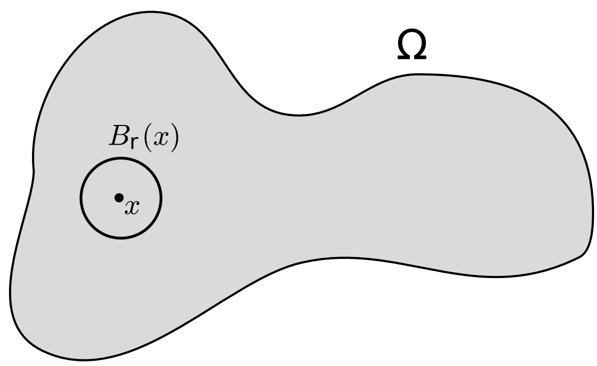
Laplace equation in bounded domains

We now want to consider the Laplace equation in bounded domains, that would be the analogous of problems in intervals in $\mathbb{R}$. The fact is that the intervals are really simple objects, while a bounded set in $\mathbb{R}^N$ can be very wild.

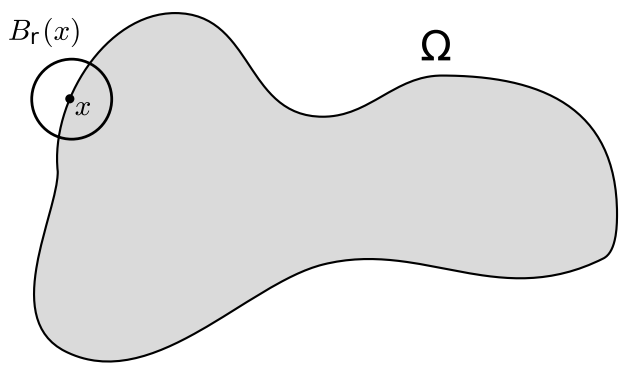
Definition 0.1. Given a set $\Omega \in \mathbb{R}^N$, we define its *complement* (in $\mathbb{R}^N$), denoted by Ω^c as the set of points that do no belong to Ω.

Definition 0.2. Given a set $\Omega \in \mathbb{R}^N$, we say that it is *bounded* if there exists a radius $R > 0$ such that Ω is contained in $B_R(0)$.

Definition 0.3. Let $\Omega \subset \mathbb{R}^N$ be a set. We say that Ω is *open* if for every point $x \in \Omega$, it is possible to find a radius $r > 0$ such that the ball $B_r(x)$ is contained in Ω.



Definition 0.4. Let $\Omega \subset \mathbb{R}^N$ be a set. We define the *boundary* of Ω, denoted by $\partial\Omega$, as the sets of points $x \in \mathbb{R}^N$ such that for every $r > 0$ the ball $B_r(x)$ intersects both Ω and its complement Ω^c.



Definition 0.5. Let $\Omega \subset \mathbb{R}^N$ be a set. We define the *closure* of Ω, denoted by $\bar{\Omega}$, as the sets $\Omega \cup \partial\Omega$.

We are finally in position to talk about the Laplace equation in a bounded domain. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. We consider the problem

$$\begin{cases} -\triangle u = 0 & \text{in } \Omega\,, \\ \quad u = g & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $g : \partial\Omega \to \mathbb{R}$ is a given function. The above is called the Dirichlet problem for the Laplace equation. When we say that u is a solution of the above problem, we mean that $u : \bar{\Omega} \to \mathbb{R}$ is a function such that

- u is continuous in $\bar{\Omega}$,
- $u \in C^2(\Omega)$,
- and u solves (1).

Indeed, since the derivative of a function is defined only in points that are *interior* points, we need the set Ω to be open, and u to be regular enough to talk about its Laplacian, namely, we need u to have (at least) two derivatives. Moreover, since we want u to attain a certain value on $\partial\Omega$, we need to make sense of that assertion. That's why we asked u to be continuous up to the boundary.

A fundamental result about the above problem is the following

Theorem 0.6 (Maximum principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $u : \bar{\Omega} \to \mathbb{R}$ be a continuous function such that*

$$\begin{cases} -\triangle u = 0 & \text{in } \Omega\,, \\ \quad u = g & \text{on } \partial\Omega\,. \end{cases}$$

Then, either u is constant, or the maximum and the minimum of u are attained only *on $\partial\Omega$.*

Remark 0.7. What the above theorem says is that a continuous function u that solves the problem (1) is either constant, or satisfies

$$u(x) < \max_{y \in \partial\Omega} u(y)\,, \qquad u(x) > \min_{y \in \partial\Omega} u(y)\,,$$

for every $x \in \Omega$.

The maximum principle allows to prove uniqueness for the solution of problem (1).

Theorem 0.8. *(Uniqueness) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Then the problem*

$$\begin{cases} -\triangle u = 0 & \text{in } \Omega\,, \\ \quad u = g & \text{on } \partial\Omega\,, \end{cases}$$

admits a unique solution.

Proof. Assume that u, v are two solutions of the above problem, and consider the function $w := u - v$. By using the linearity of the second derivatives, it holds that

$$-\triangle w = -\triangle(u - v) = -\triangle u + \triangle v = 0\,.$$

Moreover, since both u and v are equal to g on $\partial\Omega$, we get that $w = 0$ on $\partial\Omega$. Thus, w solves the problem

$$\begin{cases} -\triangle w = 0 & \text{in } \Omega\,, \\ \quad w = 0 & \text{on } \partial\Omega\,. \end{cases}$$

By applying the maximum principle, we get that either one of the following two cases is true: either w is constant, and in this case $w = 0$ in Ω, since $w = 0$ on $\partial\Omega$, or

$$w(x) < \max_{y \in \partial\Omega} w(y)\,, \qquad w(x) > \min_{y \in \partial\Omega} w(y)\,,$$

for every $x \in \Omega$. But we have that

$$w(x) < \max_{y \in \partial\Omega} w(y) = \max_{y \in \partial\Omega} g(y) = 0\,,$$

and that means

$$u(x) < v(x)\,, \qquad \text{for every } x \in \Omega\,,$$

and

$$w(x) > \min_{y\in\partial\Omega} w(y) = \min_{y\in\partial\Omega} g(y) = 0\,,$$

and that means

$$u(x) > v(x)\,, \qquad \text{for every } x \in \Omega\,,$$

This is clearly a contradiction. Thus, the first case holds, and thus $u = v$.

□