Fourier series - solution of the wave equation

We would like to justify the solution of the wave equation in a bounded domain we found by using the separation of variable technique. Let us consider the following problem

$$\begin{cases} u_{tt} - Du_{xx} = 0 & \text{in } [0, L] \times (0, \infty)\,, \\ u(x, 0) = g(x) & \text{in } [0, L]\,, \\ u_t(x, 0) = h(x) & \text{in } [0, L]\,, \\ u(0, t) = u(L, t) = 0 & \text{for } > 0\,. \end{cases} \tag{1}$$

The solution we were able to find was

$$u(x,t) := \sum_{n=1}^{\infty} \left[g_n \cos\left(\frac{n\pi}{L} ct\right) + \frac{L}{n\pi c} h_n \sin\left(\frac{n\pi}{L} ct\right) \right] \sin\left(\frac{n\pi}{L} x\right)\,, \tag{2}$$

by assuming the following sine Fourier series expansion of the initial data g and h:

$$\sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L} x\right)\,, \qquad \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{L} cx\right)\,.$$

In order to prove that the function u above is the solution of our problem, we cannot differentiate term-by-term the series defining u. We will instead use the reflection method: we consider the odd $2L$-periodic extensions of g and h, namely we first extend g and h in $[-L, L]$ in an odd way ($g(x) = -g(x)$ for $x \in [-L, 0)$ and same for h), and then we take the periodic extensions $\widetilde{g}$ and $\widetilde{h}$ of these functions. Let us now consider the wave equation in the whole space

$$\begin{cases} v_{tt} - Dv_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty)\,, \\ v(x, 0) = \widetilde{g}(x) & \text{in } \mathbb{R}\,, \\ v_t(x, 0) = \widetilde{h}(x) & \text{in } \mathbb{R}\,. \end{cases}$$

It is easy to see that $v(0) = v(L) = 0$. Thus, we have that v restricted to the interval $[0, L]$ solves problem (1). But we have an explicit formula for v:

$$v(x,t) = \frac{1}{2}\left[\widetilde{g}(x - ct) + \widetilde{g}(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \widetilde{h}(y)\, \mathrm{d}y\,.$$

Since we are using the sine Fourier series for g and h, these are the Fourier series of $\widetilde{g}$ and $\widetilde{h}$ (since they are odd in $[-L, L]$). By inserting these two expansion in the above formula, we get

$$v(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{L}(x - ct)\right) + \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{L}(x + ct)\right) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{L} y\right)\,.$$

We now use the identities

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\,,$$

and

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

combined with the fact that we can integrate term-by-term the Fourier series, to obtain

$$v(x,t) = \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{L} x\right)\cos\left(\frac{n\pi}{L} ct\right) + \frac{1}{c}\sum_{n=1}^{\infty} \frac{L}{n\pi} h_n \sin\left(\frac{n\pi}{L} x\right)\sin\left(\frac{n\pi}{L} ct\right)\,,$$

that is the expression defining u in (2). Thus, the function we found by using the separation of variables technique is a solution of the problem (1).