Laplace equation in the 2D ball - the Poisson formula

We want to solve the problem

$$\begin{cases} -\triangle u = 0 & \text{in } B_{\bar r}(0)\,, \\ \quad u = g & \text{on } \partial B_{\bar r}(0)\,, \end{cases} \tag{1}$$

where

$$B_{\bar r}(0) := \{\,(x,y)\in\mathbb{R}^2 \,:\, \|(x,y)\| < \bar r\,\}\,,$$

is the (open) ball of radius $\bar r$ centered at the origin,

$$\partial B_{\bar r}(0) := \{\,(x,y)\in\mathbb{R}^2 \,:\, \|(x,y)\| = \bar r\,\}\,,$$

is the circumference of radius $\bar r$ and $g : \partial B_{\bar r}(0) \to \mathbb{R}$ is a given function. Since the set has a spherical symmetry, it is natural to set the above problem in the space of the polar coordinates. In polar coordinates, the set $B_{\bar r}(0)$ is identified by the sets of points

$$R := \{\,(r,\theta) \,:\, r\in(0,\bar r)\,,\,\theta\in[0,2\pi)\,\}\,,$$

while the set $\partial B_{\bar r}(0)$ is identified by the sets of points

$$\partial R = \{\,(\bar r,\theta) \,:\, \theta\in[0,2\pi)\,\}\,.$$

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So, it is possible to see g as a function $\widetilde g : [0,2\pi) \to \mathbb{R}$ as follows:

$$\widetilde g(\varphi) := g(\bar r\cos\varphi, \bar r\sin\varphi)\,.$$

Finally, we recall that the Laplacian in polar coordinates writes as

$$\triangle = \partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_{\theta\theta}\,.$$

Let us consider the function

$$v(r,\theta) := u\left(r\cos\theta, r\sin\theta\right)\,.$$

Then, the function u solves the problem (1) if and only if the function v solves the following

$$\begin{cases} \partial^2_{rr}v + \frac{1}{r}\partial_r v + \frac{1}{r^2}\partial^2_{\theta\theta}v = 0 & \text{in } R\,, \\ \qquad\qquad\qquad\qquad u = \widetilde g & \text{on } \partial R\,, \end{cases}$$

This problem has an advantage with respect to the previous one: it is settle in a rectangle! So, the idea is to use the technique of separation of variables in order to solve it. Namely, we look for solutions v of the form

$$v(r,\theta) = R(r)\Theta(\theta)\,.$$

If we insert such a function in the equation we want to solve, and by giving the common denominator, we get

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0\,.$$

By dividing everything by $R\Theta$, we get

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta}\,.$$

Since the left-hand side is a function of the sole r, while the right-hand side one depends only on θ, and they have to be equal for all $r\in[0,\bar r)$ and $\theta\in[0,2\pi)$, we deduce that both sides has to be constant. So, we get the two equations

$$r^2R'' + rR' - \lambda R = 0\,,$$

and

$$\Theta'' = -\lambda\Theta\,,$$

for some $\lambda\in\mathbb{R}$. We now have to impose the boundary conditions for R and Θ. Let us start with the latter one. The function Θ describes the angular behavior of the function

u. Thus, it is natural to ask Θ to be periodic, as well as for its derivatives (since we want to function u to be regular!), that is we require

$$\begin{cases} \Theta(0) = \Theta(2\pi)\,, \\ \Theta'(0) = \Theta'(2\pi)\,. \end{cases}$$

On the other hand, for the function R we cannot impose any boundary condition at $r = \bar{r}$ in the equation, since we need to let it free in order to match the boundary conditions. Moreover, since the function R is *not* defined at $r = 0$, we cannot ask R to take a specific value at that point. We simply ask it to be finite when approaching zero, namely we ask for

$$\lim_{r\to 0} |R(r)| < \infty\,.$$

So, we have the problems

$$\begin{cases} \Theta'' = -\lambda\Theta & \text{in } (0, 2\pi)\,, \\ \Theta(0) = \Theta(2\pi)\,, \\ \Theta'(0) = \Theta'(2\pi)\,. \end{cases}$$

and

$$\begin{cases} r^2 R'' + rR' - \lambda R = 0 & \text{in } (0, \bar{r})\,, \\ \lim_{r\to 0} |R(r)| < \infty\,. \end{cases}$$

The first problem has the following general solution

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)\,,$$

where $\lambda = n^2$, for $n = 0, 1, 2, 3, \ldots$. Let us now consider, for each $n \in \mathbb{N}$, the corresponding problem for R. In the case $n = 0$, the equation for R reduces to

$$r^2 R'' + rR' = 0\,,$$

that is, by dividing by r,

$$0 = rR'' + R' = \left(rR'\right)'\,.$$

Thus, we get $rR' = C_0$, for some constant $C_0 \in \mathbb{R}$, and in turn

$$R(r) = C_0 \log r + D_0\,,$$

for some constant $D_0 \in \mathbb{R}$. By imposing the limiting behavior at $r = 0$, we get $C_0 = 0$. Thus, $R_0 = D_0$.

Let us now consider the problem for R in the case $n = 1, 2, 3, \ldots$. The equation is

$$r^2 R'' + rR' - n^2 R = 0\,.$$

Such a type of equations are called *Euler type* of equation: they are ODEs where the coefficient of the k^{th} derivative is r^k, *i.e.* of the form

$$\sum_{k=1}^{D} \left(r^k \frac{\mathrm{d}^k}{\mathrm{d}r^k} u \right) = 0\,.$$

The idea is to look for solutions of the form (and in one line you'll get why!)

$$R(r) = r^\alpha\,,$$

for some $\alpha \in \mathbb{C}$ we have to find (we also want to consider sine and cosine). If we plug in this expression in the above equation, we get

$$\left(\alpha(\alpha - 1) + \alpha - n^2\right) r^\alpha = 0\,,$$

that is we reduced an ODE to an algebraic equation. In particular, we get $\alpha = \pm n$. So, the general solution of the above equation is given by

$$R_n(r) = A_n r^n + \frac{B_n}{r^n}\,.$$

If we now impose that R_n stays bounded as r approaches zero, we get $B_n = 0$.

So, we obtain the solution (by using just one name for the constants)

$$v_n(r,\theta) = \frac{A_0}{2},$$

in the case $n = 0$, and

$$v_n(r,\theta) = r^n\,(\,A_n\cos(n\theta) + B_n\sin(n\theta)\,)\;.$$

So, as usual, we consider the function

$$v(r,\theta) := \sum_{n=0}^{\infty} v_n(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n\,(\,A_n\cos(n\theta) + B_n\sin(n\theta)\,)\;.$$

We now need to impose the boundary condition at $r = \bar{r}$: we want

$$\widetilde{g}(\theta) = v(\bar{r},\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \bar{r}^n\,(\,A_n\cos(n\theta) + B_n\sin(n\theta)\,)\;.$$

Thus, we assume g to be of the form

$$\widetilde{g}(\theta) = \frac{\widetilde{g}_0^1}{2} + \sum_{n=1}^{\infty} \bar{r}^n\left(\,\widetilde{g}_n^1\cos(n\theta) + \widetilde{g}_n^2\sin(n\theta)\,\right)\,,$$

where

$$\widetilde{g}_n^1 = \frac{1}{\pi}\int_0^{2\pi}\widetilde{g}(\varphi)\cos(n\varphi)\mathrm{d}\varphi\,,\qquad \widetilde{g}_n^2 = \frac{1}{\pi}\int_0^{2\pi}\widetilde{g}(\varphi)\sin(n\varphi)\mathrm{d}\varphi\,,$$

are the Fourier coefficients of g. So, we get

$$A_n = \frac{\widetilde{g}_n^1}{\bar{r}^n}\,,\qquad B_n = \frac{\widetilde{g}_n^2}{\bar{r}^n}\,,$$

and the solution is

$$v(r,\theta) = \frac{\widetilde{g}_0^1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{\bar{r}}\right)^n\left(\,\widetilde{g}_n^1\cos(n\theta) + \widetilde{g}_n^2\sin(n\theta)\,\right)\;.$$

You would also be satisfied with this writing of the solution. But sometimes, you need more. And in this case, you can get more! Indeed, it is remarkable that the above series can be summed *explicitly*! So

$$\begin{aligned}
v(r,\theta) &= \frac{\widetilde{g}_0^1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{\bar{r}}\right)^n\left(\,\widetilde{g}_n^1\cos(n\theta) + \widetilde{g}_n^2\sin(n\theta)\,\right)\\
&= \frac{1}{\pi}\int_0^{2\pi}\widetilde{g}(\varphi)\mathrm{d}\varphi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{\bar{r}}\right)^n\left(\int_0^{2\pi}\widetilde{g}(\varphi)\cos(n\varphi)\cos(n\theta)\mathrm{d}\varphi + \int_0^{2\pi}\widetilde{g}(\varphi)\sin(n\varphi)\sin(n\theta)\mathrm{d}\varphi\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\widetilde{g}(\varphi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{\bar{r}}\right)^n\cos\left(n(\theta-\varphi)\right)\right]\mathrm{d}\varphi\,,
\end{aligned}$$

where in the last step we took the series inside the integral and we used the formula

$$\cos(\alpha-\beta) = \frac{1}{2}\left(\cos\alpha\cos\beta + \sin\alpha\sin\beta\right)\;.$$

Now, we repeat a computation we did when we proved the pointwise convergence theorem for the Fourier series:

$$\begin{aligned}
1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{\bar{r}}\right)^n\cos\left(n(\theta-\varphi)\right) &= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{\bar{r}}\right)^n e^{in(\theta-\varphi)} + \sum_{n=1}^{\infty}\left(\frac{r}{\bar{r}}\right)^n e^{-in(\theta-\varphi)}\\
&= 1 + \sum_{n=1}^{\infty}\left(\frac{re^{i(\theta-\varphi)}}{\bar{r}}\right)^n + \sum_{n=1}^{\infty}\left(\frac{re^{-i(\theta-\varphi)}}{\bar{r}}\right)^n,
\end{aligned}$$

where in the first step we used the Euler formula

$$e^{i\alpha} = \cos\alpha + i\sin\alpha\,,$$

and the fact that sin is an odd function. By recalling that

$$\sum_{n=1}^{\infty} q^n = \frac{1}{1-q} - 1\,,$$

if $|q| < 1$, by setting (for having a lighter notation in the forthcoming computations)

$$p := \frac{r}{\bar r}\,, \qquad t := e^{i(\theta-\varphi)}\,,$$

we get

$$1 + \sum_{n=1}^{\infty}(pt)^n + \sum_{n=1}^{\infty}\left(\frac{p}{t}\right)^n = 1 + \frac{1}{1-pt} - 1 + \frac{1}{1-\frac{p}{t}} - 1 = t\frac{1-p^2}{(1-pt)(t-p)}\,,$$

and thus

$$1 + \sum_{n=1}^{\infty}\left(\frac{re^{i(\theta-\varphi)}}{\bar r}\right)^n + \sum_{n=1}^{\infty}\left(\frac{re^{-i(\theta-\varphi)}}{\bar r}\right)^n = e^{i(\theta-\varphi)}\frac{1-\left(\frac{r}{\bar r}\right)^2}{(1-\frac{r}{\bar r}e^{i(\theta-\varphi)})(e^{i(\theta-\varphi)}-\frac{r}{\bar r})}$$
$$= \frac{\bar r^2 - r^2}{\bar r^2 + r^2 - 2r\bar r\cos(\theta-\varphi)}\,.$$

We would like to rewrite the denominator. By looking at the figure,
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we realize that, by applying the generalized Pythagorean theorem[1] we get

$$\bar r^2 + r^2 - 2r\bar r\cos(\theta-\varphi) = |P - \bar P|^2\,, \tag{2}$$

where P and $\bar P$ are the points given by

$$\begin{cases} x = r\cos\theta\,,\\ y = r\sin\theta\,, \end{cases} \qquad \begin{cases} x = \bar r\cos\varphi\,,\\ y = \bar r\sin\varphi\,, \end{cases}$$

respectively. Hence, we get

$$v(r,\theta) = \frac{\bar r^2 - r^2}{2\pi}\int_0^{2\pi}\frac{\widetilde g(\varphi)}{\bar r^2 + r^2 - 2r\bar r\cos(\theta-\varphi)}\mathrm{d}\varphi\,. \tag{3}$$

We would like to have the expression for the function u. Basically, we just have to make a change of variable (to rescale) the integral: instead of an integral on the circumference of radius 1, we want an integral on the circumference of radius $\bar r$. The difference between the length of the two circumferences is $\bar r$. Thus, by recalling (2), we get

$$\boxed{u(P) = \frac{\bar r^2 - \|P\|^2}{2\pi\bar r}\int_{\partial B_{\bar r}(0)}\frac{g(Q)}{\|P-Q\|^2}\mathrm{d}s(Q)\,,}$$

where the integral is just a way to write (3). This formula is called the **Poisson formula**. As you may notice, the above formula is *not* defined for points in $\partial B_{\bar r}(0)$. Nevertheless, the above function is a solution of our problem, because it is harmonic inside the ball, and it is possible to prove that

$$\lim_{P\to Q} u(P) = g(Q)\,,$$

for every point Q on the circumference of radius $\bar r$, where the points P vary in the ball.

The above formula is important for several reasons: first of all it is an explicit solution of the Laplace equation in the sphere. Moreover, we can use it to (finally) justify the

[1] Also called the cosine rule, but it is really better to called it the other way, since, well, that is its geometric meaning!

name *harmonic* for functions satisfying the Laplace equation. Indeed, take P to be the origin. The above expression for u tells us that

$$u(0) = \frac{1}{2\pi\bar{r}} \int_{\partial B_{\bar{r}}(0)} g(Q) \mathrm{d}s(Q) \,,$$

namely, $u(0)$ is the average of its values on the circumference. Of course, this holds for *every* point and *every* circumference around that point. By using polar coordinates, it is also possible to prove that

$$u(0) = \frac{1}{\pi\bar{r}^2} \int_{B_{\bar{r}}(0)} u(Q) \mathrm{d}s(Q) \,,$$

that is, $u(0)$ is also the average of its values in a ball centered at that point (again, this holds for *every* point and *every* ball around that point). Finally, it is also true that the above property characterizes harmonic functions, *i.e.*, if $u : \mathbb{R}^N \to \mathbb{R}$ is a C^2-function such that

$$u(x) = \frac{1}{\pi r} \int_{B_r(x)} u(Q) \mathrm{d}s(Q) \,,$$

for every $x \in \mathbb{R}^N$ and every $r > 0$, then u is harmonic.