The Laplace and the Poisson equations in the whole space.

We would like to consider the heat and the wave equation in higher dimensions, meaning, when the region where the phenomena takes place is not one dimensional. Think, for instance, to the vibration of a drum head: the set that is vibration is the membrane, that can be thought as a two dimensional set. The heat equation in one dimension is

$$u_t - Du_{xx} = 0\,.$$

Thus, we have to consider the analogous of the second derivative in higher dimension. The correct notion is given by the following object

Definition. Let $u : \mathbb{R}^N \to \mathbb{R}$ be a function of class C^2. We define the *Laplacian* of u, denoted by $\triangle u : \mathbb{R}^N \to \mathbb{R}$, as the function given by

$$\triangle u(x) := \sum_{i=1}^{N} \partial^2_{x_i x_i} u(x)\,,$$

where x_i is the i^{th} coordinate in $\mathbb{R}^N$.

Example: if $u : \mathbb{R}^2 \to \mathbb{R}$, we have that

$$\triangle u = \partial^2_{xx} u + \partial^2_{yy} u\,,$$

while if $u : \mathbb{R}^3 \to \mathbb{R}$, we have that

$$\triangle u = \partial^2_{xx} u + \partial^2_{yy} u + \partial^2_{zz} u\,,$$

and so on.

It is of great interest to study the following equation:

$$-\triangle u = 0\,, \qquad \text{in } \mathbb{R}^N\,, \tag{1}$$

called the **Laplace equation**. A function satisfying the above equation is called **harmonic**. This name will be justified later. The above equation has **no** unique solution. Nevertheless we can look for a specific kind of solution, namely one that depends only on the modulus of the point, *i.e.*, we look for solutions of (1) of the form

$$u(x) = v(|x|) = v(r)\,,$$

where $v : [0, \infty) \to \mathbb{R}$ is a function, and $r(x) := |x|$. We want to understand what equation v has to satisfy in order for u to solve the Laplace equation. By applying the chain rule, we have that

$$\partial_{x_i} u(x) = \partial_{x_i}\left(v(r(x))\right) = v'(r(x))\frac{\partial r(x)}{x_i} = v'(r(x))\frac{x_i}{r}\,,$$

$$\partial^2_{x_i x_i} u(x) = \partial_{x_i}\left(v'(r(x))\frac{x_i}{r}\right) = v''(r(x))\frac{x_i^2}{r^2} + v'(r(x))\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\,.$$

Thus, by using the fact that

$$r^2(x) = |x|^2 = \sum_{i=1}^{N} x_i^2\,,$$

we get that

$$\triangle u(x) = \sum_{i=1}^{N} \partial^2_{x_i x_i} u(x) = \sum_{i=1}^{N} \left[v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right]$$
$$= v''(r) + \frac{N-1}{r} v'(r)\,.$$

so, u solves (1) if and only if v solves

$$v''(r) + \frac{N-1}{r} v'(r) = 0\,.$$

The above equation can be written as

$$\big(\log(|v'|)\big)' = -\frac{N-1}{r}\,.$$

By integrating both sides, we get

$$\log(|v'|) = \log\left(\frac{1}{rN-1}\right)\,,$$

and thus

$$v' = \frac{C}{r^{N-1}}\,,$$

for some constant C. We now have to consider two cases: if $N = 2$, the above equation writes as

$$v' = \frac{C}{r}\,,$$

and thus the solution is

$$v(r) = C\log r\,.$$

In the case $N \geq 3$, we have that

$$v(r) = \frac{C}{r^{N-2}}\,.$$

We thus define (the choice of the constants below have a reason!) the so called **fundamental solution** of the Laplace equation as

$$\Gamma(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } N = 2\,, \\ \frac{1}{N(N-2)\omega_N}\frac{1}{|x|^{N-2}} & \text{if } N \geq 3\,, \end{cases}$$

where ω_N is the measure of the unit ball $\{x \in \mathbb{R}^N \,:\, |x| < 1\}$.

Remark. Notice that the function Γ is **not** defined for $x = 0$! However, for every $x \neq 0$, it holds $-\triangle\Gamma(x) = 0$.

The Poisson equation.
The reason why we care about the fundamental solution is that, similarly to what we did with the heat kernel, with that function we can construct a solution of the inhomogeneous Laplace equation

$$-\triangle u = g\,, \qquad \text{in } \mathbb{R}^N\,,$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is a given function. The above equation is called the **Poisson equation**. It holds that the function

$$u(x) := \int_{\mathbb{R}^N} g(y)\Gamma(x-y)\,\mathrm{d}y\,,$$

solves the above equation.