Fourier series - computation of the coefficients

Let us consider a function $g : [0, L] \to \mathbb{R}$ and assume that it is possible to write g as

$$g(x) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right),$$

for every $x \in (0, L)$, where $g_n \in \mathbb{R}$. We would like to find to find the coefficients g_n's. For, let us recall the identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)].$$

Then, we have that

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi}{L}x\right)\,\mathrm{d}x - \frac{1}{2}\int_0^L \cos\left(\frac{(n+m)\pi}{L}x\right)\,\mathrm{d}x.$$

So, in the case $n = m$, we find that

$$\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \frac{L}{2} - \frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}x\right)\Big|_0^L = \frac{L}{2},$$

while, in the case $n \neq m$, we get

$$\begin{aligned}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x &= \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi}{L}x\right)\,\mathrm{d}x - \frac{1}{2}\int_0^L \cos\left(\frac{(n+m)\pi}{L}x\right)\,\mathrm{d}x \\ &= \frac{L}{2(n-m)\pi}\sin\left(\frac{(n-m)\pi}{L}x\right)\Big|_0^L - \frac{L}{2(n+m)\pi}\sin\left(\frac{(n+m)\pi}{L}x\right)\Big|_0^L = 0.\end{aligned}$$

Thus,

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \begin{cases}\frac{L}{2} & \text{if } n = m, \\ 0 & \text{if } n \neq m.\end{cases}$$

We now argue as follows: for all $n \geq 1$, it holds that

$$\begin{aligned}g_n &= \frac{2}{L}\int_0^L g_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \frac{2}{L}\sum_{m=1}^{\infty}\int_0^L g_m \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x \\ &= \frac{2}{L}\int_0^L \left(\sum_{m=1}^{\infty} g_m \sin\left(\frac{m\pi}{L}x\right)\right)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x,\end{aligned}$$

where the previous to last step (taking the series inside the integral) is not completely justified (but we believe it!).

The bottom line is the following: if it is possible to write a function $g : [0, L] \to \mathbb{R}$ as

$$g(x) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right),$$

then we must have

$$g_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\,\mathrm{d}x.$$

With a similar argument, it is possible to find the coefficients in the case of the cosine or the full Fourier series (see HM4-Ex.8).