The diffusion equation in bounded domains: separation of variables

Let us consider the diffusion equation in an interval $[0, L]$:

$$\begin{cases} u_t - Du_{xx} = 0 & \text{in } [0, L] \times (0, \infty)\,, \\ u(x, 0) = g(x) & \text{in } [0, L]\,, \\ u(0, t) = 0 & \text{for } t > 0\,, \\ u(L, t) = 0 & \text{for } t > 0\,, \end{cases} \tag{1}$$

Notice that, besides the initial condition $u(x, 0) = g(x)$, we have some boundary conditions, telling us what is going on at the boundary points $x = 0$ and $x = L$. The particular conditions we are considering here are the **homogeneous** Dirichlet conditions

$$u(0, t) = u(L, t) = 0,\,,$$

for $t > 0$. The general strategy we are going to develop now is valid for *any* kind of **homogeneous** boundary conditions (like the Neumann BCs, the Robin BCs,...). Moreover, we will see that the same procedure allows to find the solution even in the case of non-homogeneous boundary conditions, as well as for the non-homogeneous heat equation. The basic idea is the following: since we don't know what the solution can be, we look for a particular kind of solution, namely one of the form:

$$u(x, t) = T(t)X(x)\,,$$

for some (one variable) functions T and X. Notice that, for the above functions, the two variables t and x are **separated**. This means that we are looking for a solution for which a fixed profile, given by X, is evolving in time, according to T. This solution is a particular kind of **self-similar** solution. In order for such a function u to solve the equation

$$u_t - Du_{xx} = 0\,,$$

we need the functions T and X to satisfy

$$T'(t)X(x) - DT(t)X''(x) = 0\,.$$

By dividing by $DT(t)X(x)$ (here we are assuming that it is possible to divide by $T(t)X(x)$. This is just a formal operation. It is possible, and we will do it later on during the course, to perform this step in a more rigorous way), we get

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}\,.$$

since the left-hand side is a function just of the variable t, while the right-hand side is a function just of the variable x, in order to have the above equality in force for every $x \in [0, L]$ and every $t > 0$, we must impose both sides to be constant. So, we are lead to the equations

$$\begin{cases} T'(t) = \lambda DT(t)\,, \\ X''(x) = \lambda X(x)\,, \end{cases}$$

for some *arbitrary* $\lambda \in \mathbb{R}$. The equation for T leads us to (up to a multiplicative constant)

$$T(t) = e^{\lambda Dt}\,. \tag{2}$$

Let us now consider the equation for X. The reason why to consider the BCs for u in the equation for X is the following: the BCs we have are

$$0 = u(0, t) = T(t)X(0)\,, \qquad 0 = u(L, t) = T(t)X(L)\,,$$

for every $t > 0$. In order to satisfy them, we can have $T \equiv 0$ (but this would implies that $u \equiv 0$, and this is a solution if and only if $g \equiv 0$), or we need to impose

$$X(0) = 0\,, \qquad X(L) = 0\,.$$

So, we have to solve the problem

$$\begin{cases} X''(x) = \lambda X(x) & \text{in } [0, L]\,, \\ X(0) = 0\,, \\ X(L) = 0\,. \end{cases} \tag{3}$$

We have to consider three cases (recall that λ is an arbitrary number!):

- $\lambda > 0$: in this case, the general solution of the above equation is

  $$X(x) = a\sinh(\sqrt{\lambda}x) + b\cosh(\sqrt{\lambda}x)\,.$$

  We now have to impose the boundary conditions. So,

  $$0 = X(0) = b\,,$$

  and hence $X(x) = a\sinh(\sqrt{\lambda}x)$. Moreover, we have to impose impose

  $$0 = X(L) = a\sinh(\sqrt{\lambda}L)\,.$$

  Since $\sinh y \neq 0$ if $y \neq 0$ (and this is the case, since $\sqrt{\lambda}L \neq 0$ - recall that we are in the case $\lambda > 0$), we get that the above equation can be satisfied only if $a = 0$. So, we obtain the trivial solution $X \equiv 0$, leading to the function

  $$u(x, t) = T(t)X(x) \equiv 0\,.$$

  So, u can be a solution of problem (1) if and only if $g \equiv 0$ (it is the only way for the null function to match the initial condition). In the case $g \not\equiv 0$, this cannot be a solution, and thus we have to exclude it.

- $\lambda = 0$, in this case, the general solution of the above equation is

  $$X(x) = ax + b\,.$$

  By imposing the boundary conditions, we get $a = b = 0$. Thus, $X \equiv 0$. By arguing as before, for a nontrivial initial data, this function cannot lead to a solution of our problem.

- $\lambda < 0$: in this case, the general solution of the equation is given by

  $$X(x) = a\cos(\sqrt{-\lambda}x) + b\sin(\sqrt{-\lambda}x)\,.$$

  By imposing the boundary conditions at $x = 0$, we get

  $$0 = X(0) = a\,,$$

  and hence $X(x) = b\sin x$. Then, by imposing the boundary conditions at $x = L$, we get

  $$0 = X(L) = b\sin(\sqrt{-\lambda}L)\,.$$

  Since for $b = 0$ we obtain the trivial solution, we want to impose $\sin(\sqrt{-\lambda}L) = 0$. By recalling that

  $$\sin x = 0 \quad \Leftrightarrow \quad x = n\pi\,,$$

  for some $n \in \mathbb{N}$, we get

  $$\sqrt{-\lambda}L = n\pi \quad \Rightarrow \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2\,.$$

  So, for every $n = 1, 2, 3, \ldots$ (because, for $n = 0$, we obtain $\lambda = 0$, and we already discussed that case), we get that the function

  $$x \mapsto b_n \sin\left(\frac{n\pi}{L}x\right)\,,$$

  where $b_n \in \mathbb{R}$ is *arbitrary*, is a solution of problem (3) with $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$.

By inserting this value of λ_n in (2), we get that, for every $n = 1, 2, 3, \dots$, the function

$$u_n(x,t) = b_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin\left(\frac{n\pi}{L}x\right)$$

satisfies the diffusion equation and matches the boundary conditions. In order to solve the problem we also have to satisfy the initial condition. Notice that

$$u_n(x,0) = b_n \sin\left(\frac{n\pi}{L}x\right) .$$

That is, the function u_n solves the problem (1) if and only if the initial data g is of the form

$$g(x) = a \sin\left(\frac{n\pi}{L}x\right) ,$$

for some $a \in \mathbb{R}$. This is too restrictive. We would like to use the above functions u_n's to build a solution for a generic initial data g. To this purpose, let us consider

$$v(x,t) := \sum_{n=1}^{N} b_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin\left(\frac{n\pi}{L}x\right) , \tag{4}$$

for some $N \in \mathbb{N}$. Since the diffusion equation is **linear**, any *finite* sum of the above functions still satisfies the diffusion equation. So, v still satisfies the equation. Moreover, since the boundary conditions are **homogeneous**, v will match them. Again, we see that

$$v(x,0) := \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi}{L}x\right) ,$$

and hence, we need to assume the initial data g to be of the form

$$g(x) = \sum_{n=1}^{N} g_n \sin\left(\frac{n\pi}{L}x\right) ,$$

for some $g_n \in \mathbb{R}$. Again, this is too restrictive.

We now do something a little crazy: we set $N = +\infty$ in (4) (making the sum a series), forgetting about asking ourselves whether it makes sense or not. We just do it!, obtaining the function

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin\left(\frac{n\pi}{L}x\right) .$$

So, we can say that **formally** (that means, if we believe it!), the above function is still a solution of the diffusion equation and it matches the boundary condition (since every *finite* sum of the u_n's does).

Then, in order for u to satisfy the initial problem, it just need to match the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = g(x) ,$$

for $x \in [0, L]$. In order to have the above equality satisfies, we will make a further assumption: we will **assume** that g is of the form

$$\sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right) , \tag{5}$$

for some $g_n \in \mathbb{R}$. Is this too restrictive? We will see, thanks to the theory of Fourier series, that it is not!

So, assume that

$$g(x) = \sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right) .$$

In order for u to match the initial condition, we need to have

$$\sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) ,$$

for every $x \in [0, L]$. But two series of functions are equal if and only if all the terms in the series are the same. That is, we need to impose

$$g_n \sin\left(\frac{n\pi}{L}x\right) = b_n \sin\left(\frac{n\pi}{L}x\right) ,$$

for every $x \in [0, L]$ and every $n = 1, 2, 3, \ldots$. But these conditions boil down to impose

$$b_n = g_n ,$$

for every $n = 1, 2, 3, \ldots$. Thus, we get that the function u defined as

$$u(x,t) := \sum_{n=1}^{\infty} g_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin\left(\frac{n\pi}{L}x\right) .$$

is a solution of the problem (1).

Summing up, we have that:

- if we assume the initial data g to be of the for (5) and
- if we believe that everything we did was correct

then **the** (since we already know uniqueness) solution of problem (1) is given by

$$u(x,t) := \sum_{n=1}^{\infty} g_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin\left(\frac{n\pi}{L}x\right) .$$

Notice. The things that we believed in the above reasonment, as well as seeing that the hypothesis on the form of the initial data g are, will be justified when we will develop the theory of Fourier series.

Asymptotic behavior of u as $t \to \infty$. Notice that, for $t \to \infty$, every term of the above series (that is, if we fix $x \in [0, L]$) tends to 0, since the exponential goes to 0, *i.e.*, for every fixed $x \in [0, L]$, we have that

$$g_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin\left(\frac{n\pi}{L}x\right) \to 0 , \qquad \text{as } t \to \infty .$$

So, (with a bit of magic!) we have that

$$u(x,t) := \sum_{n=1}^{\infty} g_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin\left(\frac{n\pi}{L}x\right) \to 0 ,$$

that is, we can say that (again, formally), for $t \to \infty$, the solution of the diffusion equation with Dirichlet boundary condition tends to the null function (independently of the initial data g). This is what we expect from a physical point of view: system (1) models the diffusion of particles (with initial density given by g) that are diffusing in the interval $[0, L]$ and such that, every time a particle reaches one of the ends, it exits the interval. So, it is clear that, after a long time, we expect no particle presents in the interval. And this is exactly what we get from the equation!