Fourier series - solution of the heat equation

We would like to justify the solution of the heat equation in a bounded domain we found by using the separation of variable technique. Let us consider the following problem

$$\begin{cases} u_t - Du_{xx} = 0 & \text{in } [0,L]\times(0,\infty)\,,\\ u(x,0) = g(x) & \text{in } [0,L]\,,\\ u(0,t) = u(L,t) = 0 & \text{for } > 0\,. \end{cases}$$

The solution we were able to find was

$$u(x,t) := \sum_{n=1}^{\infty} g_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} u_n(x,t)\,, \tag{1}$$

by assuming the following sine Fourier series expansion of the initial data g:

$$\sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right)\,.$$

In order to prove that the function u above is the solution of our problem, we need to recall the following theorem about derivation of a series of functions.

Theorem. Let us consider functions $f_n : [a,b] \to \mathbb{R}$. Define, for every $N \in \mathbb{N}$, the function

$$F_N(x) := \sum_{n=1}^{N} f_n(x)\,,$$

and assume that there exists a point $\bar{x} \in [a,b]$ for which

$$\sum_{n=1}^{\infty} f_n(\bar{x}) \in \mathbb{R}\,.$$

Let us consider, for every $N \in \mathbb{N}$, the function

$$G_N(x) := \sum_{n=1}^{N} f'(x)\,.$$

Assume that G_N converges *uniformly* in $[a,b]$ to

$$G(x) := \sum_{n=1}^{\infty} f'(x)\,.$$

Then F_N converges *uniformly* in $[a,b]$ to

$$f(x) := \sum_{n=1}^{\infty} f_n(x)\,,$$

and

$$f'(x) = \sum_{n=1}^{\infty} f'(x)\,.$$

Bottom line: in order to differentiate term-by-term a series of functions, we need the series of the derivatives to converge uniformly!

We would like to apply the above theorem in order to differentiate the function u in (1). For, let us suppose that the sine Fourier series of g converges uniformly in $[0,L]$ to g. We now that this is true if g is a function of class C^1 with $g(0) = g(L) = 0$.

So, let us consider the functions

$$v_N(x,t) := \sum_{n=1}^{N} \partial_x u_n(x,t) = \sum_{n=1}^{N} g_n \frac{n\pi}{L} \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt},$$

$$w_N(x,t) := \sum_{n=1}^{N} \partial_{xx} u_n(x,t) = \sum_{n=1}^{\infty} u_n(x,t) - \sum_{n=1}^{N} g_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt},$$

$$z_N(x,t) := \sum_{n=1}^{N} \partial_t u_n(x,t) = -\sum_{n=1}^{N} g_n \sin\left(\frac{n\pi}{L}x\right) \left(\frac{n\pi}{L}\right)^2 De^{-\left(\frac{n\pi}{L}\right)^2 Dt}.$$

If we fix $t > 0$, we see that, for n large enough,

$$\left(\frac{n\pi}{L}\right)^2 e^{-\left(\frac{n\pi}{L}\right)^2 Dt} < 1.$$

Thus, since the Fourier series of g convergence uniformly, we obtain that w_N converges uniformly to

$$w(x,t) := -\sum_{n=1}^{\infty} g_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}.$$

Since $v_N\left(\frac{L}{2}, t\right) = 0$, by applying the above theorem, we have that, for every $t > 0$, v_N converges uniformly in $[0, L]$ to

$$v(x,t) := \sum_{n=1}^{\infty} g_n \frac{n\pi}{L} \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt},$$

and that

$$w(x,t) = \partial x v(x,t),$$

Now, since the since Fourier series of g is finite at the point $x = 0$, by applying again the theorem, we have that

$$v(x,t) = \partial_x u(x,t),$$

and that the series defining u converges uniformly in $[0, L]$ for every $t > 0$. Thus $w(x,t) = \partial_{xx} u(x,t)$. With a similar argument, we see that, for every fixed $t > 0$, z_N converges uniformly to

$$z(x,t) := -\sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right) \left(\frac{n\pi}{L}\right)^2 De^{-\left(\frac{n\pi}{L}\right)^2 Dt},$$

and that

$$z(x,t) = \partial_t u(x,t).$$

So, it is possible to differentiate term-by-term the series defining u, Hence, we have that

$$u_t - Du_{xx} = -\sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right) \left(\frac{n\pi}{L}\right)^2 De^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$
$$+ D\sum_{n=1}^{\infty} g_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt} = 0.$$

That is, the function $u$we found by using the separation of variables technique, really solves the heat equation!