THE FOURIER TRANSFORM

1) Complex form of the Fourier series:

To a function $f:[-L,L]\to\mathbb{R}$ we associate its Fourier series

$$F(x):=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi}{L}x\right)+B_n\sin\left(\frac{n\pi}{L}x\right)\right].$$

For many applications, it is more convenient to use another writing of the above series.
By recalling that

$$\cos\alpha=\frac{e^{i\alpha}+e^{-i\alpha}}{2}\,,\qquad \sin\alpha=\frac{e^{i\alpha}-e^{-i\alpha}}{2i}\,,$$

we can use the functions $e^{\pm i\frac{n\pi}{L}x}$ in place of $\cos\left(\frac{n\pi}{L}x\right)$ and $\sin\left(\frac{n\pi}{L}x\right)$.
We would thus like to consider the series:

$$*)\quad F(x):=\sum_{n=-\infty}^{+\infty}C_n\,e^{in\frac{\pi}{L}x}.$$

We notice that:

- $$\int_{-L}^{L} e^{\pm i n \frac{\pi}{L} x} e^{\pm i m \frac{\pi}{L} x} dx = 0 \quad \forall n, m \in \mathbb{N}$$

- $$\int_{-L}^{L} e^{\pm i n \frac{\pi}{L} x} e^{\mp i m \frac{\pi}{L} x} dx = \begin{cases} 2L & n = m \\ 0 & n \neq m \end{cases}$$

Thus, by using the same strategy we adopted to obtain the Fourier coefficients for the classical Fourier series, we get

$$c_n := \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i n \frac{\pi}{L} x} dx.$$

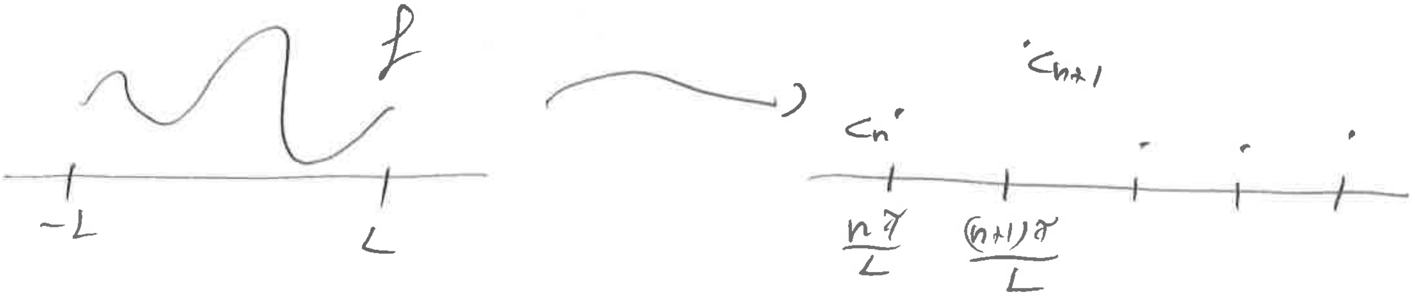
Notice that the series (*) is made by complex functions.

2) Meaning of the Fourier series:

- periodic function $\rightleftarrows$ Fourier coefficients

  by using the convergence thm

- the idea is that it is possible to decompose a periodic function into simple oscillating functions, namely, to see a periodic phenomena as a superposition of oscillations, with a discrete (countable) set of frequences $\left(\frac{n\pi}{L}\right)$ and whose amplitude is given by the corresponding Fourier coefficient (c_n). The convergence thm says that it is also possible to reconstruct the phenomena by adding together these oscillations.



space of frequences

3) Non-periodic phenomena:

- The question now is whether it is possible to perform a similar analysis for a function $f: \mathbb{R} \to \mathbb{R}$ that is not necessarely $2L$-periodic. We give here some heuristics.

Consider $f: \mathbb{R} \to \mathbb{R}$ restricted to $[-\frac{L}{2}, \frac{L}{2}]$, for some $L > 0$, and extend it in a periodic way to the whole $\mathbb{R}$. For such a function it is possible to write it [by assuming that we can!] as:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\frac{2\pi}{L}x} \qquad x \in [-\tfrac{L}{2}, \tfrac{L}{2}].$$

By pluggin in the definition of the c_n's we get:

$$f(x) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(y) e^{-\frac{i2\pi y n}{L}} dy \right] e^{\frac{in2\pi x}{L}}$$

by setting $K := \frac{n}{L}$

$$= \sum_{n=-\infty}^{+\infty} \left[\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(y) e^{-2\pi i k y} dy \right] e^{2\pi i k x}.$$

The above writing looks similar to a Riemann sum, even if it is not.
By sending $L \to +\infty$, with a bit of magic, we get:

$$f(x) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(y)\, e^{-2\pi i k y}\, dy \right] e^{2\pi i k x}\, dK.$$

4) The Fourier transform:

- Def.: Let $f: \mathbb{R} \to \mathbb{C}$. We write $f \in L^1(\mathbb{R})$ if

$$\int_{\mathbb{R}} |f(x)|\, dx < +\infty,$$

where $|f(x)|$ denotes the norm of $f(x)$.
We say that f is absolutely integrable.

· <u>Def</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be s.t. $f \in L^2(\mathbb{R})$.
We define its <u>Fourier transform</u> $\hat{f}: \mathbb{R} \to \mathbb{C}$ as:

$$\hat{f}(K) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i K x}\, dx.$$

· <u>Def</u>: Let $g: \mathbb{R} \to \mathbb{C}$ be s.t. $g \in L^2(\mathbb{R})$.
We define its <u>Fourier anti-transform</u> (or <u>Fourier syntesis</u>) $\check{g}: \mathbb{R} \to \mathbb{C}$ as

$$\check{g}(x) := \int_{\mathbb{R}} g(K)\, e^{2\pi i K x}\, dK.$$

· Notice that the two definitions change only in the sign of the exponent of the exponential.

The heuristics we gave suggest us the following result:

- <u>Thm</u>: (<u>inversion thm</u>)

  Let $f \in L^1(\mathbb{R})$ be s.t. $\hat{f} \in L^1(\mathbb{R})$.

  Then:

$$f = (\hat{f})^\vee.$$

  Namely:

$$f(x) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(y) e^{-2\pi i k y} dy \right] e^{2\pi i k x} dk.$$

- <u>Meaning of the above formula</u>:

  it is possible to reconstruct a non-periodic phenomena by adding together oscillating functions, where in this case we consider <u>all</u> the frequences $k \in \mathbb{R}$, weighted with $\hat{f}(k)$, that plays the role of the Fourier coefficient.

• Properties:

i) linearity: both the transform and the anti-transform are linear. Namely:

$$(\alpha f+\beta g)^{\wedge} = \alpha \hat{f}+\beta \hat{g} \qquad \begin{array}{l}\forall \alpha,\beta\in\mathbb{R}\\ \forall f,g:\mathbb{R}\to\mathbb{R}\end{array}$$

ii) Fourier transform of the derivative:

• $(f')^{\wedge}(k) = (2\pi i k)\hat{f}(k)$ $\quad\left|\begin{array}{l}\text{if } f\in L^1(\mathbb{R})\\ \text{with } f'\in L^1(\mathbb{R})\\ f \text{ piece } C^1\end{array}\right.$

Indeed:

$$(f')^{\wedge}(k) = \int_{\mathbb{R}} f'(x)e^{-2\pi i k x}\,dx$$

$$\underset{\text{by parts}}{=} \underbrace{f(x)e^{-2\pi i k x}\Big|_{-\infty}^{+\infty}}_{\substack{\text{it is possible}\\ \text{to prove that}\\ \text{this is zero}}} + 2\pi i k\underbrace{\int_{\mathbb{R}} f(x)e^{-2\pi i k x}\,dx}_{=\hat{f}(k)}$$

• by induction, we get:

$$(f^{(m)})^{\wedge}(k) = (2\pi i k)^m \hat{f}(k)$$

iii) convolution:

Given two functions $f, g: \mathbb{R} \to \mathbb{R}$, we define their convolution $f*g: \mathbb{R} \to \mathbb{R}$ as:

$$(f*g)(x) := \int_{\mathbb{R}} f(x-y)\, g(y)\, dy.$$

It holds that: $f*g = g*f$.
We want to compute the Fourier transform of the convolution product:

$$(f*g)^{\wedge}(k) = \int_{\mathbb{R}} (f*g)(x)\, e^{-2\pi i k x}\, dx$$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x-y)\, g(y)\, dy \right] e^{-2\pi i k x}\, dx$$

switching the order of integration

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x-y)\, e^{-2\pi i k x}\, dx \right] g(y)\, dy$$

$x - y = z$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(z)\, e^{-2\pi i k (z+y)}\, dz \right] g(y)\, dy$$

$$= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(z)\, e^{-2\pi i z k}\, dz \right] e^{-2\pi i k y}\, g(y)\, dy$$

$$= \left[\int_{\mathbb{R}} f(z)\, e^{-2\pi i k z}\, dz\right]\left[\int_{\mathbb{R}} g(y)\, e^{-2\pi i k y}\, dy\right]$$

the inner integral is independent of y

$$= \hat{f}(k)\, \hat{g}(k)$$

$$\Longrightarrow \boxed{(f * g)^{\wedge} = \hat{f} * \hat{g}}$$

iv) let's compute the Fourier transform of $f(x) := e^{-\frac{\alpha^2 x^2}{2}}$

We have that:

$$\int_{\mathbb{R}} e^{-\frac{\alpha^2 x^2}{2}} e^{-2\pi i k x}\, dx = \int_{\mathbb{R}} e^{-\frac{1}{2}\left(\alpha^2 x^2 + 4\pi i k x\right)}\, dx$$

$$= \left[\int_{\mathbb{R}} e^{-\frac{1}{2}\left(\alpha x + \frac{2\pi i k}{\alpha}\right)^2} dx\right] e^{-\frac{2\pi^2 k^2}{\alpha^2}}$$

$$= \frac{\sqrt{2\pi}}{\alpha}\, e^{-\frac{2\pi^2 k^2}{\alpha^2}}.$$

$y = \alpha x + \frac{2\pi i k}{\alpha}$

not rigorous!

5) Solving PDEs by using the Fourier transform:

Notice the following:

If $u: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $(x,t) \mapsto u(x,t)$, we can take the Fourier transform of

$x \mapsto u(x,t)$ when $t\in\mathbb{R}$ is fixed.

Namely, we compute:

$$\hat{u}(k,t) := \int_{\mathbb{R}} u(x,t)\, e^{-2\pi i k x}\, dx.$$

It holds that the derivatives with respect to the variable t commute with the Fourier transform w.r.t. x. That is:

$$(u_t)^\wedge = \hat{u}_t,$$

$$(u_{tt})^\wedge = \hat{u}_{tt}.$$

- **The heat equation:**

Let us consider the heat equation:

$$\begin{cases} u_t - D u_{xx} = f(x,t) & \text{in } \mathbb{R}\times(0,+\infty), \\ u(x,0) = g(x) & \text{in } \mathbb{R}. \end{cases}$$

By taking the Fourier Transform w.r.t. x of the above equation, we get:

$$\begin{cases} \hat{u}_t = -4k^2\pi^2 D u + \hat{f} \\ \hat{u}(k,0) = \hat{g}(k) \end{cases}$$

The above is an ODE problem for every fixed $k\in\mathbb{R}$, whose solution is given by:

$$\hat{u}(k,t) = \hat{g}(k) e^{-4\pi^2 D t k^2} + \int_0^t \hat{f}(k,s) e^{-4\pi^2 D k^2 (t-s)}\, ds.$$

To obtain the solution $(x,t)\mapsto u(x,t)$ of the original problem, we take the Fourier anti-transform of $\hat{u}$, getting:

$$u(x,t) = \left(\hat{g}(k) e^{-4\pi^2 D t k^2}\right)^{\vee} + \int_0^t \left[\hat{f}(k,s) e^{-4\pi^2 D k^2 (t-s)}\right]^{\vee} ds.$$

By recalling that:

- $$\left(e^{-\beta^2 x^2}\right)^{\vee} = \frac{\sqrt{\pi}}{\beta} e^{-\frac{\pi^2 x^2}{\beta^2}}$$

- $$\left(\hat{f}\hat{g}\right)^{\vee} = f * g$$

we get:

$$u(x,t) = \int_{\mathbb{R}} g(y)\,\Gamma_0(x-y,t)\,dy + \int_0^t \left[\int_{\mathbb{R}} f(y,s)\,\Gamma_0(x-y,t-s)\,dy\right] ds.$$

6) The Heisenberg uncertainity principle:

- An important property of the Fourier transform is the following:

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(k)|^2 dk,$$

called Parseval's equality.

- In the following, we will need the following inequality, called Cauchy-Schwarz's inequality:

$$\left|\int_{\mathbb{R}} f(x) g(x)\, dx\right| \leq \left[\int_{\mathbb{R}} |f(x)|^2 dx\right]^{1/2} \left[\int_{\mathbb{R}} |g(x)|^2 dx\right]^{\frac{1}{2}}$$

- Let us take a function $f: \mathbb{R} \to \mathbb{R}$ s.t.

  - $f \in L^2(\mathbb{R})$

  - $\int_{\mathbb{R}} |f(x)|^2 dx = 1$

  - $f' \in L^2(\mathbb{R})$

• Let us make the following computation:

i) $$\int_{\mathbb{R}} x f(x) f'(x)\, dx =$$

$$\underset{\text{by parts}}{=} \underbrace{\frac{1}{2} x (f(x))^2 \Big|_{-\infty}^{+\infty}}_{\substack{=0 \text{ by using the hypoth.}\\ \text{[believe it!]}}} - \frac{1}{2} \underbrace{\int_{\mathbb{R}} (f(x))^2\, dx}_{=1 \text{ by hypothesis}}$$

$$= -\frac{1}{2}$$

ii) $$\left| \int_{\mathbb{R}} (x f(x)) (f'(x))\, dx \right| \leq$$

$$\underset{\text{by Cauchy-Schwarz}}{\leq} \left[\int_{\mathbb{R}} |x f(x)|^2 dx \right]^{1/2} \left[\int_{\mathbb{R}} |f'(x)|^2 dx \right]^{1/2}$$

$$\underset{\text{Parseval equality}}{=} \left[\int_{\mathbb{R}} |x f(x)|^2 dx \right]^{1/2} \left[\int_{\mathbb{R}} |(f')^{\wedge}(k)|^2 dk \right]^{1/2}$$

$$\underset{(f')^{\wedge}(k) = 2\pi i k \hat{f}(k)}{=} \left[\int_{\mathbb{R}} |x f(x)|^2 dx \right]^{1/2} \left[\int_{\mathbb{R}} 4\pi^2 |k \hat{f}(k)|^2 dk \right]^{1/2}$$

By combining i) & ii) we get:

$$\left[\int_{\mathbb{R}} (x f(x))^2 dx\right]^{1/2} \left[\int_{\mathbb{R}} |k \hat{f}(k)|^2 dk\right]^{1/2} \geq \frac{1}{4\pi}.$$

The physical interpretation of the above result is the following: [quantum mechanics]

if the variable x denotes the position of a particle whose wave function is f, then the variable k represents the momentum.

The expected value for x is

$$\bar{x}^2 := \int_{\mathbb{R}} |x f(x)|^2 dx$$

and the one for k is

$$\bar{k}^2 := \int_{\mathbb{R}} |k \hat{f}(k)|^2 dk.$$

We proved that:

$$\bar{x}\,\bar{k} \geq \frac{1}{4\pi},$$

that is, we cannot precisely determine both the position and the momentum of the particle. This is known as the Heisenberg uncertainty principle.