Review on Fourier series and general Fourier expansion

When we apply the technique of separation of variables in order to solve the heat or the wave equation in a bounded domain, *i.e.*, when seeking for solutions of the form $u(x,t) = T(t)X(x)$, we have to solve, for the function X, a problem of the form

$$\begin{cases} X''(x) = \lambda X(x)\,, \\ \text{boundary conditions} \\ \text{at } x=0 \text{ and } x=L\,. \end{cases}$$

Usually, these kind of systems admit a countably many solutions $(X_n)_n$, relative to different λ_n's. Then, if for each n we solve the equation for the function T relative to the value λ_n, we find a function u_n that satisfies the equation and matches the boundary conditions. In order to obtain a solution of the original problem, we need to satisfy also the initial condition. The main idea of the separation of variable technique is to use the family $(u_n)_n$ to build a function

$$u(x,t) := \sum_{n=n_0}^{\infty} u_n(x,t)\,,$$

for some n_0, that solves the problem. If we forget for a moment the technical difficulties of differentiate a series of functions, the main question is the following: how general can the initial data be in order for a function u like the one above to match them? For instance, let us consider the heat equation, where the only initial condition is $u(x,0) = g(x)$. The question is: how much freedom do we have in the choice of the initial data g, if we ask it to be of the form

$$\sum_{n=n_0}^{\infty} u_n(x,0)\,?$$

The (classical) theory of Fouries series tells us that the data g can be very general (more or less, according to the boundary conditions we have). Moreover, the so called L^2-theory of Fourier series allows us to choose the initial data in a broader class of functions, if we are willing to pay the price of *relaxing* the meaning of u matching the initial conditions. But this is another story...

We now review the three main Fourier series expansions (relative to the so called *natural* boundary conditions) and we will present the generalization of the theory in the case of more general boundary conditions.

The cosine Fourier series.
Given a function $f : [0,L] \to \mathbb{R}$, we define its cosine Fourier series as

$$F(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)\,,$$

where

$$a_n := \frac{2}{L}\int_0^L f(y)\cos\left(\frac{n\pi}{L}y\right)\,\mathrm{d}y\,.$$

We have that:

- this expansion corresponds to the case of **Neumann** boundary conditions, *i.e.*, when we have to solve the problem

$$\begin{cases} X''(x) = \lambda X(x)\,, \\ X'(0) = X'(L) = 0\,. \end{cases}$$

- the following **pointwise** convergence theorem holds:
if f and f' are piecewise continuous, then

$$F_N(x) \to \frac{1}{2}\left[\, f^+(x) + f^-(x) \,\right] ,$$

for every $x \in (0, L)$ (we are **excluding** the boundary points!), where

$$F_N(x) := \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{L}x\right) .$$

- the following **uniform** convergence theorem holds:
if f and f' are continuous (in brief, $f \in C^1([0, L])$), and $f'(0) = f'(L) = 0$, then

$$F_N \to f ,$$

uniformly for $x \in [0, L]$, as $N \to \infty$ (we are **including** the boundary points!).

- if we have a function $f : [-L, L]$, and we want to obtain similar results as the ones above, that is, we want the cosine Fourier series to converge to f, we need f to be **even**! In this case, the Fourier cosine coefficients of f in $[-L, L]$ are exactly the ones defined above, and the two previous convergence results hold.

The sine Fourier series.
Given a function $f : [0, L] \to \mathbb{R}$, we define its sine Fourier series as

$$F(x) := \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) ,$$

where

$$b_n := \frac{2}{L}\int_0^L f(y) \sin\left(\frac{n\pi}{L}y\right) \mathrm{d}y .$$

We have that:

- this expansion corresponds to the case of **Dirichelt** boundary conditions, *i.e.*, when we have to solve the problem

$$\begin{cases} X''(x) = \lambda X(x) , \\ X(0) = X(L) = 0 . \end{cases}$$

- the following **pointwise** convergence theorem holds:
if f and f' are piecewise continuous, then

$$F_N(x) \to \frac{1}{2}\left[\, f^+(x) + f^-(x) \,\right] ,$$

for every $x \in (0, L)$ (we are **excluding** the boundary points!), where

$$F_N(x) := \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi}{L}x\right) .$$

- the following **uniform** convergence theorem holds:
if f and f' are continuous (in brief, $f \in C^1([0, L])$), and $f(0) = f(L) = 0$, then

$$F_N \to f ,$$

uniformly for $x \in [0, L]$ as $N \to \infty$ (we are **including** the boundary points!).

- if we have a function $f : [-L, L]$, and we want to obtain similar results as the ones above, that is, we want the sine Fourier series to converge to f, we need f to be **odd**! In this case, the Fourier sine coefficients of f in $[-L, L]$ are exactly the ones defined above, and the two previous convergence results hold.

The full Fourier series.

Given a function $f : [-L, L] \to \mathbb{R}$, we define its full Fourier series as

$$F(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right],$$

where

$$a_n := \frac{1}{L} \int_0^L f(y) \cos\left(\frac{n\pi}{L}y\right) \mathrm{d}y,$$

and

$$b_n := \frac{1}{L} \int_0^L f(y) \sin\left(\frac{n\pi}{L}y\right) \mathrm{d}y.$$

We have that:

- this expansion corresponds to the case of **periodic** boundary conditions, *i.e.*, when we have to solve the problem

$$\begin{cases} X''(x) = \lambda X(x), \\ X(0) = X(L), \\ X'(0) = X'(L). \end{cases}$$

- the following **pointwise** convergence theorem holds:
  if f and f' are piecewise continuous, then

$$F_N(x) \to \frac{1}{2} \left[f^+(x) + f^-(x) \right],$$

  for every $x \in (-L, L)$ (we are **excluding** the boundary points!), where

$$F_N(x) := \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right],$$

- the following **uniform** convergence theorem holds:
  if f and f' are continuous (in brief, $f \in C^1([0, L])$), and it holds $f(-L) = f(L)$, $f'(-L) = f'(L)$, then, as $N \to \infty$

$$F_N \to f,$$

  uniformly for $x \in [-L, L]$, as $N \to \infty$ (we are **including** the boundary points!).

- notice that in the case of the full Fourier series, we **need** to take the function on $[-L, L]$ in order to have the above expansion[1]. This is because

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) \mathrm{d}x \neq 0,$$

  while

$$\int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) \mathrm{d}x = 0,$$

  for every $n, m \in \mathbb{N}$. Moreover, we don't need to assume any parity condition on f, since the sum of odd and even functions can be whatever!

[1] **Warning:** this is in order to have the expansion as above! It is possible to expand a function $f : [a, b] \to \mathbb{R}$ satisfying $f(a) = f(b)$ and $f'(a) = f'(b)$ with the *appropriate* full Fourier series expansion!

General Fourier expansion.
Now, the question follows naturally: what about the case of different boundary conditions? Can we still have some kind of expansion of a function f in a series of functions like the ones above?

Let us recall the way we ended up talking about Fourier series: by using the separation of variable technique in order to solve the heat or the wave equation, we had to solve the following problem

$$\begin{cases} X''(x) = \lambda X(x)\,, \\ \text{boundary conditions} \\ \text{at } x = 0 \text{ and } x = L\,. \end{cases} \tag{1}$$

Assume there exist a sequence of countably many values $(\lambda_n)_n$ and a sequence of countably many functions $(X_n)_n$ such that the following system is satisfied for every n:

$$\begin{cases} X_n''(x) = \lambda_n X_n(x)\,, \\ \text{boundary conditions} \\ \text{at } x = 0 \text{ and } x = L\,. \end{cases}$$

We would like to look at the above problem from a different perspective: let us consider the space

$$\mathcal{A} := \{\, u \in C^2([0,L]) \,:\, u \text{ satisfies the boundary conditions of the problem } \}\,,$$

and the linear operator $\mathcal{L} : \mathcal{A} \to C^0([0,L])$ defined as

$$\mathcal{L}u := u''\,.$$

Then, the above problem can be written as

$$\mathcal{L}u = \lambda u\,.$$

Thus, in analogy with linear algebra, we call the sequence $(X_n)_n$ the **eigenfunctions** of $\mathcal{L}$, and the sequence $(\lambda_n)_n$ the **eigenvalues** of $\mathcal{L}$. The idea then, is to consider an expansion of a general function $f : [0,L] \to \mathbb{R}$ in terms of the eigenfunctions of $\mathcal{L}$:

$$F(x) := \sum_{n=0}^{\infty} f_n X_n(x)\,.$$

for some n_0. We would like to find the coefficients f_n in such a way that (formally)

$$F(x) = f(x)\,,$$

for every x (again, we do not specify in what set). In order to find the coefficients, we reason in a similar way as we did in the previous cases: we consider, in the case $\lambda_n \neq 0$, we take $n \neq m$, and we consider

$$\int_0^L X_n X_m \,\mathrm{d}x = \frac{1}{\lambda_n}\int_0^L (\lambda_n X_n) X_m \,\mathrm{d}x = \frac{1}{\lambda_n}\int_0^L X_n'' X_m \,\mathrm{d}x\,,$$

thanks to the equation satisfied by X_n. Now, by using integration by parts twice, we get

$$\begin{aligned} \frac{1}{\lambda_n}\int_0^L X_n'' X_m \,\mathrm{d}x &= \frac{1}{\lambda_n}\left[X_n' X_m \Big|_0^L - \int_0^L X_n' X_m' \,\mathrm{d}x \right] \\ &= \frac{1}{\lambda_n}\left[X_n' X_m \Big|_0^L - X_n X_m' \Big|_0^L + \int_0^L X_n X_m'' \,\mathrm{d}x \right] \\ &= \frac{1}{\lambda_n}\left[X_n' X_m \Big|_0^L - X_n X_m' \Big|_0^L \right] + \frac{\lambda_m}{\lambda_n}\int_0^L X_n X_m \,\mathrm{d}x\,, \end{aligned}$$

where in the last step we used the equation satisfied by X_m. Thus, we have

$$\int_0^L X_n X_m \,\mathrm{d}x = \frac{1}{\lambda_n}\left[\left(X_n' X_m - X_n X_m'\right)\Big|_0^L \right] + \frac{\lambda_m}{\lambda_n}\int_0^L X_n X_m \,\mathrm{d}x \,.$$

Then, we have that

$$\int_0^L X_n X_m \,\mathrm{d}x = 0 \quad \Leftrightarrow \quad \left(X_n' X_m - X_n X_m'\right)\Big|_0^L = 0 \,.$$

We then say that the eigenfunctions of $\mathcal{L}$ satisfy **symmetric** boundary conditions, if

$$\left(X_n' X_m - X_n X_m'\right)\Big|_0^L = 0 \,,$$

for every n and m. In this case, we have that

$$\int_0^L X_n X_m \,\mathrm{d}x = 0 \,,$$

whenever $n \neq m$. By defining

$$\|X_n\|_{L^2}^2 := \int_0^L X_n^2 \,\mathrm{d}x \,,$$

we have that

$$\begin{aligned} f_n = \frac{f_n}{\|X_n\|_{L^2}^2}\int_0^L X_n^2 \,\mathrm{d}x &= \frac{1}{\|X_n\|_{L^2}^2}\sum_{m=0}^{\infty}\int_0^L f_m X_m X_n \,\mathrm{d}x \\ &= \frac{1}{\|X_n\|_{L^2}^2}\int_0^L \left(\sum_{m=0}^{\infty} f_m X_m\right) X_n \,\mathrm{d}x = \frac{1}{\|X_n\|_{L^2}^2}\int_0^L f X_n \,\mathrm{d}x \,. \end{aligned}$$

Thus, let us define the **generalized** Fourier series of f with respect to the eigenfunctions X_n's, as

$$F(x) := \sum_{n=0}^{\infty} f_n X_n(x) \,,$$

where

$$f_n := \frac{1}{\|X_n\|_{L^2}^2}\int_0^L f(x) X_n(x) \,\mathrm{d}x \,.$$

Now, the big question is: when is it possible to have convergence theorems like the ones we had in the case of the cosine, sine and full Fourier series? The answer is the following:

Uniform convergence theorem for general Fourier expansion.
Let $f : [0, L] \to \mathbb{R}$. Assume that

(i) the eigenfunctions of (1) satisfy *symmetric* boundary conditions,
(ii) f, f' and f'' exists and are continuous, in brief $f \in C^2([0, L])$,
(iii) f satisfies the boundary conditions of (1).

For every $N \in \mathbb{N}$, let

$$F_N(x) := \sum_{n=0}^{N} f_n X_n(x) \,,$$

the f_n's are defined as above. Then

$$F_N \to f$$

uniformly in $[0, L]$, as $N \to \infty$.

Remark: for the classical Fourier series we did **not** assume the existence of f''!