PDEs

General review

1) Transport equation:

i)

$$\begin{cases} u_t + c u_x = f(x,t) & \text{in } \mathbb{R}\times(0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

There exists a unique solution, given by:

$$u(x,t) = \underbrace{g(x-ct)}_{\substack{\text{the initial data}\\ \text{is transported}\\ \text{along the}\\ \text{characteristic lines}\\ \text{[i.e., solution when}\\ f\equiv 0]}} + \underbrace{\int_0^t f(x-(t-s)c, s)\,ds}_{\substack{\text{Duhamel's principle:}\\ \text{superposition of all}\\ \text{the effects of } f.\\ \text{For every } s\in[0,t],\\ x\mapsto f(x,s)\\ \text{acts like an initial data.}}}.$$

ii)

$$\begin{cases} a(x,t)\, u_t(x,t) + b(x,t)\, u_x(x,t) = 0 & \text{in } \mathbb{R}\times(0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

the characteristic curves are the solutions $s \overset{z}{\longmapsto} (z_1(s), z_2(s))$ to

$$(CS)\ \begin{cases} \dot z_1(s) = a(z(s)) \\ \dot z_2(s) = b(z(s)) \end{cases}$$

It can be the case that, for a point $(x_0, t_0) \in \mathbb{R}^2$

- no characteristics pass through it
  [freedom in the definition of $u(x_0, t_0)$]

- more than one characteristic passes through it
  [$\to$ problem!] [• either the initial data is nice, or we have an issue!]

Notice that, when $a(x,t) \neq 0$, it's easier to consider:

$$u_t + \frac{b(x,t)}{a(x,t)} u_x = 0$$

and to look for curves of the form

$$t \mapsto (p(t), t),$$

where p solves:

$$\dot{p}(t) = \frac{b(p(t), t)}{a(p(t), t)}.$$

2) The heat equation:

i) in the whole space:

$$(HEU)\begin{cases} u_t - D u_{xx} = f(x,t) & \text{in } \mathbb{R}\times(0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R}. \end{cases}$$

- in general the above pb has more than one solution. Uniqueness holds in a particular class of functions.
- let

$$u(x,t) := \int_{\mathbb{R}} g(y)\,\Gamma_D(x-y,t)\,dy + \int_0^t \left[\int_{\mathbb{R}} f(y,s)\,\Gamma_D(x-y,t-s)\,dy\right] ds$$

where:

$$\Gamma_D(x,t) := \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}}$$

is the heat kernel.

Assume:
- $\int_{\mathbb{R}} |g(x)|\,dx < +\infty$,
- f, f_t, f_{xx} are continuous & bounded in $\mathbb{R}\times[0,+\infty)$,

then u solves (HEU).

Notice that the value of the initial data g in a pt y will affect the value of the solution u in any $(x,t)\in\mathbb{R}\times(0,\infty)$ → infinite speed of propagation

ii) bounded domains:

[f depends only on x & t, not on u!

$$(\text{HEB})\begin{cases} u_t - Du_{xx} = \underbrace{f(x,t)} & \text{in } (0,L)\times(0,+\infty), \\ u(x,0) = g(x) & x\in\mathbb{R}, \\ \text{BCs at } x=0, x=L & \forall t\geq 0. \end{cases}$$

↳ they have to be linear in u!

- Maximum principle:
  In the case $f\equiv 0$ & any BCs it holds:
  either u is constant, or it attains its maximum and its minimum only in $(\mathbb{R}\times\{t=0\})\cup(\{0\}\times[0,+\infty))$
  $\cup(\{L\}\times[0,+\infty))$.

- Uniqueness:
  The solution of (HEB) is unique.

- How to find the solution of (HEB):

  a) let v be any function satisfying the inhomogeneous BCs, and consider
  $$\tilde{u} := u - v.$$
  Then $\tilde{u}$ solves:

$$(\text{HEB})^*\begin{cases} \tilde{u}_t - D\tilde{u}_{xx} = \tilde{f}(x,t) & \text{in } \mathbb{R}\times[0,+\infty), \\ \tilde{u}(x,0) = \tilde{g}(x) & \text{in } \mathbb{R} \\ \text{homogeneous BCs} & \forall t\geq 0, \end{cases}$$

  for some $\tilde{f}$ and $\tilde{g}$ depending on f, g and v.
  [in a linear way!]

b) Consider the pb [the homogeneous one]

$$\begin{cases} w_t - D w_{xx} = 0 & \text{in } \mathbb{R} \times [0, +\infty), \\ w(x,0) = g(x) & \text{in } \mathbb{R}, \\ \text{homogeneous BCs} & \forall t > 0. \end{cases}$$

By using the separation of variable technique, it is possible to solve for w, and write it as:

$$w(x,t) = \sum_n a_n(t) X_n(x), \quad \rightsquigarrow \quad \boxed{\begin{cases} X_n'' = -\lambda_n X_n \\ \text{BCs for } X_n \end{cases}}$$

where the functions X_n's depends on the BCs.

c) Assume $\tilde{u}$ of the form:

$$\tilde{u}(x,t) = \sum_n b_n(t) X_n(x).$$

Moreover, write: [by using the generalized Fourier series]

- $\tilde{f}(x,t) = \sum_n \tilde{f}_n(t) X_n(x),$
- $\tilde{g}(x) = \sum_n \tilde{g}_n X_n(x).$

By plugging in these expressions in (HEB)* and by using the uniqueness of the Fourier expansion, we get the family of ODEs:

$$(\text{FODEs}) \begin{cases} b_n'(t) + D\lambda_n b_n(t) = \tilde{f}_n(t), \\ b_n(0) = \tilde{g}_n. \end{cases}$$

Thus, we find:

$$b_n(t) = \tilde{g}_n e^{-D\lambda_n t} + \int_0^t \tilde{f}_n(s) e^{-D\lambda_n (t-s)} ds.$$

d) By using the linearity of the Fourier coefficients, it is possible to write each $\tilde{f}_n(s)$ and $\tilde{g}_n$ in terms of $f_n(s)$, g_n and v [notice that v depends only on the inhomogeneous BCs]. Thus, we find:

$$u(x,t) = \tilde{u}(x,t) + v(x,t),$$

written in terms of f, g and the inhomogenous BCs only.

- <u>Notice</u>:
  - the same strategy holds also for variants of the heat equation in bounded domains,
  - in the case of variants of the heat equation in the <u>whole space</u>, it is possible to reduce to the classical heat equation by using a change of variable.

3) The wave equation:

1) in the whole space:

$$(WEU) \quad \begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{in } \mathbb{R} \times (0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R}, \\ u_t(x,0) = h(x) & \text{in } \mathbb{R}. \end{cases}$$

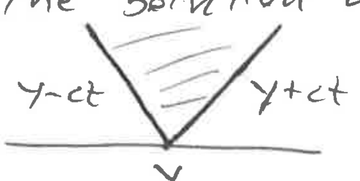
The solution of the above problem is unique and it's given by the d'Alembert Formula

$$u(x,t) = \frac{1}{2}\left[g(x-ct) + g(x+ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy.$$

Notice that:

- $u(x,t) = F(x-ct) + G(x+ct)$, is the sum of a waves moving on the right with speed c, F, and a wave moving on the left with speed c, G;

- $u(x,t)$ depends only on the initial data in $(x-ct, x+ct)$. In particular, the value of the initial data in a pt y will affect the value of the solution u only in



$\rightarrow$ Finite speed of propagation

b) <u>bounded domains</u>:

$$(WEB) \begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{in } (0,L)\times[0,+\infty), \\ u(x,0) = g(x) & \text{in } (0,L), \\ u_t(x,0) = h(x) & \text{in } (0,L), \\ \text{BCs at } x=0,\ x=L & \forall t>0. \end{cases}$$

For the above pb, the maximum principle does **not** hold!

However, by using the <u>energy method</u>, it is possible to prove the <u>uniqueness</u> for the solution.
[for good BCs, like Dirichlet or Neumann]

- <u>How to find the solution of (WEB)</u>:

The strategy is the adapted version of the one we used for (HEB). The big difference is in the family of ODEs we get:

$$\begin{cases} b_n''(t) + \lambda_n^2 c^2 b_n(t) = \tilde{f}_n(t), \\ b_n(0) = \tilde{g}_n, \\ b_n'(0) = \tilde{h}_n. \end{cases}$$

The solution is given by:

$$b_n(t) = \left[\tilde{g}_n - \frac{1}{\lambda_n c}\int_0^t \tilde{f}_n(s)\sin(\lambda_n s)\,ds\right]\cos(\lambda_n t)$$

$$+ \left[\frac{1}{\lambda_n c}\tilde{h}_n + \frac{1}{\lambda_n c}\int_0^t \tilde{f}_n(s)\cos(\lambda_n s)\,ds\right]\sin(\lambda_n t).$$

Thus:

$$u(x,t) = v(x,t) + \sum_n b_n(t) X_n(x),$$

where the X_n's are those from the solution of the homogeneous pb. Moreover, it is possible to express the b_n's in terms of the data $f, g, h, \varphi_1, \varphi_2$.

• <u>Notice</u>:
- the same strategy applies to variants of the wave eq. in bounded domains,
- in the case of variants of the wave eq. in the <u>whole</u> space, it is possible to reduce to the classical wave eq. by using a change of variable.

4) Fourier series:

- **Def:** a function $f: [a,b] \to \mathbb{R}$ is said to be <u>piece-wise continuous</u> if there exist $a = x_0 < x_1 < \dots < x_K = b$ s.t. f restricted to each $[x_i, x_{i+1}]$ is continuous.
  In particular, for every index i there exist

$$f^-(x_i) := \lim_{y \to x_i^-} f(y) \quad \to y < x_i$$

$$f^+(x_i) := \lim_{y \to x_i^+} f(y) \quad \to y > x_i$$

- **Def:** a sequence of functions $(f_n)_{n \in \mathbb{N}}$ is said to <u>convergence pointwise</u> to f in $[a,b]$ if

$$\lim_{n \to +\infty} f_n(x) = f(x) \qquad \forall x \in [a,b].$$

→ the above can be written as follows:

$$\forall \varepsilon > 0 \quad \exists\, \bar{n}(x) \in \mathbb{N} \text{ s.t.}$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\forall n \geq \underbrace{\bar{n}(x)}.$$

meaning that this $\bar{n}$ <u>depends</u> on $x \in [a,b]$

- Def: a sequence of functions $(f_n)_{n\in\mathbb{N}}$ is said to converge uniformly to a function f in $[a,b]$ if

$$\forall \varepsilon > 0 \quad \exists\, \bar{n} \in \mathbb{N} \text{ (independent of } x \in [a,b]\text{)}$$

s.t.

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall x \in [a,b] \quad \forall n \geq \bar{n}.$$

- Notice that uniform convergence implies pt-wise convergence, while the opposite is false.

- For all the other properties, see the handouts on Review on Fourier series and FS-properties

5) The Laplace and the Poisson equation:

a) in the whole space:

- $$-\Delta u = 0 \quad \text{in } \mathbb{R}^N, \qquad \text{Laplace eq.}$$

  - $\leadsto$ u is called harmonic
  - the above equation represents the static phenomena, e.g., a steady wave.
  - non uniqueness

- $$-\Delta u = f(x) \quad \text{in } \mathbb{R}^N, \qquad \text{Poisson eq.}$$

  - non uniquess
  - a solution is given by

$$u(x) := \int_{\mathbb{R}} \Gamma(x-y) f(y)\, dy,$$

where Γ is the fundamental solution for the Laplace eq., given by:

$$\Gamma(x) := \begin{cases} -\frac{1}{2\pi} \log(\|x\|) & N = 2, \\ \frac{1}{N(N-2)\omega_N} \frac{1}{\|x\|^{N-2}} & N \geq 3, \end{cases}$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$.

b) in bounded domains:

- by using Greens second identity

$$\int_{\Omega} (u \Delta v - v \Delta u) \, dx = \int_{\partial\Omega} (u \partial_\nu v - v \partial_\nu u)$$

and the fact that:

$$\text{"} -\Delta \Gamma(x) = \delta_0(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \text{"},$$

we get the formula:

$$u(x) = \int_{\Omega} \Gamma(y-x)(-\Delta u(y)) \, dy + \int_{\partial\Omega}$$

- Let us now consider the problem:

$$(BP) \begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open bounded set with smooth boundary.
Since we do not know $\partial_\nu u$ on $\partial\Omega$, we need to modify Γ on $\partial\Omega$ in such a way to get rid of that term.

So, for every $x \in \partial\Omega$, let Φ^x be a function s.t.

$$\begin{cases} -\Delta \Phi^x = 0 & \text{in } \Omega, \\ \Phi^x(y) = \Gamma(y-x) & \text{on } \partial\Omega \end{cases}$$

and define:

$$G(x,y) := \Gamma(y-x) - \Phi^x(y).$$

the Greens function [for $-\Delta$ in Ω w.r.t. Dirichlet BCs]

Then, the <u>unique</u> solution of (BP) is given by:

$$\left[\ u(x) = \int_\Omega G(x,y)\, f(y)\, dy + \int_{\partial\Omega} g(y)\, \underbrace{\nabla_y G(x,y)}_{\text{gradient w.r.t. } y \text{ [keeping } x \text{ fixed]}} \cdot \nu .\right.$$

Notice that, even if we don't know explicitly the function G, it is still possible to prove properties of the solution u.

- <u>Maximum principle</u>:
  Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of $\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$
  Then either $\min_{\partial\Omega} g < u(x) < \max_{\partial\Omega} g$ for every $x \in \Omega$

  or u is constant.

c) <u>Special domains</u>:

In the case Ω has a special geometry, it is possible to use the technique of separation of variables to solve the problem. The general strategy is the following:

i) use the linearity of the pb to handle one inhomogeneous BC per time, i.e., write the solution u as $u = \sum v^i$, where each v^i solves $-\Delta v^i = 0$, but satisfies all homogenous BCs, but one,

ii) for every v^i, look for separated solutions

iii) use the equation for the variable with homogeneous BCs to get the eigenvalues and substitute them in the other equations

iv) sum all the functions found in iii) ; $\sum c_n f_n$

v) find the coefficients of the series, c_n, by imposing the inhomogeneous BC.

- Examples of special domains are rectangles, both in euclidean or polar/spherical coordinates.

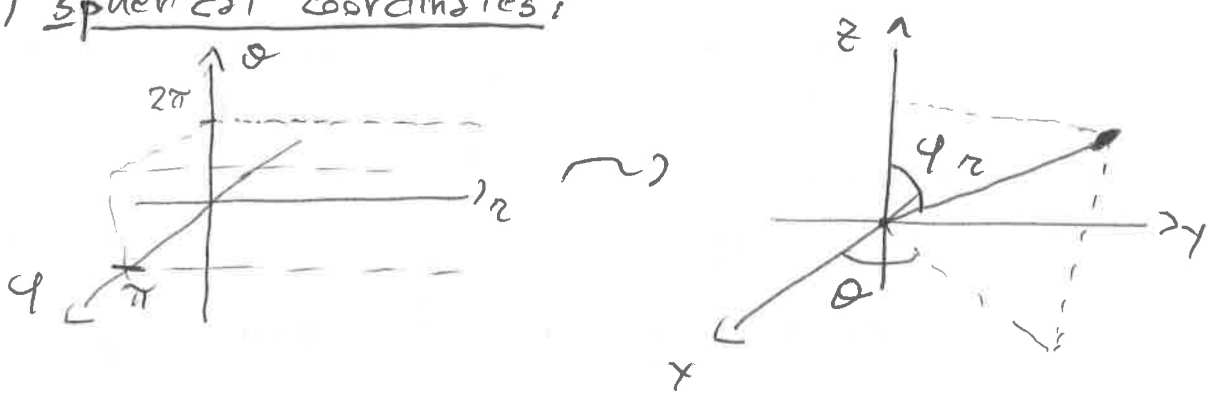
d) Laplacian in polar and spherical coordinates:

i) polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\left[\ \partial^2_{xx} + \partial^2_{yy} = \partial^2_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial^2_{\theta\theta}\right.$$

ii) spherical coordinates:



$$\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\cos\varphi\sin\theta \\ z = r\sin\varphi \end{cases}$$

$$\left[\partial^2_{xx}+\partial^2_{yy}+\partial^2_{zz}=\partial^2_{rr}+\frac{2}{r}\partial_r+\frac{1}{r^2}\left[\partial^2_{\varphi\varphi}+\cot g\,\varphi\,\partial_\varphi+\frac{1}{\sin^2\varphi}\partial^2_{\theta\theta}\right]\right.$$