Forcing term in the wave equation in bounded domains

We want to solve the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{in } (0,L)\times(0,\infty)\,, \\ u(x,0) = g(x) & \text{in } (0,L)\,, \\ u_t(x,0) = h(x) & \text{in } (0,L)\,, \\ u(0,t) = 0 & \text{for } t>0\,, \\ u(L,t) = 0 & \text{for } t>0\,, \end{cases} \tag{1}$$

where $f : (0,L)\times(0,\infty) \to \mathbb{R}$ is a given function. The above problem model, for instance, an oscillating string, where the term f describes the effects of the sources of oscillation.

We apply the same strategy we used for the inhomogeneous heat equation. We know that the solution in the case $f \equiv 0$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L}x\right)\,.$$

Let us assume the following sine Fourier expansions:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi}{L}x\right)\,,$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right)\,,$$

$$h(x) = \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi}{L}x\right)\,.$$

Then, we obtain the family of problems

$$\begin{cases} b_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 b_n(t) = f_n(t)\,, \\ b_n(0) = g_n\,, \\ b_n'(0) = h_n\,. \end{cases}$$

In order to solve the above problem, we use the variation of coefficients method of ODEs. Let us recall (see the handout *Review on ODEs*) that the solution of the problem

$$\begin{cases} y''(t) + B(t)y'(t) + C(t)y(t) = F(t)\,, \\ y(0) = y_0\,, \\ y'(0) = y_1\,, \end{cases}$$

is given by

$$y(t) = \lambda_1(t)y_1(t) + \lambda_2(t)y_2(t)\,,$$

where y_1 and y_2 are independent solutions of the equation

$$y''(t) + B(t)y'(t) + C(t)y(t) = 0\,,$$

namely, such that $y_1(t)y_2'(t) - y_2(t)y_1'(t) \neq 0$ for all $t>0$, and

$$\lambda_1(t) := \lambda_1(0) - \int_0^t \frac{y_2(s)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)} F(s)\,\mathrm{d}s\,,$$

and

$$\lambda_2(t) := \lambda_2(0) + \int_0^t \frac{y_1(s)}{y_1(s)y_2'(s) - y_2(s)y_1'(s)} F(s)\,\mathrm{d}s\,,$$

where the values $\lambda_1(0)$ and $\lambda_1(0)$ can be derived by the initial conditions satisfied by y. In our case, we have that

$$B(t) \equiv 0\,, \qquad C(t) \equiv c^2 \left(\frac{n\pi}{L} \right)^2 , \qquad F(t) := f_n(t)\,.$$

In particular, two independent solutions of the homogeneous problem

$$b_n''(t) + c^2 \left(\frac{n\pi}{L} \right)^2 b_n(t) = 0\,,$$

are

$$y_1(t) := \cos\left(\frac{n\pi c}{L} t \right) , \qquad y_2(t) := \sin\left(\frac{n\pi c}{L} t \right) .$$

So, we have that

$$y_1(s) y_2'(s) - y_2(s) y_1'(s) = \frac{n\pi c}{L}\,,$$

and thus

$$\lambda_1(t) = \lambda_1(0) - \frac{L}{n\pi c} \int_0^t f(s) \sin\left(\frac{n\pi c}{L} s \right) \mathrm{d}s\,,$$

$$\lambda_2(t) = \lambda_2(0) + \frac{L}{n\pi c} \int_0^t f(s) \cos\left(\frac{n\pi c}{L} s \right) \mathrm{d}s\,.$$

So, let

$$b_n(t) := \lambda_1(t) \cos\left(\frac{n\pi c}{L} t \right) + \lambda_2(t) \sin\left(\frac{n\pi c}{L} t \right) .$$

Finally, by using the initial conditions, we need to find $\lambda_1(0)$ and $\lambda_2(0)$. We have that

$$g_n = b_n(0) = \lambda_1(0)\,,$$

and

$$h_n = b_n'(0) = \lambda_1'(0) \cos 0 - \frac{n\pi c}{L} \lambda_1(0) \sin 0 + \lambda_2'(0) \sin 0 + \frac{n\pi c}{L} \cos 0 \lambda_2(0) = \frac{n\pi c}{L} \lambda_2(0)\,.$$

Thus, find that

$$b_n(t) = \left[g_n - \frac{L}{n\pi c} \int_0^t f_n(s) \sin\left(\frac{n\pi c}{L} s \right) \mathrm{d}s \right] \cos\left(\frac{n\pi c}{L} t \right) + \left[\frac{L}{n\pi c} h_n + \frac{L}{n\pi c} \int_0^t f_n(s) \cos\left(\frac{n\pi c}{L} s \right) \mathrm{d}s \right] \sin\left(\frac{n\pi c}{L} t \right) .$$

Notice that these b_n's are the same that we would have obtained in the case $f \equiv 0$ *perturbed* by two additional terms due to the presence of the forcing term f. Hence, the solution of problem (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[\left(g_n - \frac{L}{n\pi c} \int_0^t f_n(s) \sin\left(\frac{n\pi c}{L} s \right) \mathrm{d}s \right) \cos\left(\frac{n\pi c}{L} t \right) + \left(\frac{L}{n\pi c} h_n + \frac{L}{n\pi c} \int_0^t f_n(s) \cos\left(\frac{n\pi c}{L} s \right) \mathrm{d}s \right) \sin\left(\frac{n\pi c}{L} t \right) \right] \sin\left(\frac{n\pi}{L} x \right) .$$

With a similar strategy it is possible to solve the wave equation with forcing term with different boundary conditions.