Fourier series - properties

Uniqueness. If two functions have the same Fourier series expansion, then the two functions are equal. If two Fourier series expansion are the same, the the Fourier coefficients must coincide, *i.e.*,

$$\sum_{n\in\mathbb{N}} a_n X_n(x) = \sum_{n\in\mathbb{N}} b_n X_n(x)\,,$$

for every $x \in I$, where $I \subset \mathbb{R}$ is an interval, if and only if $a_n = b_n$ for all $n \in \mathbb{N}$.

Linearity. Let us assume we have two functions $f, g : [-L, L] \to \mathbb{R}$, and define the function $h : [-L, L] \to \mathbb{R}$ as $h := \lambda f + g$, for some $\lambda \in \mathbb{R}$. Consider the full Fourier series of h

$$H(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] ,$$

and the full Fourier series of f

$$F(x) := \frac{c_0}{2} + \sum_{n=1}^{\infty} \left[c_n \cos\left(\frac{n\pi}{L}x\right) + d_n \sin\left(\frac{n\pi}{L}x\right) \right] ,$$

and of g

$$G(x) := \frac{e_0}{2} + \sum_{n=1}^{\infty} \left[e_n \cos\left(\frac{n\pi}{L}x\right) + f_n \sin\left(\frac{n\pi}{L}x\right) \right] ,$$

Then, it is clear from the definition of the coefficients, that

$$a_n = \lambda c_n + e_n\,, \qquad b_n = \lambda d_n + f_n\,.$$

The same conclusion holds true for the cosine and the sine Fourier series.

Fourier series of the derivative of a function. Let $f : [-L, L] \to \mathbb{R}$ be a $2L$-periodic function of class C^1, and consider its Fourier series

$$F(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] .$$

Let us also consider the Fourier series of f',

$$G(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left[c_n \cos\left(\frac{n\pi}{L}x\right) + d_n \sin\left(\frac{n\pi}{L}x\right) \right] .$$

We would like to understand the relationship between the coefficients of the two series. Fist of all, we notice that

$$c_0 = \frac{1}{L}\int_{-L}^{L} f'(x)\,\mathrm{d}x = \frac{1}{L}\left(f(L) - f(-L) \right) = 0\,.$$

Moreover, by integrating by parts, we have that (for $n \neq 0$)

$$\begin{aligned} c_n &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x \\ &= \frac{1}{L}\left[f(x)\cos\left(\frac{n\pi}{L}x\right)\Big|_{-L}^{L} + \frac{n\pi}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x \right] = \frac{n\pi}{L} b_n\,, \end{aligned}$$

where in the last step we used the fact that $f(-L) = f(L)$. In a similar way, we find that

$$d_n = -\frac{n\pi}{L} a_n \,.$$

So, we obtain

$$G(x) = \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} b_n \cos\left(\frac{n\pi}{L}x\right) - \frac{n\pi}{L} a_n \sin\left(\frac{n\pi}{L}x\right) \right] .$$

Notice that the fact that the Fourier series for f converges do **not** imply that the Fourier series for f' does! This is because we have no hypothesis on $(f')'$.

Finally, we would like to notice that the coefficients we obtained are the one that we would have obtained by performing a (not justified!) differentiation term-by-term of the Fourier series of f:

$$\begin{aligned} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] \right] \\ &= \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] \\ &= \sum_{n=1}^{\infty} \left[-a_n \frac{n\pi}{L} \cos\left(\frac{n\pi}{L}x\right) + b_n \frac{n\pi}{L} \sin\left(\frac{n\pi}{L}x\right) \right] . \end{aligned}$$

The previous to last step is **not** justified!!! It holds under some particular conditions.

Fourier series of the primitive of a function. Let $f : [-L, L] \to \mathbb{R}$ be a $2L$-periodic function of class C^1, and consider its Fourier series

$$F(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] .$$

Let us also consider the Fourier series of the function

$$g(x) := \int_{-L}^{x} f(t)\,\mathrm{d}t \,,$$

and its Fourier series

$$G(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left[c_n \cos\left(\frac{n\pi}{L}x\right) + d_n \sin\left(\frac{n\pi}{L}x\right) \right] .$$

By using the above result, we obtain that

$$c_n = -\frac{L}{n\pi} b_n \,, \qquad d_n = \frac{L}{n\pi} a_n \,.$$

Thus,

$$G(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left[-\frac{L}{n\pi} b_n \cos\left(\frac{n\pi}{L}x\right) + \frac{L}{n\pi} a_n \sin\left(\frac{n\pi}{L}x\right) \right] .$$

Notice that, in this case, the convergence of the Fourier series of f implies the convergence of the Fourier series of its primitive, since F is continuous with piecewise constant derivative.

By arguing as in the previous case, we see that the coefficients of the primitive of f are the ones that we would obtain by integrating term-by-term the Fourier series of f.

Bottom line. Differentiating term-by-term a Fourier series is delicate, and needs particular hypothesis, while integrating term-by-term a Fourier series is allowed.