the case $f \equiv 0$. The particular form of the X_n's depends on the *boundary conditions* of the above problem. By assuming

$$f(x,t) = \sum_n f_n(t) X_n(x)\,, \qquad g(x) = \sum_{n=1}^{\infty} g_n X_n(x)\,,$$

we have that the T_n's are given by

$$T_n(t) = e^{-D\left(\frac{n\pi}{L}\right)^2 t}\left[g_n + \int_0^t f_n(s) e^{D\left(\frac{n\pi}{L}\right)^2 s}\,\mathrm{d}s\right],$$

and thus, the solution of (2) is given by

$$u(x,t) = \sum_{n=1}^{\infty} e^{-D\left(\frac{n\pi}{L}\right)^2 t}\left[g_n + \int_0^t f_n(s) e^{D\left(\frac{n\pi}{L}\right)^2 s}\,\mathrm{d}s\right] X_n(x)\,.$$

So, once we know the solution of the homogeneous problem, we also know how to solve the inhomogeneous one.

Notice. We will see in the exercises that, once we are able to solve the above problem, we are also able to solve problems of the type

$$\begin{cases} u_t - Du_{xx} = f(x,t) & \text{in } (0,L)\times(0,\infty)\,, \\ u(x,0) = g(x) & \text{in } (0,L)\,, \\ u(0,t) = \varphi_1(t) & \text{for } t > 0\,, \\ u(L,t) = \varphi_1(t) & \text{for } t > 0\,, \end{cases}$$

where $f : (0,L)\times(0,\infty) \to \mathbb{R}$, $\varphi_1 : (0,\infty) \to \mathbb{R}$ and $\varphi_2 : (0,\infty) \to \mathbb{R}$ are given functions.