4.3) The homogeneous global Cauchy pb:

We want to solve:

$$(C) \quad \begin{cases} u_t - D u_{xx} = 0 & \mathbb{R} \times (0, \infty), \\ u(x,0) = g(x) & \mathbb{R}. \end{cases}$$

Let us make a couple of observations:

i) the equation is <u>linear</u>; if u_1, u_2 solve $u_t - D u_{xx} = 0$, then $u_1 + u_2$ solves it as well.

ii) if u is a solution of $u_t - D u_{xx} = 0$, then $\forall y \in \mathbb{R}$ $\tilde{u}(x) := u(x-y)$ is a solution of the same eq..

Thanks to the above observations, we can heuristically reason as follows: <u>assume</u> that the solution <u>exists</u> and is <u>unique</u>.
The function g represents the initial density of mass/temperature.
The function $\Gamma_D(x-y,t)$ describes the evolution of a unitary mass at the pt y at time $t=0$.
By linearity, we thus expect

$$(*)\quad \left[\; u(x,t) := \int_{\mathbb{R}} g(y)\,\Gamma_D(x-y,t)\,dy = \frac{1}{\sqrt{4\pi D t}}\int_{\mathbb{R}} g(y)\,e^{-\frac{(x-y)^2}{4Dt}}\,dy\right.$$

to be the solution of (C).

- Let us now discuss the issues of existence and uniqueness.
The formula above could not make sense: indeed, let us suppose $g(y) = c\,e^{Ay^2}$. Then:

$$\int_{\mathbb{R}} g(y)\,\Gamma_D(x-y,t)\,dy = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi D t}}\,c\,e^{Ay^2 - \frac{(x-y)^2}{4Dt}}\,dy$$

$$\underset{\text{assume } x=0}{\overset{\text{for simplicity}}{=}} \frac{c}{\sqrt{4\pi D t}}\int_{\mathbb{R}} e^{y^2\left(A - \frac{1}{4Dt}\right)}\,dy.$$

If $t > \dfrac{1}{4AD}$, then the above integral is $+\infty$!

With the above example in mind, it is not surprising that:

- **Thm:** (existence)
  Let g be a function with a finite number of discontinuity and such that $|g(y)| \le c e^{Ay^2}$ $y \in \mathbb{R}$, for some $c, A > 0$.

Then, the function u defined in $(*)$ is well defined for all $t \in (0, \frac{1}{4DA})$. Moreover, in such an interval it is differentiable up to any order, and solves the eq.

$$u_t - D u_{xx} = 0.$$

Finally, if $x_0 \in \mathbb{R}$ is a pt where g is continuous, we have

$$\lim_{\substack{t \to 0 \\ y \to x_0}} u(y,t) = g(x_0).$$

Furthermore, $\exists\, c_1, c_2 > 0$ s.t. $|u(x,t)| \le c_1 e^{c_2 x^2}$ $\forall x \in \mathbb{R}, \forall t < \frac{1}{4DA}$

- Remarks:
  - the growth at infinity of g determines the time interval where u is well-defined.
  - the diffusion process is a regularizing process; if we start with a rough data, at any $t>0$ we have a smooth one.
  - $u(x,t) > 0$ $\forall t>0$ [if $g \geq 0$] $\forall x \in \mathbb{R}$
    $\to$ instantaneous diffusion of the mass
  - if $\int_{\mathbb{R}} |g(y)| \, dy < +\infty$, then u is well-defined and solves the pb (C) $\forall t>0$. Moreover, $\forall t>0$,

$$\int_{\mathbb{R}} u(x,t)\, dx = \int_{\mathbb{R}} g(x)\, dx \rightsquigarrow \text{the mass is conserved!}$$

Let us now discuss uniqueness. Consider the pb:

$$\begin{cases} u_t - D u_{xx} = 0 & \mathbb{R} \times (0, +\infty), \\ u(x,0) = 0. \end{cases}$$

Clearly $u \equiv 0$ is a solution of the above pb.
But also the function:

$$u(x,t) := \sum_{k=0}^{\infty} \frac{h^{(k)}(t)}{(2k)!} x^{2k},$$

[this example is due to Tychonov]

where: $h(t) := \begin{cases} e^{-t^{-\alpha}} & t > 0, \\ 0 & t \leq 0, \end{cases}$ with $\alpha > 1$, is a solution.

Thus, in general, we cannot expect uniqueness.
But, if we restrict ourselves to a special class of functions, it does.

- **Thm**: [maximum principle]
Suppose u is a solution of (C) such that, u, u_x, u_t, u_{xx} continuous in $(0,T)$, and such that $|u(x,t)| \leq C e^{Ax^2}$, $C, A > 0$, $x \in \mathbb{R}$, $t \in [0,T]$.
Then:

$$\sup_{\mathbb{R} \times [0,T]} |u(x,t)| = \sup_{\mathbb{R}} g.$$

- **Thm**: [uniqueness]
Suppose g is s.t. $|g(y)| \leq C e^{Ay^2}$ $\forall y \in \mathbb{R}$.
Then, there exists a unique u s.t.
  i) u solves (C) in $\mathbb{R} \times (0,T)$
  ii) $|u(x,t)| \leq C_1 e^{C_2 x^2}$, $C_1, C_2 > 0$, for $x \in \mathbb{R}$, $t \in (0,T)$.

↳ it would correspond to $t = 0$