Review on ODEs

First order linear equations.
Let us consider the initial value problem

$$\begin{cases} y'(t) = f(t)y(t) + g(t)\,, \\ y(0) = y_0\,, \end{cases}$$

where $f, g : (0,\infty) \to \mathbb{R}$ are given functions. Then, the **unique** solution of the above problem is given by

$$y(t) := y_0 e^{\int_0^t f(s)\,\mathrm{d}s} + \int_0^t g(s) e^{\int_s^t f(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\,.$$

Second order homogeneous linear equations with constant coefficients.
Let us consider the initial value problem

$$\begin{cases} ay''(t) + by'(t) + cy(t) = 0 \\ y(0) = y_0\,, \\ y'(0) = y_1\,, \end{cases} \tag{1}$$

for some $a, b, c, y_0, y_1 \in \mathbb{R}$. The theory of ODEs tells us that the set of solutions of

$$ay''(t) + by'(t) + cy(t) = 0\,, \tag{2}$$

is a vector space of dimension 2. Thus, if v_1 and v_2 are two linearly independent solutions of (2), then the general solution of (2) is of the form

$$y(t) = c_1 v_1(t) + c_2 v_2(t)\,,$$

for some constants $c_1, c_2 \in \mathbb{R}$. So, once we find such a base of the vector space of the solutions of (2), we know the general form of the solution. Then, by using the initial conditions, we can find **the** solution of (1) (since the solution of (1) is **unique**!).

The idea in order to solve the above problem is the following: let us find solutions of the form

$$y(y) = e^{\mu t}\,, \tag{3}$$

for some $\mu \in \mathbb{R}$ that we have to determine. If we plug in the above form of u in the equation (2), we obtain

$$e^{\mu t}\left(a\mu^2 + b\mu + c\right) = 0\,.$$

Thus, we consider the so called **characteristic polynomial**

$$a\mu^2 + b\mu + c = 0\,. \tag{4}$$

We have that: the function u in (3) solves the equation (2) if and only if μ is a solution of the characteristic polynomial (4). Then, the general solution will be of the form

$$y(t) = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t}\,, \tag{5}$$

where $\mu_1 \neq \mu_2$ are the two **distinct** solutions of (4), or

$$y(t) = c_1 e^{\mu_1 t} + c_2 t e^{\mu_1 t}\,, \tag{6}$$

in the case they coincide. Notice that μ_1 and μ_2 can also be complex numbers. So:

(i) if $\mu_1 \neq \mu_2$ and they are **real** numbers, then the general solution is given by

$$y(t) = c_1 e^{\mu_1 t} + c_2 e^{\mu_1 t}\,,$$

or, equivalently by

$$y(t) = c_1 \cosh(\mu_1 t) + c_2 \sinh(\mu_1 t)\,,$$

(ii) if $\mu_1 = \mu_2$ (and, of course, they are both real!), then the general solution is given by
$$y(t) = c_1 e^{\mu_1 t} + c_2 t e^{\mu_1 t}\,,$$

(iii) if the solution of (4) is of the form
$$\alpha \pm i\sqrt{\beta}\,,$$
for some $\alpha, \beta \in \mathbb{R}$, then the general solution of (2) is given by
$$y(t) = e^{\alpha t}\left[c_1 \cos(\sqrt{\beta}t) + c_2 \sin(\sqrt{\beta}t)\right]\,.$$

Finally, by using the initial conditions, we can find the values of c_1 and c_2 for which the function u above is a solution of (1).

General second order linear equations - variation of coefficients. We now want to solve the problem
$$\begin{cases} y''(t) + b(t)y'(t) + c(t)y(t) = f(t)\,, \\ y(0) = y_0\,, \\ y'(0) = y_1\,, \end{cases} \tag{7}$$
where b, c and f are given functions. Notice that the coefficient in from of y'' is 1.

The idea to find the solution of (7) is the following: let y_1 and y_2 be a base for the space of solutions of the homogeneous problem
$$y''(t) + b(t)y'(t) + c(t)y(t) = 0\,. \tag{8}$$
Notice that we have, in general, no clue how to find y_1 and y_2. But let us assume that we know them. Then, we would like to find solutions of the equation
$$y''(t) + b(t)y'(t) + c(t)y(t) = f(t) \tag{9}$$
of the form
$$y(t) = \lambda_1(t)y_1(t) + \lambda_2(t)y_2(t)\,, \tag{10}$$
where λ_1 and λ_2 are functions that we have to find. For such a function y, we have that
$$y' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'\,.$$
Since we know nothing about the functions λ_1 and λ_2, we **impose** that
$$\lambda_1' y_1 + \lambda_2' y_2 = 0\,.$$
Then, we obtain
$$y' = \lambda_1 y_1' + \lambda_2 y_2'\,,$$
and
$$y'' = \lambda_1' y_1' + \lambda_1 y_1'' + \lambda_2' y_2' \lambda_2 y_2''\,.$$
Then, in order for the function y as in (10) to solve the equation (9), we need to impose (after a bit of rearrangement)
$$\lambda_1\left(y_1''(t) + b(t)y_1'(t) + c(t)y_1(t)\right) + \lambda_2\left(y_2''(t) + b(t)y_2'(t) + c(t)y_2(t)\right) + \lambda_1' y_1' + \lambda_2' y_2' = f(t)\,.$$
since y_1 and y_2 are solutions of (8), we have that the first two terms in the above equality are zero. Thus, we need λ_1 and λ_2 to satisfy
$$\begin{cases} \lambda_1' y_1 + \lambda_2' y_2 = 0\,, \\ \\ \lambda_1' y_1' + \lambda_2' y_2' = f(t)\,. \end{cases}$$
Let us recall that, since y_1 and y_2 are linearly independent, we have that
$$y_1(s)y_2'(s) - y_2(s)y_1'(s) \neq 0\,,$$
for every s (in technical terms, the Wronskian never vanishes).

Thus, we obtain

$$\lambda_1 = \lambda_1(0) - \int_0^t \frac{y_2(s)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)} f(s)\,\mathrm{d}s\,,$$

and

$$\lambda_2 = \lambda_2(0) + \int_0^t \frac{y_1(s)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)} f(s)\,\mathrm{d}s\,.$$

Finally, in order to find the values of $\lambda_1(0)$ and $\lambda_2(0)$, we use the initial conditions for y. Thus, we obtain the solution of problem (7) as

$$y(t) = \lambda_1(t)y_1(t) + \lambda_2(t)y_2(t)\,,$$

where λ_1 and λ_2 are as above.

Separation of variables technique.

In order to solve a differential equation of the form

$$y'(t) = f(y)g(t)\,, \tag{11}$$

we perform the following steps (they are only a mnemonic way to remember how to solve it. There is a solid theory behind that justifies all we are going to do!):

(i) separate the variables

$$\frac{\mathrm{d}y}{f(y)} = g(t)\mathrm{d}t\,,$$

(ii) integrate both sides

$$\int \frac{\mathrm{d}y}{f(y)} = \int g(t)\mathrm{d}t\,,$$

(iii) we obtain something like

$$F(y) = G(t)\,.$$

In theory the function F is (locally) invertible where $f \neq 0$. If we are able to invert it explicitly, then we can find the solution of (11) as

$$y(y) = F^{-1}\left(G(t)\right)\,.$$

If we also have to incorporate the initial condition, we write $G(t)+k$ in step (iii) above, for some constant k that will be determined by means of the initial condition.