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Final Thesis

# THE BERNSTEIN PROBLEM IN THE EUCLIDEAN AND SUB-RIEMANNIAN SETTING

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*Dedicata a mio padre*

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# Introduction

The Bernstein Problem is an important problem in the setting of minimal surface theory. Consider a $C^2$ function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$; the area of its graph is given by

$$\mathcal{A}(u;\Omega) := \int_\Omega \sqrt{1+|Du|^2}\, \mathrm{d}\mathcal{L}^n$$

Since the area functional $\mathcal{A}$ is strictly convex, a function $u$ is a minimum for the area functional $\mathcal{A}$ in $\Omega$ if and only if $u$ satisfied the Eulero equation for $\mathcal{A}$ in $\Omega$, the so called *minimal surface equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \qquad \text{in } \Omega \tag{1}$$

In 1915 S.Bernstein (see [Ber17]) proved that the affine functions are the only functions that satisfied (1) in $\Omega = \mathbb{R}^2$. The classical Bernstein Problem for $n > 2$ asks whether the only solutions of (1) in the whole $\mathbb{R}^n$ are the affine functions. This is what we called the Bernstein Problem. Different proofs of Bernstein's theorem were found later by several authors (see, for instance, Chapter 1), but none of those techniques can be extended to dimension $n > 2$.

The suitable technique for higher dimensions turned out to be the ones of *geometric measure theory* (GMT). In particular the pioneering notion of *perimeter measure*, introduced by E. De Giorgi in 1954, had several applications in the topic of minimal surfaces and, more generally, in GMT setting.

Let us recall that, if $E \subset \mathbb{R}^m$ is a measurable set and $A \subset \mathbb{R}^m$ is open, the perimeter measure of $E$ in $A$ is denoted by $|\partial E|(A)$ and defined by

$$|\partial E|(A) := \sup\left\{\int_E \operatorname{div}(\phi) \mid \phi \in C^1_c(A;\mathbb{R}^m),\, |\phi| \le 1\right\} \tag{2}$$

(see Chapter 5).

The perimeter measure plays an important role in the Bernstein problem. Indeed, if $u \in C^2(\Omega)$, then

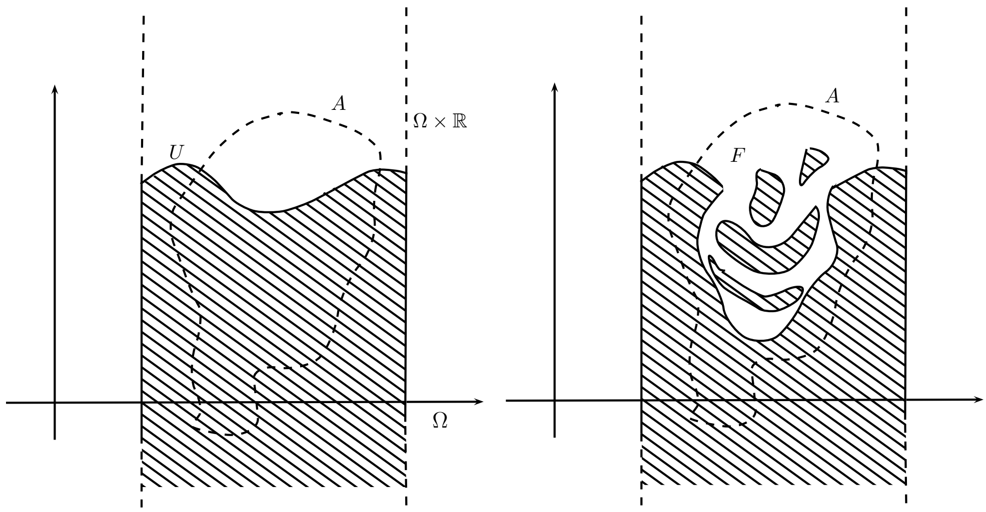
$$|\partial U|(\Omega \times \mathbb{R}) = \int_\Omega \sqrt{1+|Du|^2}\, dx$$

where $U$ denotes the subgraph in $\mathbb{R}^{n+1}$ induced by $u$, i.e.

$$U := \left\{ x = (x', x_{n+1}) \in \Omega \times \mathbb{R} : \, x_{n+1} < \, u(x') \, \right\} .$$

Moreover a function $u \in C^2(\Omega)$ satisfies (1) if and only if $U$ locally minimizes the perimeter measure in the cylinder $\Omega \times \mathbb{R}$, that is, for each open set $A \Subset \Omega \times \mathbb{R}$ and measurable set $F \subset \mathbb{R}^{n+1}$ such that $F \Delta \, U := \; (F \backslash U) \cup (U \backslash F) \Subset A$, it holds that

$$|\partial U|(A) \leq \, |\partial F|(A) \, .$$




As a consequence, an equivalent formulation of the Bernstein problem in $\mathbb{R}^n$ can be stated asking wether the only (locally) minimizing subgraphs $U$ in $\mathbb{R}^{n+1} = \, \mathbb{R}^n \times \mathbb{R}$, induced by functions $u \in C^2(\mathbb{R}^n)$, must be half-spaces. This equivalent formulation has the advantage that the theory of sets of finite perimeter (also called Caccippoli's sets, devoleped by De Giorgi in the 1950s, see Chapter 5) can be applied to the Bernstein problem.

The new idea, suitable for solving the Bernstein problem in higher dimensions, was introduced by W. Fleming in 1962 (see [Fle62]), who gave a new proof of Bernstein's theorem. Roughly speaking, Fleming idea was the following. Let $u \in C^2(\mathbb{R}^n)$ be a function which induces a locally perimeter minimizing set $U$ in $\mathbb{R}^{n+1}$. We can consider the sequence of sets

$$U_j := \left\{ x \in \mathbb{R}^{n+1} : \, j\, x \in U \right\} \quad j \in \mathbb{N}$$

and show that, up to a subsequence, it converges to a locally perimeter minimizing set $C$. Fleming then proved that $C$ is a cone and that its boundary $\partial C$ is a hyperplane if and only if $u$ was an affine function. In other words, the existence of non trivial entire minimal graphs in $\mathbb{R}^n$ implies the existence of singular minimal cones in $\mathbb{R}^n$. Eventually Fleming proved there are no minimal cones in $\mathbb{R}^3$, whence a new proof of Bernstein's theorem.

De Giorgi (see [DG65]) improved the result in 1965 proving that if there is no minimal cone in $\mathbb{R}^{n-1}$ then the analogue of Bernstein's theorem is true in $\mathbb{R}^{n-1}$, which in particular implies that it is true in $\mathbb{R}^3$.

F. Almgren (see [AJ65]) showed in 1966 there are no minimal cones in $\mathbb{R}^4$, thus extending Bernstein's theorem to $\mathbb{R}^4$.

J. Simons (see [Sim68]) extended the result in 1969 proving that there are no minimal cones in $\mathbb{R}^n$ up to $n \leq 7$. Thus he extended Bernstein's theorem up to $\mathbb{R}^n$ with $n \leq 7$. He also conjectured that the cone

$$C_S := \left\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : \, |x|^2 < \, |y|^2 \right\}$$

was minimal in $\mathbb{R}^8$.

E. Bombieri, De Giorgi & E. Giusti (see [BDGG69]) showed in 1969 that Simons cone $C_S$ is indeed of locally minimal perimeter in $\mathbb{R}^8$, and showed that in $\mathbb{R}^{n+1}$ for $n \geq 8$ there are graphs that are minimal but not hyperplanes. Combined with the result of Simons, this shows that the analogue of Bernstein's theorem is true in dimensions up to 7, and false in higher dimensions.

Therefore the Bernstein Problem for the Euclidean case is completely solved, and his solution can be summarize in the following

**Theorem 1.** 1. *If $n \leq 7$ every $C^2$ solution $u$ of (1) in $\mathbb{R}^n$ is an affine function. If $n \geq 8$ there are analytic functions $u$, solving (1), that are not affine.*

2. *Suppose that $U$ is a subgraph of a $C^2$ function $u : \mathbb{R}^n \to \mathbb{R}$ that locally minimize the perimeter in $\mathbb{R}^n \times \mathbb{R}$. Then either $n \geq 8$ or $\partial U$ is an hyperplane.*

In the last part of this thesis we propose an introduction to the Bernstein problem in the setting of the simplest sub-Riemannian metric structure, namely the Heisenberg group . We are going to briefly introduce the sub-Riemannian Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ and then the Bernstein problem in this setting. Moreover we will show the features that the problem shares with the Euclidean one as well as the main differences involved and the questions still open.

We will call sub-Riemannian Heisenberg group, denoted $\mathbb{H}^n$, the set $\mathbb{R}^{2n+1}$ equipped with the following algebraic, differentiable, metric and measure structures.

The **algebraic structure** is introduced in $\mathbb{R}^{2n+1}$ by the following group law

$$P \cdot Q := (x + x', y + y', t + t' - 2\langle x, y'\rangle + 2\langle x', y\rangle).$$

where $P = (x, y, t)$, $Q = (x', y', t')$. $(\mathbb{R}^{2n+1}, \cdot)$ turns out to be a Lie group, not Abelian. Moreover we equip $\mathbb{R}^{2n+1}$ with a 1- parameter group of automorphims $\delta_\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ $(\lambda > 0)$, called *intrinsic dilations*, defined by

$$\delta_\lambda(P) := (\lambda x, \lambda y, \lambda^2 t)$$

The **differentiable structure** is introduced by the following vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$, $T$ on $\mathbb{R}^{2n+1}$ defined by

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}\,, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}\,,\ i = 1, \dots, n$$

$$T := \frac{\partial}{\partial t}$$

which are a basis for the Lie algebra $\mathfrak{h}_n$ associated to $(\mathbb{R}^{2n+1}, \cdot)$. Sometimes we will write $X_i := Y_{i-n}$ if $i = n + 1.\dots, 2n$.

Let us observe that the only non-vanishing commutator is given by

$$[X_i, Y_i] = -4\,T \quad \forall i = 1, \dots, n\,.$$

We will also equip $\mathfrak{h}_n$ by a scalar product $\langle \cdot, \cdot \rangle$ which respect to the vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$, $T$ are orthonormal.

Then we introduce a subbundle $H\mathbb{H}^n$of the tangent bundle $T(\mathbb{R}^{2n+1})$, called *horizontal bundle*, whose each fiber is defined by

$$H_P\mathbb{H}^n := \operatorname{span}\{X_1(P), \dots, X_{2n}(P)\}\ .$$

The vector fields of the horinzontal bundle $H\mathbb{H}^n$ are called *horizontal* while the vectot field $T$ is called *vertical.* The horizontal vector fileds will play the role, in the sub-Riemannian setting, of admissible vector fields along which the differentiation is allowed. In particular the role of *intrinsic gradient* is played in this setting by the section of $H\mathbb{H}^n$

$$\nabla_{\mathbb{H}} f = \sum_{i=1}^{2n} X_i f\, X_i \equiv (X_1 f, \dots, X_{2n} f) \quad \text{ if } f \in C^1(\mathbb{R}^{2n+1})$$

and it is called *horizontal gradient.* As well, the notion of *intrinsic divergence* for a regular section $\phi = \sum_{i=1}^{2n} \phi_i\, X_i : \mathbb{R}^{2n+1} \to H\mathbb{H}^n$ is defined by

$$\operatorname{div}_{\mathbb{H}}(\phi) := \sum_{i=1}^{2n} X_i \phi_i$$

and it is called *horizontal* or $\mathbb{H}$- *divergence.*

The **metric structure** is introduced by a so-called *homogeneous metric* $d$ on $\mathbb{R}^{2n+1}$, which is a metric well-behaved either with respect to the left-translations of the group and the intrinsic dilations, that is, the metric $d$ satisfies

$$d(P \cdot Q,\ P \cdot R) = d(Q,\ R) \quad \forall\, P, Q, R \in \mathbb{R}^{2n+1}\,, \tag{3}$$

$$d(\delta_\lambda(P),\, \delta_\lambda(Q)) = \lambda\, d(P,\, Q) \quad \forall\, P, Q \in \mathbb{R}^{2n+1}, \forall\, \lambda > 0\,. \tag{4}$$

A relevant homogeneus metric in the sub-Riemannian setting is the so-called *Carnot-Carathéodory metric* associated to the subbundle $H\mathbb{H}^n$ and denoted by $d_c$. Its definition is reminiscent of the Riemannian metric definition when the tangent bundle $T(\mathbb{R}^{2n+1})$ is replaced with the horizontal subbundle $H\mathbb{H}^n$. More precisely it is defined as follows. We say that a Lipschitz continous curve $\gamma : [0,T] \to \mathbb{R}^{2n+1}$ is a subunit path (briefly s.p.) if for almost every $t \in [0,T]$

$$\dot\gamma(t) \in H_{\gamma(t)}\mathbb{H}^n\,, \qquad |\dot\gamma(t)|_{\gamma(t)} \le 1$$

Then we define the Carnot-Carathéodory distance $d_c$ between the points $P, Q \in \mathbb{H}^n$ as

$$d_c(P,Q) := \inf\Big\{T \ge 0 \mid \exists\, \gamma : [0,T] \to \mathbb{R}^{2n+1} \text{ s.p., with } \gamma(0) = P,\ \gamma(T) = Q\Big\}$$

Since the family $(X_1, \ldots, X_{2n})$ Lie generate the whole tangent space, from a theorem due to Chow (see Chapeter 12) we know that $d_c$ is actually a distance on $\mathbb{H}^n$, i.e. $d_c(P,Q)$ is finite for each pair of points $P, Q \in \mathbb{H}^n$. Since the distance $d_c$ is not explicit, it is convenient to consider an equivalent but explicit homogenous distance, the infinity distance $d_\infty$, defined as follows

$$d_\infty(P,Q) := \|P^{-1} \cdot Q\|_\infty$$

where $\|P\|_\infty := \max\{|(x,y)|, |t|^{\frac{1}{2}}\}$. Then it can be proved that $d_c$ and $d_\infty$ are equivalent, that is there exists a constant $\alpha > 1$ such that

$$\frac{1}{\alpha}\, d_\infty(P,Q) \le\, d_c(P,Q) \le\, \alpha\, d_\infty(P,Q) \quad \forall\, P, Q \in \mathbb{R}^{2n+1} \tag{5}$$

and that the bounded sets in the metric space $(\mathbb{R}^{2n+1}, d)$ coincide with the ones of $(\mathbb{R}^{2n+1}, |\cdot|)$ where $d =\, d_c$ or $d_\infty$ and $|\cdot|$ denotes the Euclidean distance in $\mathbb{R}^{2n+1}$ (see Chapter 12). Instead of they are not Riemannian, meaning that they are not equivalent to the Euclidean distance. Indeed it holds that, for each bounded set $\Omega \subset \mathbb{R}^{2n+1}$, there exists a constant $c = c(\Omega) > 1$ such that

$$\frac{1}{c}\, |P - Q| \le\, d_\infty(P,Q) \le\, c\, \sqrt{|P - Q|} \quad \forall\, P, Q \in \mathbb{R}^{2n+1}\,. \tag{6}$$

Because $d_c$ and $d_\infty$ are equivalent, $d_c$ also satisfies (6) for a suitable constant $c$. On the other hand, by (5) and (6), it follows that $(\mathbb{R}^{2n+1}, d)$ with $d = d_c$ or $d_\infty$ and $(\mathbb{R}^{2n+1}, |\cdot|)$ are topologically equivalent, that is they are homeomorphic by means of the identity map.

Eventually the **measure structure** is introduced by means of an intrinsic notion of volume measure. The volume measure is simply represented by the (2n+1)-Lebesgue dimensional measure $\mathcal{L}^{2n+1}$. Indeed it can be proved that $\mathcal{L}^{2n+1}$ is the *Haar measure* of the group $(\mathbb{R}^{2n+1}, \cdot)$, that is a Radon measure such that

$$\mathcal{L}^{2n+1}(P \cdot A) = \mathcal{L}^{2n+1}(A) \quad \forall\, P \in \mathbb{R}^{2n+1},\, A \subset \mathbb{R}^{2n+1}. \tag{7}$$

Moreover $\mathcal{L}^{2n+1}$ is also homogeneuous of order $2n+2$ with respect to the intrinsic dilations, that is

$$\mathcal{L}^{2n+1}(\delta_\lambda(K)) = \lambda^{2n+2}\,\mathcal{L}^{2n+1}(K) \quad \text{for each compact } K \subset \mathbb{R}^{2n+1}, \lambda > 0\,. \tag{8}$$

As a consequence of (8), it can be proved that the metric dimension of $(\mathbb{R}^{2n+1}, d)$ with $d = d_{cc}$ or $d_\infty$ is $2n+2$, instead of its topological dimension which is $2n+1$ (see Chapter 12). This feature is another evidence of the different behaviour with respect to a Riemannian manifold, where the two dimensions coincide.

Now we are going to introduce the Bernstein problem in the Heisenberg group $\mathbb{H}^n$. We need to introduce the notions of *hyperplane*, *graph* and *area* of a graph in this setting.

The notion of intrinsic hyperplane in $\mathbb{H}^n$ arises in a natural way on taking into account Pansu's differentiability theorem in Carnot groups (see [Pan89]): a function $f : \mathbb{H}^n \to \mathbb{R}$ which is Lipschitz with respect to the metric $d_\infty$ can be approximated a.e. by an intrinsic differential, i.e. by a homogeneous linear function $L : \mathbb{H}^n \to \mathbb{R}$. Such a function $L$ must be of the form

$$L(x, y, t) = \langle a, x\rangle + \langle b, y\rangle$$

for some $a, b \in \mathbb{R}^n$. Then it is natural to define a *vertical plane* $V$ in $\mathbb{H}^n$ as a level set of $L$

$$V = \Big\{(x, y, t) \in \mathbb{H}^n \mid \langle a, x\rangle + \langle b, y\rangle = c\Big\}$$

for some $c \in \mathbb{R}$. Moreover we call $f : \mathbb{H}^n \to \mathbb{R}$ an intrinsic affine function if $f$ is of the form

$$f(x, y, t) = \langle a, x\rangle + \langle b, y\rangle + c$$

There are two natural notions of $2n$-dimensional graph in the setting of $\mathbb{H}^n$.

The former is the one of graph with respect to the vertical vector field $T$, called **$t$-graph**. The latter is the one of graph with respect to the horizontal vector field $X_i$ for fixed $i = 1, \ldots, 2n$, called **intrinsic** or **$X_i$- graph**.

Let $\Pi := \big\{(z, t) \in \mathbb{R}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R} : z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,\, t = 0\big\} \equiv \mathbb{R}^{2n}$ and let $e_1, \ldots, e_{2n+1}$ denote the canonical basis of $\mathbb{R}^{2n+1}$. Then a set $S \subset \mathbb{R}^{2n+1}$ is a $t$- graph in $\mathbb{H}^n$ if there exist a set $\mathcal{U} \subset \Pi$ and a function $u : \mathcal{U} \to \mathbb{R}$ such that

$$S = \{(z, 0) \cdot u(z)\, e_{2n+1} = (z, u(z)) : z \in \mathcal{U}\}\,.$$

We call $t$-subgraph in $\mathbb{H}^n$ the set

$$E_u^t := \{(z,t) \in \mathcal{U} \times \mathbb{R} : \, z \in \mathcal{U}, \, t < \, u(z) \, \} \tag{9}$$

where $u : \mathcal{U} \subset \Pi \to \mathbb{R}$.

We observe that the notion of $t$-graph coincides with the one of Euclidean $t$-graph in $\mathbb{R}^{2n+1}$.

Let $\mathbb{W} := \left\{(x,y,t) \in \mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \, x_1 = 0\right\} \equiv \mathbb{R}^{2n}$. Then a set $S \subset \mathbb{R}^{2n+1}$ is a $X_1$- graph in $\mathbb{H}^n$ if there exist a set $\omega \subset \mathbb{W}$ and a function $\phi : \omega \to \mathbb{R}$ such that

$$S = \{A \, \cdot \, \phi(A) \, e_1 \, : \, A \in \omega\} \, .$$

We call $X_1$-subgraph in $\mathbb{H}^n$ the set

$$E_\phi := \{A \cdot \, s \, e_1 \in \omega \cdot \mathbb{R} \, e_1 \, : \, A \in \omega, \, s < \, \phi(A) \, \} \tag{10}$$

where $\phi : \omega \to \mathbb{R}$ and $\omega \cdot \mathbb{R} \, e_1 := \{A \cdot \, s \, e_1 \, : \, s \in \mathbb{R}\, \}$. Similar definitions for the intrinsic $X_i$-graphs for $i = 2, \ldots, 2n$.

Let us now introduce the notion of intrinsic area for $t$- and $X_1$-graph in $\mathbb{H}^n$. We are going to define it as intrinsic perimeter of their respective subgraphs.

Firstly, let us introduce the intrinsic perimeter measure, called $\mathbb{H}$-perimeter, in the setting of $\mathbb{H}^n$. If $E \subset \mathbb{R}^{2n+1}$ is a measurable set and $\Omega \subset \mathbb{R}^{2n+1}$ is open, the $\mathbb{H}$-perimeter measure of $E$ in $\Omega$ is denoted by $|\partial E|_{\mathbb{H}}(\Omega)$ and defined by

$$|\partial E|_{\mathbb{H}}(\Omega) := \sup \left\{ \int_E \mathrm{div}_{\mathbb{H}}(\phi) \mid \, \phi \in C_c^1(\Omega; H\mathbb{H}^n), \, |\phi(P)|_P \leq \, 1 \, \forall \, P \in \Omega \right\}$$

(see Chapter 12).

Let $\mathcal{U} \subset \Pi \equiv \mathbb{R}^{2n}$ and $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be bounded open sets, then it holds that (see [BASCV07])

$$|\partial E_u^t|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) = \int_{\mathcal{U}} |\nabla u + \, X^*| \, \mathrm{d}\mathcal{L}^{2n} := \, \mathcal{A}_t(u) \quad \forall \, u \in C^2(\overline{\mathcal{U}})$$

where $X^* : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the map defined by $X^*(z) := \, 2\,(-y,x)$ if $z = (x,y) \in \mathcal{U}$, and

$$|\partial E_\phi|_{\mathbb{H}}(\omega \cdot \, \mathbb{R} \, e_1) = \int_\omega \sqrt{1 + |W^\phi \phi|^2} \, \mathrm{d}\mathcal{L}^{2n} := \, \mathcal{A}_{\mathbb{W}}(\phi) \quad \forall \, \phi \in C^2(\overline{\omega}) \tag{11}$$

where $W^\phi$ is defined as follows

$$W^\phi \phi := \begin{cases} (X_2\phi, \ldots, X_n\phi, Y_1\phi - 2T(\phi^2), Y_2\phi, \ldots, Y_n\phi) & , n \geq 2 \\ Y_1\phi - 2T(\phi^2) & , n = 1 \end{cases}$$

The functionals $\mathcal{A}_t : C^2(\overline{\mathcal{U}}) \to \mathbb{R}$ and $\mathcal{A}_{\mathbb{W}} : C^2(\overline{\omega}) \to \mathbb{R}$ are respectively called $t$-area and $X_1$-area functionls.

Making a simple first variation of the area formula (11) we obtain the so called *minimal surface equation for $X_1$-graphs*

$$W^\phi \cdot \frac{W^\phi \phi}{\sqrt{1+|W^\phi \phi|^2}} = 0 \tag{12}$$

It turns out that intrinsic affine functions satisfied equation (12), and that parametrize, in the sense of $X_1$-graphs, exactly vertical planes; moreover their $X_1$-subgraphs locally minimize the $\mathbb{H}$-perimeter.

The Bernstein Problem in $\mathbb{H}^1$ for $C^2$ t-graphs has been studied in [GP], [CDG94], [DGN], [GN96], [DGNPa], [DGN07], [DGNPb], [Pau04], [CHMY05]. A suitable minimal surface equation for $u$ has been obtained and its solution have been called *H-minimal.* In particular it turns out that there exists H-minimal functions $u : \mathbb{R}^2 \to \mathbb{R}$ whose t-graph is not an affine plane. On the other han, $C^2$ regular entire H-minimal solutions $u$ for which its t-subgraph is a minimizer have been characterized in [CHMY05] and in [RR08].

In this thesis we will only deal with the Bernstein Problem for $X_1$-graphs.

So with this notions of hyperplanes and subgraphs we can give this two formulations in $\mathbb{H}^n$ of the Bernstein Problem:

**(B1) - Bernstein Problem in $\mathbb{H}^n$ - version I**: Are there entire $C^2$ solutions of the minimal surface equation for $X_1$-graphs (12) wich do not parametrize vertical planes?

**(B2) - Bernstein Problem in $\mathbb{H}^n$ - version II**: Let $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ be such that its $X_1$-subgraph $E_\phi$ locally minimize the $\mathbb{H}$-perimeter in $\mathbb{H}^n$. It is true that $\partial E_\phi$ is a vertical plane?

A main difference from the Euclidean case is that this two formulations are not equivalent! In fact in [DGN08] it has been obtained the existence of a $C^2$ function $\phi : \mathbb{R}^2 \to \mathbb{R}$ that is solution of the minimal surface equation (12), but such that whose subgraph $E_\phi$ is not a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^1$ and it is not a vertical plane. Such a function provided a positive answer to Problem $(B1)$. The function $\phi$ is defined as

$$\phi(\eta, \tau) := -\frac{\alpha\eta\tau}{1+2\alpha\eta^2}$$

for $\alpha > 0$.

The main result for the Bernstein Problem for $X_1$-graphs in the Heisenberg group, obtained in [BASCV07], is the following

**Theorem 2.** *1. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$ function, and let $E, S \subset \mathbb{H}^1$ be respectively the $X_1$-graph and the $X_1$-subgraph of $\phi$. Let us suppose that $E$ is a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^1$. Then $S$ is a vertical plane, i.e. $\phi(\eta\tau) = w\eta + c$ for all $(\eta, \tau) \in \mathbb{R}^2$ for some constants $w, c \in \mathbb{R}$.*

*2. If $n \geq 5$ there exists functions $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ that satisfied (12) but that are not intrinsic affine. Moreover their $X_1$-subgraph locally minimizes the $\mathbb{H}$-perimeter in $\mathbb{H}^n$.*

The assumption that $\phi$ is a $C^2$ function is crucial for the above result, because in [RSCV08] it has been found a counterexample to the above result if we drop that assumption. The Bernstein Problem for $X_1$-graphs in the Heisenberg group $\mathbb{H}^n$ remains still open in the cases $n = 2, 3, 4$.

The structure of the thesis is the following. In Chapter 1 we present a simple proof of the Bernstein Theorem due to Nische in dimension $n = 2$ (see [Nit67]).

In Chapter 2 we introduce some basic tools of measure theory that we will use through the thesis: in particular we prove the classical Vitali's covering Theorem and Besicovitch's covering Theorem (Section 2.6), that we will use to prove the Differentiation Theorem for Radon measures in $\mathbb{R}^n$ (Section 2.7). Finally we prove the Riesz Representation Theorem (Section 2.7) and we study the weak convergence for Radon measure in $\mathbb{R}^n$ (Section 2.9).

Chapter 3 is dedicate to introduce the Hausdorff measures in a metric space, and to prove their basic properties: in particular we will prove the isodiametric inequality (Theorem 3.2.5), the fact that $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$ (Theorem 3.2.6) and we will study the density properties of Hausdorff measures (Section 3.1.2).

In Chapter 4 we introduce some particular metric space in which we can generalize the covering theorems presented in Chapter 2: we define the notion of homogeneous spaces (Section 4.1), that allows to extend Vitali's covering Theorem, and the notion of directionally metric space (Section 4.2), that allows to extend Besicovitch's covering Theorem.

Chapter 5 is dedicated to the introduction of the space of functions of bounded variation and Caccioppoli sets. In particular we will prove the semicontinuity of the total variation (Theorem 5.1.4), Anzellotti-Giaquinta's approximating theorem (Theorem 5.2.1), the existence of minimal surfaces (Theorem 5.3.3) and the isoperimetric inequalities (Theorem 5.4.2).

In Chapter 6 we introduce the reduced boundary of a Caccioppoli set, and we prove the foundamental Theorem of De Giorgi (Theorem 6.3.2) that state that the reduced boundary of a Caccioppoli set is rectifiable, i.e. is, up to a set of zero perimeter, a countable union of compact subsets of $C^1$ hypersurfaces.

In Chapter 7 we define the trace of a $BV$ function on the boundary of a Lipschitz bounded open set (Theorem 7.2.2); in particular this notion allows us to extend the classical Gauss-Green formula to $BV$ functions.

In Chapter 8 we prove some important inequalities concerning minimal sets, that allow us to give a lower and an upper estimate of the perimeter of a minimal set in a boundary point, and a lower and an upper estimate of the Lebesgue measure of a minimal set in a ball centered in a boundary point (Section 8.2).

Chapter 9 is dedicated to the regularity of the minimal surfaces: in particular we prove the non existence of minimal cones in $\mathbb{R}^n$ for $n \leq 7$ (Sub-Section 9.3.3) and that Simons cone $\mathcal{C}_S$ is a minimal set in $\mathbb{R}^8$ (Section 9.4).

In Chapter 10 we deal with the Dirichlet problem for the area functional in an open set $\Omega \subset \mathbb{R}^n$. In Section 10.1 we solved the Dirichlet problem in a classical method: under some assumption on the curvature of $\partial\Omega$ we prove the existence of a minimum for the area functional among all Lipschitz continous functions with a prescribed datum on $\partial\Omega$. We will also prove that the hypothesis on the curvature of the boundary is necessary. In Section 10.2 we study a relaxed formulation of the Dirichlet problem in the setting of $BV$ spaces. Then, in Section 10.2.2, we prove the connection between parametric and non-parametric minimal surfaces.

In Chapter 11 we present the solution of the Bernstein Problem in the Euclidean case for dimension $n \geq 3$.

Chapter 12 is dedicated to the introduction of the sub-Riemannian Heisenberg group $\mathbb{H}^n$, and to the introduction of the principal notions and results useful to state the Bernstein Problem in $\mathbb{H}^n$.

Finaly in Chapter 13 we state two formulations of the Bernstein Problem for intrinsic $X_1$-graphs, and we present the solutions obtained so far.

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# Notation

| | |
|---|---|
| $\Subset$ | compactly contained |
| $\triangle$ | symmetric difference of sets |
| $\mathcal{P}(X)$ | family of subsets of $X$ |
| $\mathrm{Card}(A)$ | Cardinality of the set $A$ |
| $\mathrm{diam}(A)$ | diameter of the set $A$ |
| $\circ$ | composition of functions |
| $\oplus$ | direct sum of vector spaces |
| $\mathrm{supp}(f)$ | support of $f$ |
| $f_{\vert A}$ | restrinction of the function $f$ to $A$ |
| $U_r(x)$ | open ball centered in $x$ with radius $r$ |
| $B_r(x)$ | closed ball centered in $x$ with radius $r$ |
| $U_r^c(x)$ | open ball centered in $x$ with radius $r$ with respect to the distance $d_c$ |
| $B_r^c(x)$ | closed ball centered in $x$ with radius $r$ with respect to the distance $d_c$ |
| | |
| $\mu \llcorner A$ | restriction of the measure $\mu$ to a set $A$ |
| $\mu \ll \nu$ | $\mu$ is absolutely continous with respect to $\nu$ |
| $\mu \perp \nu$ | $\mu$ and $\nu$ are mutually singular |
| $\vert\mu\vert$ | total variation of the measure $\mu$ |
| $\mathrm{supp}(\mu)$ | support of the measure $\mu$ |
| $\mathcal{H}^k$ | $k$-dimensional Hausdorff measure |
| $\mathcal{S}^k$ | $k$-dimensional spherical Hausdorff measure |
| $\mathcal{H}^k_\infty$ | $k$-dimensional Hausdorff measure induced by $d_\infty$ |
| $\mathcal{S}^k_\infty$ | $k$-dimensional spherical Hausdorff measure induced by $d_\infty$ |
| | |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space |
| $\langle x, y\rangle$ | standard Euclidean scalar product of $x, y \in \mathbb{R}^n$ |
| $\vert x\vert$ | Euclidean norm of $x \in \mathbb{R}^n$ |
| $f_x, \frac{\partial f}{\partial x}$ | partial derivate of $f$ with respect to $x$ |
| $D_i f$ | $i$-th partial derivate of $f$ |
| $Df, \nabla f$ | gradient of $f$ |
| div | divergence |
| $f * g$ | convolution of $f$ and $g$ |
| $\chi_E$ | characteristic function of a measureable set $E \subset \mathbb{R}^n$ |
| $\mathcal{L}^n$ | Lebesgue measure in $\mathbb{R}^n$ |
| $\omega_n$ | Lebesgue measure of the unit ball in $\mathbb{R}^n$ |
| $\fint$ | average integral |
| $\varepsilon_{ij}$ | Kronecker's symbol |

| | |
|---|---|
| $\lvert Df \rvert$ | total variation of $f$ |
| $\lvert \partial E \rvert$ | total variation of $\chi_E$, perimter measure of $E$ |
| $\nu_E$ | outer normal to $E$ |
| $\partial^* E$ | reduced boundary of $E$ |
| | |
| $\mathbb{G}$ | a Carnot group |
| $\mathfrak{g}$ | Lie algebra of $\mathbb{G}$ |
| $x \cdot y$ | group product between $x, y \in \mathbb{G}$ |
| $TM$ | tangent boundle to a manifold $M$ |
| $[X, Y]$ | commutator of $X$ and $Y$ |
| $dc$ | Carnot-Carathéodory distance |
| $l_x$ | left translation by an element $x \in \mathbb{G}$ |
| $\delta_r$ | homogeneous dilatation of $r$ in $\mathbb{G}$ |
| $\star$ | convolution on groups |
| $\mathbb{H}^n$ | $n$-th Heisenberg group |
| $\mathfrak{h}$ | Lie algebra of $\mathbb{H}^n$ |
| $\nabla_{\mathbb{H}}$ | Heisenberg gradient |
| $\mathrm{div}_{\mathbb{H}}$ | $\mathbb{H}$-divergence |
| $H\mathbb{H}^n$ | horizzontal subboundle to $\mathbb{H}^n$ |
| $\lVert \cdot \rVert_\infty$ | infinity norm |
| $d_\infty$ | infinity distance |
| | |
| $C^k(\Omega)$ | continously $k$-differentiable real functions in $\Omega$ |
| $C^k_c(\Omega)$ | functions in $C^k(\Omega)$ with compact support |
| $BV(\Omega)$ | functions of bounded variation in $\Omega$ |
| $C^1_{\mathbb{H}}(\Omega)$ | continously $\nabla_{\mathbb{H}}$-differentiable functions in $\Omega$ |

# Chapter 1

# The Bernstein Problem in $\mathbb{R}^2$

In this chapter we present the result due to Bernstein, i.e. an entire solution of the minimal surface equation in the plane is an affine function, that makes rise the problem of the validity of this result in higher dimension, that is what we called the Bernstein Problem. The result is the following one:

**Theorem 1.0.1** (**Bernstein, ∼1915**). *Let $u : \mathbb{R}^2 \to \mathbb{R}$ is a solution of the minimal surface equation in the plane. Then the graph of $u$ is an affine plane.*

To prove this result we do not follow the original proof, but we present a proof due to Nitsche (see [Nit67]), thet uses a diffeomorphism introduced by Lewy. First of all we observe that the minimal surface equation in the plane

$$\frac{\partial}{\partial x}\frac{u_x}{\sqrt{1+{u_x}^2+{u_y}^2}}+\frac{\partial}{\partial y}\frac{u_y}{\sqrt{1+{u_x}^2+{u_y}^2}}=0$$

is equaivalent to

$$(1+{u_y}^2)u_{xx}-2u_xu_yu_{xy}+(1+{u_x}^2)u_{yy}=0$$

In 1955 Heinz noted that, if $u : \mathbb{R}^2 \to \mathbb{R}$, then the matrix

$$A := \frac{1}{\sqrt{1+{u_x}^2+{u_y}^2}}\begin{pmatrix} 1+{u_x}^2 & u_xu_y \\ u_xu_y & 1+{u_y}^2 \end{pmatrix}$$

has $\det A = 1$ and also satisfied: $A$ is an hessian matrix if and only if $u$ is a solution of the minimal surface equation.
In fact:

$$\frac{\partial}{\partial y}A_{1,1}=\frac{2u_xu_{xy}(1+{u_x}^2+{u_y}^2)-(1+{u_x}^2)(u_xu_{xy}+u_yu_{yy})}{(1+{u_x}^2+{u_y}^2)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial x}A_{1,2} = \frac{(u_{xx}u_y + u_x u_{xy})(1+{u_x}^2+{u_y}^2) - u_x u_y(u_x u_{xx} + u_y u_{xy})}{(1+{u_x}^2+{u_y}^2)^{\frac{3}{2}}}$$

Now $\frac{\partial}{\partial y}A_{1,1} = \frac{\partial}{\partial x}A_{1,2}$ if and only if

$$\begin{aligned} &2u_x u_{xy}(1+{u_x}^2+{u_y}^2) - (1+{u_x}^2)(u_x u_{xy} + u_y u_{yy}) \\ =&(u_{xx}u_y + u_x u_{xy})(1+{u_x}^2+{u_y}^2) - u_x u_y(u_x u_{xx} + u_y u_{xy}) \end{aligned}$$

that is

$$-u_y((1+{u_y}^2)u_{xx} - 2u_x u_y u_{xy} + (1+{u_x}^2)u_{yy}) = 0$$

which is equiavalent to the minimal surface equation in the plane, thanks to the observation made above. Same calculation for the other equality to check.

So we have obtained that $u$ is a solution of the minimal surface equation if and only if there exists a $C^2$ map $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $H\phi = A$, where $A$ is defined as above. Such a $\phi$ has $\det H\phi = 1$.
Now, thanks to the following result due to Jorgens in 1954, we obtain our desidered theorem.

**Theorem 1.0.2.** *Let $v : \mathbb{R}^2 \to \mathbb{R}$ a $C^2$ map with $\det Hv \equiv 1$. Then $v$ is a polyminial of degree two.*

*Proof.* Since $1 = \det Hv = v_{xx}v_{yy} - {v_{xy}}^2$ we have that $v_{xx}v_{yy} \geq 0$; so we can suppose that $v_{xx}, v_{yy} > 0$, that is $v$ is a convex function. Now we introduce the following change of variables:

$$\psi : \begin{cases} \xi := x + v_x \\ \eta := y + v_y \end{cases}$$

So we obtain

$$\det(J\psi) = \det\begin{pmatrix} 1+v_{xx} & v_{xy} \\ v_{xy} & 1+v_{yy} \end{pmatrix} = 2 + v_{xx} + v_{yy} > 2$$

So $\psi$ defines an open map. Since $\psi$ is convex, it holds:

$$\langle Dv(x_2,y_2) - Dv(x_1,y_1), (x_2-x_1, y_2-y_1)\rangle \ \geq 0$$

for each $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$. Equivalently:

$$0 \leq (x_2-x_1)[v_x(x_2,y_2) - v_x(x_1,y_1)] + (y_2-y_1)[v_y(x_2,y_2) - v_y(x_1,y_1)]$$

If we substitute the change of variable in the last inequality, we obtain, using the Cauchy-Schwarz inequality:

$$|(x_2,y_2) - (x_1,y_1)| \leq |(x_2-x_1, y_2-y_1)| \cdot |(\xi_2-\xi_1, \eta_2-\eta_1)|$$

and so $\psi$ is a closed map. Since $\psi$ is open and closed, $\psi$ is a diffeomorphism. Introducing the complex variable $z := \xi + i\eta$, we define the function:

$$f(z) := (x - v_x) - i(y - v_y)$$

that comes out to be holomorphic. In fact, if in the definition of $\psi$ we derive to respect $\xi$ and $\eta$ we obtain the system:

$$\begin{cases} 1 &= \dfrac{\partial x}{\partial \xi}(1 + v_{xx}) + \dfrac{\partial y}{\partial \xi} v_{xy} \\ 0 &= \dfrac{\partial x}{\partial \eta}(1 + v_{xx}) + \dfrac{\partial y}{\partial \eta} v_{xy} \\ 0 &= \dfrac{\partial y}{\partial \xi}(1 + v_{yy}) + \dfrac{\partial x}{\partial \xi} v_{xy} \\ 1 &= \dfrac{\partial y}{\partial \eta}(1 + v_{yy}) + \dfrac{\partial x}{\partial \eta} v_{xy} \end{cases}$$

that has as solution

$$\begin{cases} \dfrac{\partial y}{\partial \xi} &= -\dfrac{v_{xy}}{2 + v_{xx} + v_{yy}} \\ \dfrac{\partial x}{\partial \xi} &= \dfrac{1 + v_{yy}}{2 + v_{xx} + v_{yy}} \\ \dfrac{\partial y}{\partial \eta} &= -\dfrac{1 + v_{xx}}{2 + v_{xx} + v_{yy}} \\ \dfrac{\partial x}{\partial \eta} &= -\dfrac{v_{xy}}{2 + v_{xx} + v_{yy}} \end{cases}$$

From these equalities we obtain that

$$\frac{\partial f}{\partial \xi} = \frac{v_{yy} - v_{xx}}{2 + v_{xx} + v_{yy}} + i \frac{2 v_{xy}}{2 + v_{xx} + v_{yy}}$$

$$\frac{\partial f}{\partial \eta} = -\frac{2 v_{xy}}{2 + v_{xx} + v_{yy}} + i \frac{v_{yy} - v_{xx}}{2 + v_{xx} + v_{yy}}$$

that is

$$\frac{\partial f}{\partial \xi} = -i \frac{\partial f}{\partial \eta}$$

and so $f$ is holomorphic.
Since

$$f'(z) = \frac{v_{yy} - v_{xx}}{2 + v_{xx} + v_{yy}} + i \frac{2 v_{xy}}{2 + v_{xx} + v_{yy}} \tag{1.1}$$

we have that

$$1 - |f'(z)| = \frac{4}{2 + v_{xx} + v_{yy}} > 0$$

and hence by Liouville's theorem $f'$ is constant, and in particular $1-|f'(z)|$ is constant. Hence $v_{xx}$ and $v_{yy}$ are constant, and by 1.1 is constant also $-v_{xx}+v_{yy}$ and $v_{xy}$. At the end we have that $v_{xx}, v_{yy}, v_{xy}$ are constant, and so $v$ is a polynomial of degree two. $\square$

Coming back to our martix $A$, and applying the theorem just proved, we obtain that $1+{u_x}^2, u_x u_y, 1+{u_y}^2$ are constant, and hence $u_x, u_y, u_{xy}$ are constant. So $u$ is an affine function.

We note that the tecnique used here are *ad hoc* for dimension two, and cannot be extended to higher dimension. In order to try to prove the validity of the Bernstein Theorem in higher dimensions we need a new idea suitable for extension in all the dimensions. We will see how to do it in Chapter 11.

**Note**: we start studing the case $n=2$, because the case $n=1$ is quite simple. In fact in one dimension the minimal surface equation becomes

$$\frac{u''(1+|u'|^2-u'|u'|)}{(1+|u'|^2)^{\frac{3}{2}}}=0$$

Since the equation $1+|u'|^2-u'|u'|=0$ has no solution, we need to impose that $u''\equiv 0$, hence obtaining that the only solutions of the minimal surface equation in one dimension are the lines. So the Bernstein Problem in $\mathbb{R}$ is trivial.

# Chapter 2

# Introduction to Measure Theory

The aim of this chapter is to introduce some basic tools on measure theory. We begin by presenting briefly, in the first five sections, some standard results on outer measures, measures, vector valued measures, and the connections between this objects. Then we will prove in Section 2.6 two important covering theorems in $\mathbb{R}^n$, Vitali's covering Theorem (Theorem 2.6.1) and Besicovitch's covering Theorem (Theorem2.6.6), that we will extend for metric spaces in Chapter 4. Section 2.7 is dedicate to study the possibility of differentiating in $\mathbb{R}^n$ a Radon measure $\mu$ with respect to another Radon measure $\nu$ (Theorem 2.7.3) obtaining $D_\nu\mu$, the "derivate of $\mu$ with respect to $\nu$ ", and then how to recover $\mu$ from $D_\nu\mu$ (Theorem 2.7.4). Finally in Section 2.8 we will study the Riesz Representation Theorem (Theorem 2.8.5), an important theorem that links functional analysis with measure theory, and makes possible to give a notion of weak convergence for Radon measures in metric spaces, that we will study in Section 2.9.

## 2.1 Outer measures and properties

We start by proving some basic properties of outer measures.

**Definition 2.1.1.** *Let $X$ be a set. A map $\mu : \mathcal{P}(X) \to [0, \infty]$ is called an **outer measure** on $X$ if:*

- $\mu(\emptyset) = 0$
- *if* $A \subset \bigcup_{i=1}^{\infty} A_i$, *then* $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$

**Definition 2.1.2.** *A set $A \subset X$ is said to be **$\mu$-measurable** if*

$$\mu(M) = \mu(M \cap A) + \mu(M \setminus A)$$

It is clear that if $\mu(A) = 0$, then $A$ is $\mu$-measurable, as every subset of $A$. Moreover, if $A$ is $\mu$-measurable, then also $X \setminus A$ is $\mu$-measurable. Measurable sets are very important in measure theorey, as it can be seen in the following theorem.

**Theorem 2.1.3.** *Let $X$ be a set, $\mu$ an outer measure on $X$, and $(A_i)_i$ be $\mu$-measurable sets. Then it hold:*

- *the sets $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ are $\mu$-measurable*
- *if $(A_i)_i$ are disjoints, then*

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i)$$

- *if $A_1 \subset A_2 \subset \dots$, then*

$$\lim_{i\to\infty} \mu(A_i) = \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big)$$

- *if $A_1 \supset A_2 \supset \dots$ and $\mu(A_1) < \infty$, then*

$$\lim_{i\to\infty} \mu(A_i) = \mu\Big(\bigcap_{i=1}^{\infty} A_i\Big)$$

**Definition 2.1.4.** *Let $X$ be a set, $\mu$ an outer measure on $X$; we denote by $\mathcal{M}(\mu)$ the $\sigma$-algebra of the $\mu$-measurable sets.*

**Definition 2.1.5.** *We say that a property $P$ holds **$\mu$-almost everywhere** ($\mu$-**a.e.**) on $X$ if there is a set $A \subset X$ such that $\mu(X \setminus A) = 0$ and property $P$ holds for all $x \in A$.*

Now we introduce some classes of outer measures:

**Definition 2.1.6.** *Let $X$ be a set.*

- *an outer measure $\mu$ on $X$ is called **regular** if for every set $A \subset X$, there exist a $\mu$-measurable set $B$ such that $B \supset A$ and $\mu(B) = \mu(A)$*
- *an outer measure $\mu$ on $X$ is called **$\sigma$-finite** if there exists $(A_i)_i \subset \mathcal{M}(\mu)$ such that $\mu(A_i) < \infty$ and $X = \bigcup_{i=0}^{\infty} A_i$*

- *an outer measure $\mu$ on a topological space $X$ is called* ***Borel outer measure*** *if every Borel set is $\mu$-measurable*

- *an outer measure $\mu$ on a topological space $X$ is called* ***Borel regular outer measure*** *if $\mu$ is a Borel outer measure, and for each set $A \subset X$ there exists a Borel set $B$ such that $B \supset A$ and $\mu(A) = \mu(B)$*

- *an outer measure $\mu$ on a metric space $(X, d)$ is called* ***locally finite*** *if for all $x \in X$ there exists $r_x > 0$ such that $\mu(B_{r_x}(x)) < \infty$*

- *an outer measure $\mu$ on a metric space $(X, d)$ is called* ***Radon outer measure*** *measure if $\mu$ is a Borel measure satisfying*

  1. *$\mu(K) < \infty$ for each compact set $K \subset X$*
  2. *$\mu(A) = \inf\{\, \mu(V) \mid V$ open , $V \supset A\}$ for all $A \subset X$*
  3. *$\mu(V) = \sup\{\, \mu(K) \mid K$ compact , $K \subset V\}$ for each open set $V \subset X$*

- *an outer measure $\mu$ on a metric space $(X, d)$ is called* ***Caratheòdory outer measure*** *if*

  $$\mu(A \cup B) = \mu(A) + \mu(B)$$

  *for every $A, B \subset X$ such that $d(A, B) > 0$*

There is some important connections from the classes of measure defined above

- if $\mu$ is a Radon measure, then $\mu$ is Borel regular

- let $(X, d)$ be a separable complete metric space; if $\mu$ is a locally finite Borel regular outer measure on $X$, then $\mu$ is a Radon outer measure.

- let $(X, d)$ be a separable complete metric space such that the closed balls are compact; let $\mu$ be a Radon outer measure on $X$. Then $\mu$ is a locally finite Borel regular outer measure

- another important connection is the following one:

  **Theorem 2.1.7** (**Caratheòdory's criterion**)**.** *A Caratheòdory outer measure is a Borel outer measure*

In particular the Caratheòdory's criterion allows to prove that a measure is a Borel outer measure, just proving its additivity on "distant" closed sets.

Now we present some important approximation properties for some classes of measures, that allow us to approximate the measure of a set with the measure of "simple" sets.

**Theorem 2.1.8.** *Let $\mu$ be an outer Borel measure on a metric space $(X, d)$. Then, for every Borel set $B \subset X$ with $\mu(B) < \infty$ and each $\varepsilon > 0$, there exists a closed set $F \subset B$ such that*

$$\mu(B \setminus F) < \varepsilon$$

*Furthermore if*

$$B \subset \bigcup_{i=1}^{\infty} V_i$$

*where each $V_i$ is an open set with $\mu(V_i) < \infty$, then for every $\varepsilon > 0$ there exists an open set $W \subset B$ such that*

$$\mu(W \setminus B) < \varepsilon$$

**Remark 2.1.9.** *There is two important particularizations of the theorem above:*

- *if the outer measure $\mu$ is Borel outer regular, then the above theorem remains true also if we only required that $B$ is $\mu$-measurable. Moreover, in this case we can approximate* ***every*** *sets from the outside with open sets, and not only the measurable one.*
- *if $\mu$ is a Radon outer measure, the approximation from the inside with closed sets can be made with compact sets*

**Definition 2.1.10.** *Let $\mu$ be an outer measure on $X$, and $A \subset X$, we denote by $\mu \llcorner A$ the function defined on the subsets $B \subset X$ by:*

$$(\mu \llcorner A)(B) := \mu(A \cap B)$$

**Theorem 2.1.11.** *It hold:*

- *$\mu \llcorner A$ is an outer mesure on $X$*
- *$\mathcal{M}(\mu) \subset \mathcal{M}(\mu \llcorner A)$*
- *if $A \in \mathcal{M}(\mu)$ and $\mu(A) < \infty$ and if $\mu$ is a Borel regular outer measure, then $\mu \llcorner A$ is Borel regular*

## 2.2 Measures

The “problem ”of measures iis that there can be non measurable sets, and so we can not apply a $\sigma$-additive property on arbitrary disjoint sets. The notion of measures solve this problem just defining it as a $\sigma$-additive sets function on a $\sigma$-algebra of sets. Clearly we will expect some connection between outer measures and measures.

**Definition 2.2.1.** *Let $X$ be a set, and $\mathcal{M}$ be a $\sigma$-algebra of subsets of $X$. A* ***measure*** *$\mu$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that*

- $\mu(\emptyset) = 0$
- *if $(A_i)_i$ is a sequence of disjoint sets in $\mathcal{M}$, then*

$$\mu\Big(\bigcup_{i=0}^{\infty} A_i\Big) = \sum_{i=0}^{\infty} \mu(A_i)$$

*The sets in $\mathcal{M}$ are called $\mu$-measurable. We call $(X, \mathcal{M}, \mu)$ a* ***measure space****.*

For measures it holds a result similar to Theorem 2.1.3.

The most important fact about measures an outer measures is the following one: from an outer measure we can obtain a measure just restricting the outer measure to its $\sigma$-algebra of measurable sets. Also the viceversa holds: from a measure we can obtain an outer measure.
The method to obtain an outer measure from a measure only required that the measure is defined on an algebra, instead that on a $\sigma$-algebra. So we need the following

**Definition 2.2.2.** *A* ***measure on an algebra*** *$\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that*

- $\mu(\emptyset) = 0$
- *if $(A_i)_i$ is a collection of subsets of $\mathcal{A}$ such that $\bigcup_{i=0}^{\infty} A_i \in \mathcal{A}$, then*

$$\mu\Big(\bigcup_{i=0}^{\infty} A_i\Big) = \sum_{i=0}^{\infty} \mu(A_i)$$

A measure on an algebra $\mathcal{A}$ generates a function $\mu^*$ defined on all subsets $E$ of $X$ in the following way:

$$\mu^*(E) := \inf\Big\{ \sum_{i=0}^{\infty} \mu(A_i) \mid E \subset \bigcup_{i=0}^{\infty} A_i,\ A_i \in \mathcal{A} \Big\}$$

**Theorem 2.2.3 (Carathéodory-Hahn Extension Theorem).** *Let $\mu$ be a measure on an algebra $\mathcal{A}$, and let $\mu^*$ be the function defined above. Then*

- *$\mu^*$ is an outer measure such that $\mu = \mu^*$ on $\mathcal{A}$*
- *$\mathcal{A} \subset \mathcal{M}(\mu^*)$*
- *if $\mathcal{M}$ is a $\sigma$-algebra such that $\mathcal{A} \subset \mathcal{M} \subset \mathcal{M}(\mu^*)$ and $\nu$ is a measure on $\mathcal{M}$ that agree with $\mu$ on $\mathcal{A}$, then $\nu = \mu^*$ on $\mathcal{M}$ provided that $\mu$ is $\sigma$-finite.*

If we start from a measure $\mu$ defined on a $\sigma$-algebra $\mathcal{A}$ there is another method to generated an outer measure. For $E \subset X$ define

$$\mu^{**}(E) := \inf\{\, \mu(B) \mid B \supset E,\ B \in \mathcal{A}\,\}$$

It holds

**Theorem 2.2.4.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then the function $\mu^{**}$ defined above is an outer measure on $X$. Moreover for every set $E \subset X$ there exists a set $B \in \mathcal{A}$ such that $B \supset E$ and*

$$\mu(B) = \mu^*(B) = \mu^*(E) = \mu^{**}(E)$$

**Important note:** thanks to the two theorems above, we can "confuse" measures and outer measures, if we work on the measurable sets. So, in what follows, we can use both the terms "measure" and "outer measure" indistinctly if we are working with measurable sets.

## 2.3 Measurable functions

Now we want to extend the notion of measurability from sets to functions introducing the concept of measurable function, which play an important role in the theory of integration. We will focus our attenction to functions $f : X \to \overline{R}$.

**Definition 2.3.1.** *Let $X$ be topological space, and let $\mu$ be a measure on $X$. We say that a function $f : X \to \overline{R}$ is **$\mu$-measurable** if $f^{-1}(U)$ is $\mu$-measurable for each open set $U \subset \overline{R}$.*

The class of measurable functions is closed under the usually elementary operations.

**Theorem 2.3.2.** *It hold:*

- *If $f, g : X \to \overline{R}$ are $\mu$-measurable, then*

$$f + g,\ fg,\ |f|,\ \min(f, g),\ \max(f, g)$$

*are $\mu$-measurable, and also $\dfrac{f}{g}$ is, provided $g \neq 0$ in $X$.*

- *If $(f_i)_i$ are $\mu$-measurable, then*

$$\inf_{i\geq 1} f_i, \quad \sup_{i\geq 1} f_i, \quad \liminf_{i\to\infty} f_i, \quad \limsup_{i\to\infty} f_i$$

*are also $\mu$-measurable.*

Now we present some theorems concerning the approximation of functions:

**Theorem 2.3.3.** *Let $f : X \to \overline{R}$ be an arbitrary function, and $\mu$ a measure on $X$. Then*

- *there exists a sequence of simple function $(f_i)_i$ such that $f_i(x) \to f(x)$ for all $x \in X$*
- *if $f$ is non negative, then the sequence can be chosen such that $0 \leq f_1 \leq f_2 \leq \cdots \leq f$*
- *if $f$ is bounded, then the sequence can be chosen such that $f_i \to f$ uniformly on $X$*
- *if $f$ is $\mu$-measurable, then the functions $f_i$ can be chosen $\mu$-measurable*

Next theorem is important because it says that a measurable function is continous, in the relative topology, on a closed set whose complementary has arbitrary small measure.

**Theorem 2.3.4 (Lusin's theorem).** *Let $\mu$ be a Borel measure on a metric space $X$, and $A \subset X$ such that $\mu(A) < \infty$. Let $f : A \to \mathbb{R}^n$ be a $\mu$-measurable fuction. Then, for every $\epsilon > 0$ there exists a closed set $F \subset A$ such that*

- $\mu(A \setminus F) < \epsilon$
- $f_{|_F}$ *is continous (in the relative topolgy!)*

**Note:** if the measure $\mu$ is a Radon measure, the set $K$ can be taken compact.

Now we introduce some notions of convergence for measurable functions

**Definition 2.3.5.** *Let $\mu$ be a measure on a space $X$, and let $(f_i)_i, f$ be $\mu$-measurable functions on $X$. We say that*

- *$f_i$ **converge pointwise almost everywhere** to $f$ if*

$$\lim_{i\to\infty} f_i(x) = f(x)$$

*for $\mu$-almost every $x \in X$.*

- $f_i$ ***converges almost uniformly*** *to* $f$ *if* $f_i$ *and* $f$ *are finite almost everywhere, and for each* $\varepsilon > 0$ *there exists a set* $A \subset X$ *such that* $\mu(X - A) < \varepsilon$ *and* $f_i$ *converge uniformly to* $f$ *on* $A$

- $f_i$ ***converge in measure*** *on* $f$ *if for every* $\varepsilon$

$$\lim_{i\to\infty} \mu\{\, x \in X \mid |f_i(x) - f(x)| \geq \varepsilon \,\} = 0$$

An important theorem that links two of this notions of convergence is the following

**Theorem 2.3.6 (Egoroff's theorem).** *Let* $\mu$ *be a measure on a space* $X$*,* $A \subset X$ *such that* $\mu(A) < \infty$*. Let* $(f_i)_i, f : A \to \mathbb{R}^n$ *be* $\mu$*-measurable functions such that* $f_i \to f$ $\mu$*-almost everywhere. Then for each* $\epsilon > 0$ *there exists a* $\mu$*-measurable set* $B \subset A$ *such that*

- $\mu(A \setminus B) < \epsilon$
- $f_i \to f$ *unifomly on* $B$

**Note:** if $X$ is a metric space, and $\mu(X) < \infty$, then the set $B$ can be taken closed.

Let $\mu$ be a measure on $X$ and let $(f_i)_i$ be a sequence of $\mu$-measurable funcions on $X$. Then the following implications hold:

- if $f_i \to f$ a.e. then
  - if $\mu(X) < \infty$ and $f_i, f$ are finite a.e., then $f_i \to f$ in measure
  - if $\mu(X) < \infty$ then $f_i \to f$ almost uniformly
- if $f_i \to f$ almost uniformly, then $f_i \to f$ a.e. and $f_i \to f$ in measure
- if $f_i \to f$ in measure, then
  - if $\mu(X) < \infty$ then there exists a subsequence $(f_{i_j})_j$ such that $f_{i_j} \to f$ almost uniformly
  - then there exists a subsequence $(f_{i_j})_j$ such that $f_{i_j} \to f$ a.e.

## 2.4 Integrals and limit theorems

In this section we introduce the concept of integral with respect to a measure, and present some important theorems related to the continuity of the integral operator.

**Definition 2.4.1.** *We say that a function $f : X \to \mathbb{R}$ is a* ***simple function*** *(briefly* s.f.*) if the range of $f$ is a countable subset of $\mathbb{R}$.*

**Definition 2.4.2.** *Let $\mu$ be a measure on $X$. We define the integral operator with respect to $\mu$ in three steps:*

- *let $f : X \to [-\infty, \infty]$ be a nonnegative and $\mu$-measurable s.f.; we define*

$$\int_X f \, \mathrm{d}\mu := \sum_{i=1}^{\infty} a_i \mu(f^{-1}\{a_i\})$$

  *where $f(X) = (a_i)_i$.*

- *let $f : X \to [-\infty, \infty]$ be a $\mu$-measurable s.f.; we define*

$$\int_X f \, \mathrm{d}\mu := \int_X f^+ \, \mathrm{d}\mu - \int_X f^- \, \mathrm{d}\mu$$

  *We say that $f$ is* ***$\mu$-integrable*** *if either $\int_X f^+ \, \mathrm{d}\mu < \infty$ or $\int_X f^- \, \mathrm{d}\mu < \infty$.*

- *let $f : X \to [-\infty, \infty]$. We define the upper integral as*

$$\int_X^* f \, \mathrm{d}\mu := \inf \left\{ \int_X g \, \mathrm{d}\mu \mid g \ \mu\text{−integrable s.f. such that } g \geq f \ \mu\text{−a.e.} \right\}$$

  *and the lower integral as*

$$\int_{*X} f \, \mathrm{d}\mu := \sup \left\{ \int_X g \, \mathrm{d}\mu \mid g \ \mu\text{−integrable s.f. such that } g \leq f \ \mu\text{−a.e.} \right\}$$

**Definition 2.4.3.** *We say that a $\mu$-measurable function $f$ is* ***$\mu$-integrable*** *if*

$$\int_{*X} f \, \mathrm{d}\mu = \int_X^* f \, \mathrm{d}\mu$$

*In this case the common value is denoted by*

$$\int_X f \, \mathrm{d}\mu$$

*If $E \subset X$ is a $\mu$-measurable set and $f$ is a $\mu$-integrable function; we define*

$$\int_E f \,\mathrm{d}\mu := \int_X \chi_E f \,\mathrm{d}\mu$$

*A function $f : X \to [-\infty, \infty]$ is called $\mu$-**summable** if it is $\mu$-integrable and*

$$\int_X |f| \,\mathrm{d}\mu < \infty$$

*A function $f : X \to [-\infty, \infty]$ is called **locally** $\mu$-**summable** if it is $\mu$-integrable and*

$$\int_K |f| \,\mathrm{d}\mu < \infty$$

*for each compact $K \subset X$.*

Now we want to ask to this question: if the functions $(f_i)_i$ converge to a function $f$ in some sense, wath can we say about $\int_X f_i \,\mathrm{d}\mu$ and $\int_X f \,\mathrm{d}\mu$? Next three theorems will answer to this important question.

**Theorem 2.4.4 (Fatou's lemma).** *Let $f_i : X \to [0, \infty]$ be $\mu$-measurable functions. Then*

$$\int_X \liminf_{i\to\infty} f_i \,\mathrm{d}\mu \;\le\; \liminf_{i\to\infty} \int_X f_i \,\mathrm{d}\mu$$

**Theorem 2.4.5 (Monotone convergence theorem - Beppo Levi).** *Let $f_i : X \to [0, \infty]$ be $\mu$-measurable functions such that $f_1 \le f_{2\le\ldots}$. Then*

$$\int_X \lim_{i\to\infty} f_i \,\mathrm{d}\mu \;=\; \lim_{i\to\infty} \int_X f_i \,\mathrm{d}\mu$$

**Theorem 2.4.6 (Dominated convergence theorem - Lebesgue).** *Let $g$ be $\mu$-summable, $f, (f_i)_i$ be $\mu$-measurable, and suppose $|f_i| \le g$ and $f_i \to f$ $\mu$-a.e. . Then*

$$\lim_{i\to\infty} \int_X |f - f_i| \,\mathrm{d}\mu = 0$$

**Remark 2.4.7.** *The converse of this last theorem holds if we pass to a suitable subsequence.*

## 2.5 Vector valued measures

In the first section we have introduced the notion of (outer) measure, that is a $\sigma$-additive function from a $\sigma$-algebra of sets of a space $X$ to $[0, \infty]$. In this section we introduce the notion of vector valued measures, that allows us to work with measures having values in $\mathbb{R}^n$.

**Definition 2.5.1.** *Let $X$ be a set, and let $\mathcal{M}$ be a $\sigma$-algebra of sets of $X$. We say that a function $\mu : \mathcal{M} \to \mathbb{R}^p$ is a **measure** if*

$$\mu(B) = \sum_{i=0}^{\infty} \mu(B_i)$$

*for each partition $B = \bigcup_{i=0}^{\infty} B_i$ where $B_i \in \mathcal{M}$. Sometimes we will omitt the reference to the $\sigma$-algebra $\mathcal{M}$.*

We note that the condition on the partition of $B$ tells us that every sum of the form

$$\sum_{i=0}^{\infty} \mu(B_i), \qquad (B_i)_i \subset \mathcal{M}$$

is absolutely convergent.

**Definition 2.5.2.** *Let $X$ be a set. Let $\mu$ be a non-negative measure on $X$ and $\lambda$ be a vector valued measure on $X$ defined on the same $\sigma$-algebra $\mathcal{M}$. We say that $\lambda$ is **absolutely continous with respect to** $\mu$,written $\lambda \ll \mu$, if for every set $E \in \mathcal{M}$*

$$\mu(E) = 0 \Rightarrow |\lambda(E)| = 0$$

*If $\mu$ and $\lambda$ are vector valued measures, we say that $\mu$ and $\lambda$ are **mutually singular**, written $\lambda \perp \mu$, if there exists a set $B \in \mathcal{M}$ such that $|\mu|(B) = 0$ and $|\lambda|(X \setminus B) = 0$.*

First of all we begin by studying a special case

**Definition 2.5.3.** *A measure $\mu : \mathcal{M} \to \mathbb{R}$ is called **signed measure**.*

If $\mu$ is a positive signed measure, then $\mu$ is a finite measure in the sense of Definition 2.1.1. Moreover a deep connection holds between signed measures and positive measures.

**Theorem 2.5.4** (**Jordan decomposition Theorem**). *Let $\mu$ be a signed measure on a $\sigma$-algebra $\mathcal{M}$. Then there exists a unique pair of mutually singular positive measures $\mu^+, \mu^- : \mathcal{M} \to [0, \infty)$ such that*

$$\mu = \mu^+ - \mu^-$$

Now we want to find out if there is also a connection between vector valued measures $\mu : \mathcal{M} \to \mathbb{R}^p$ with $p > 1$, and finite positive measures. To do this we need some definitions and some results.

**Definition 2.5.5.** *Let $\mu : \mathcal{M} \to \mathbb{R}^p$ be a vector valued measure. The **(total) variarion** $|\mu|$ of the measure $\mu$ is the function $|\mu| : \mathcal{M} \to [0, \infty]$ defined by*

$$|\mu|(B) := \sup\Big\{ \sum_{i=0}^{\infty} |\mu(B_i)| \mid B = \bigcup_{i=0}^{\infty} B_i,\ B_i \in \mathcal{M},\ B_i \textit{ disjoints} \Big\}$$

It holds:

**Theorem 2.5.6.** *The function $|\mu|$ is a finite measure on $X$, $\sigma$-additive on $\mathcal{M}$. Moreover $|\mu|$ is the smallest measure $\nu$ such that $|\mu(B)| \leq \nu(B)$ for each $B \in \mathcal{M}$.*

*Proof.* First of all we prove the $\sigma$-subadditivity of $|\mu|$ on $\mathcal{M}$: let $(E_k)_k \subset \mathcal{M}$ and $E \in \mathcal{M}$ such that $E \subset \bigcup_{k=1}^{\infty} E_k$. Set $E'_1 := E_1$ and for each $k > 1$ define the set $E'_k := E_k \setminus \bigcup_{h=0}^{k-1} E_h$. Let $(F_j)_j$ be a countable partition of $E$. Then for each $j$ we have that $(E'_k \cap F_j)_k$ is a countable partition of $F_j$. Hence

$$\begin{aligned}
\sum_{j=1}^{\infty} |\mu(F_j)| &= \sum_{j=1}^{\infty} \Big| \sum_{h=1}^{\infty} \mu(E'_h \cap F_j) \Big| \\
&\leq \sum_{h=1}^{\infty} \sum_{j=1}^{\infty} |\mu(E'_h \cap F_j)| \leq \sum_{h=1}^{\infty} |\mu|(E'_h)
\end{aligned}$$

Since the partition $(F_j)_j$ is arbitrary we conclude that $|\mu|$ is $\sigma$-subsdditive. To prove the superadditivity reason as follows: fix $\varepsilon > 0$; let $(E_h)_h \subset \mathcal{M}$ be a partition of a set $E \in \mathcal{M}$, and for each $h$ let $(F_k^h)_k \subset \mathcal{M}$ be a partition of the set $E_h$ such that

$$|\mu|(E_h) \leq \sum_{k=1}^{\infty} |\mu(F_k^h)| + \frac{\varepsilon}{2^j}$$

Then we have that

$$\sum_{h=1}^{\infty} |\mu|(E_h) \leq \sum_{h,k=1}^{\infty} |\mu(F_k^h)| + \varepsilon \leq |\mu|\Big( \bigcup_{k=1}^{\infty} E_k \Big)$$

Since $\varepsilon$ is arbitrary, we can conclude that $|\mu|$ is $\sigma$-additive on $\mathcal{M}$.

Now we prove that $|\mu|(X) < \infty$. If for absurd $|\mu|(X) = \infty$, let $(X_h)_h \subset \mathcal{M}$ be a partition of $X$ and let $n$ be an integer such that

$$\sum_{h=1}^{n} |\mu(X_h)| > 2(|\mu(X)| + 1)$$

Hence there exists a set $E$ such that $|\mu(E)| > |\mu(X)| + 1$. Let $F := X \setminus E$; hence

$$|\mu(F)| = |\mu(X) - \mu(E)| \geq |\mu(X)| - \mu(E) > 1$$

Now, since $|\mu|$ is additive, we have that $|\mu|(E) = \infty$ or $|\mu|(F) = \infty$; suppose $|\mu|(F) = \infty$, and set $E_1 := E$. Now we can repeat the above argument to $F$, and find a partition of $F$ in two sets $E_2$ and $F_1$ such that $|\mu(E_2)| > 1$ and $|\mu|(F_1) = \infty$. Iterating this process we find a sequence of sets $(E_j)_j \subset \mathcal{M}$ such that $|\mu(E_j)| > 1$ for each $j$. But this imply that the series $\sum_{j=1}^{\infty} \mu(E_j)$ is not convergent. But this is a contraddiction since $\mu$ is a measure. Hence $|\mu|(X) < \infty$.

For the last assertion: let $\nu$ be a positive measure such that $|\mu(B)| \leq \nu(B)$ for each $B \in \mathcal{M}$. Fix $B \in \mathcal{M}$, and let $(B_i)_i \subset \mathcal{M}$ be a partition of $B$; then

$$\sum_{i=0}^{\infty} |\mu(B_i)| \leq \sum_{i=0}^{\infty} \nu(B_i) = \nu(B)$$

Hence $|\mu|(B) \leq \nu(B)$ for each $B \in \mathcal{M}$. □

The connection between a signed measure and its total variation is the following one

**Theorem 2.5.7.** *Let $\mu$ be a signed measure on $X$. Then if $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of $\mu$, it holds*

$$|\mu| = \mu^+ + \mu^-$$

*Hence*

$$\mu^+ = \frac{|\mu| + \mu}{2}, \qquad \mu^- = \frac{|\mu| - \mu}{2}$$

The variation measure $|\mu|$ allows us to define a notion of support of a vector valued measure

**Definition 2.5.8.** *Let $\mu$ be a measure on the Borel $\sigma$-algebra of a metric space $X$; we define the* ***support*** *of the measure $\mu$,* $\operatorname{supp}(\mu)$ *as the minimal closed set $C \subset X$ such that $|\mu|(X \setminus C) = 0$.*

In general, it is **not true** that

$$\mu(X \setminus \operatorname{supp}(\mu)) = 0$$

To have this result we need that $X$ is separable. Moreover, in this case, it holds that

$$\operatorname{supp}(\mu) = \{x \in U \mid |\mu|(B_r(x)) > 0 \text{ for each ball } B_r(x) \subset U\}$$

Now we want to define some integral with respect to a vector valued measure

**Definition 2.5.9.** *Let $\mu : X \to \mathbb{R}^n$ be a vector valued mesure on $X$, and let $f : X \to \overline{\mathbb{R}}$ be a $|\mu|$-measurable function. We define*

$$\int_X f \,\mathrm{d}\mu := \Big( \int_X f \,\mathrm{d}\mu_1, \ldots, \int_X f \,\mathrm{d}\mu_n \Big)$$

*Let $\mu$ be a positive measure on $X$, and let $f : X \to \mathbb{R}^n$ be a $|\mu|$-measurable function. We define*

$$\int_X f \,\mathrm{d}\mu := \Big( \int_X f_1 \,\mathrm{d}\mu, \ldots, \int_X f_n \,\mathrm{d}\mu \Big)$$

*Let $\mu : X \to \mathbb{R}^n$ be a vector valued mesure on $X$ and let $f : X \to \mathbb{R}^n$ be a $|\mu|$-measurable function. We define*

$$\int_X f \cdot \,\mathrm{d}\mu := \sum_{i=1}^{n} \int_X f_i \,\mathrm{d}\mu_i$$

*If $E \subset X$ is a $\mu$-measurable set, we define*

$$\int_E f \,\mathrm{d}\mu := \int_X \chi_E f \,\mathrm{d}\mu$$

*for $\mu$ and $f$ as in both cases above.*

Now we show how, given a measure, generate a lots of measures.

**Definition 2.5.10.** *Let $\mu$ be a non-negative measure and $f \in L^1(X, \mu; \mathbb{R}^p)$. Define the measure $f\mu$ as follows*

$$(f\mu)(E) := \int_E f \,\mathrm{d}\mu = \Big( \int_E f_1 \,\mathrm{d}\mu, \ldots, \int_E f_p \,\mathrm{d}\mu \Big)$$

For the variation of this kind of measures it holds

**Theorem 2.5.11.** *Let $\mu$ be a non-negative measure and $f \in L^1(X, \mu; \mathbb{R}^p)$. Then*

$$|f\mu| = |f|\mu$$

*Proof.* For the inequality $|f\mu| \leq |f|\mu$: let $B \in \mathcal{M}$, and let $(B_i)_i \subset \mathcal{M}$ be a partition of $B$. Then

$$\sum_{i=0}^{\infty} \Big| \int_{B_i} f \, \mathrm{d}\mu \Big| \leq \sum_{i=0}^{\infty} \int_{B_i} |f| \, \mathrm{d}\mu = \int_B |f| \, \mathrm{d}\mu$$

Hence $|f\mu| \leq |f|\mu$.
For the other one let $D := (x_n)_n$ be a dense subset of $B_1 \subset \mathbb{R}^p$. Fix $B \in \mathcal{M}$, and for each $\varepsilon > 0$ define

$$\sigma(x) := \min\{\, n \mid \langle f(x), x_n \rangle \geq (1 - \varepsilon)|f(x)| \,\}$$

and let $B_n := \sigma^{-1}(\{n\}) \cap B$. Then

$$\begin{aligned} (1-\varepsilon) \int_B |f| \, \mathrm{d}\mu &= \sum_{n=0}^{\infty} (1-\varepsilon) \int_{B_n} |f| \, \mathrm{d}\mu \\ &\leq \sum_{n=0}^{\infty} \langle (f\mu)(B_n), x_n \rangle \leq \sum_{n=0}^{\infty} |(f\mu)(B_n)| \leq |f\mu|(B) \end{aligned}$$

Since $\varepsilon$ is arbitrary we obtain the desired result. □

It is clear that $(f\mu) \ll \mu$. Next theorem says that every measure $\nu \ll \mu$ can be express as above. We state two version of the theorem, corresponding to the two notions of measures we have introduced.

**Theorem 2.5.12.** ***(Radon-Nikodym Theorem - version I)***
*Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space, and let $\nu$ be a $\sigma$-finite signed measure on $\mathcal{M}$ that is absolutely continous with respect to $\mu$. Then there exists a measurable function $f$ such that either $f^+$ or $f^-$ is integrable and*

$$\nu(E) = \int_E f d\mu$$

*for each $E \in \mathcal{M}$.*

**Theorem 2.5.13.** ***(Radon-Nikodym Theorem - version II)***
*Let $\lambda$ be a vector valued measure on $X$, and let $\mu$ be a non negative scalar measure on $X$, such that $\lambda \ll \mu$. Then there exists a function $f \in L^1(X, \mu; \mathbb{R}^p)$ such that*

$$\lambda = f\mu$$

An important decomposition of a measure is the following one:

**Theorem 2.5.14.** ***(Lebesgue decomposition Theorem - version I)***
*Let $\mu$ and $\nu$ be two $\sigma$-finite measures defined on the measure space $(X, \mathcal{M})$. Then there exists a decomposition of $\nu$*

$$\nu = \nu_0 + \nu_1$$

*such that $\nu_0 \ll \mu$ and $\nu_1 \ll \mu$. The measures $\nu_0$ and $\nu_1$ are unique.*

**Theorem 2.5.15.** ***(Lebesgue decomposition Theorem - version II)***
*Let $\nu$ be a vector valued measures defined on $X$ and $\mu$ be a non negative scalar measure on $X$. Then there exists a function $f \in L^1(X, \mu; \mathbb{R}^p)$ and a vector valued function $\nu_s$ such that*

$$\nu = f\mu + \nu_s \qquad \textit{with } \nu_s \perp \mu$$

*The measures $f\mu$ and $\nu_s$ are unique.*

**Note**: Thorem 2.5.12 and Theorem 2.5.13 say that if $\lambda \ll \mu$ we can express a measure $\lambda$ in terms of $\mu$ integrating a function $f$, that can be seen as the "density" of $\lambda$ with respect to $\mu$. The problem is obviosly to calculate this such $f$. We will see in Section 2.7 and in chapter some cases in wich we can identify the function $f$ with another function, that can be computed.

Now we can state the connection between a vector valued measure and its variation. Let $\mu : \mathcal{B} \to \mathbb{R}^p$ be a vector-valued measure; then

$$\mu = (\mu_1, \ldots, \mu_p)$$

The problem is that we have $p$ different measures to manage. In order to solve this problem we reasone as follows: from Theorem 2.5.15 we can find a function $\sigma \in L^1(X, |\mu|; \mathbb{R}^p)$ such that

$$\mu = \sigma|\mu|$$

since if a measure $\nu$ is singular with respect to $|\mu|$, then $\nu$ is also singular with respect to $\mu$. Moreover from Teorem 2.5.11 we obtain that $|\sigma| = 1$ $|\mu|$-a.e..
Hence if $f = (f_1, \ldots, f_p)$ is a $\mu$-measurable function we have that

$$\int_X f \,\mathrm{d}\mu = \int_E \langle f, \sigma\rangle \,\mathrm{d}|\mu|$$

## 2.6 Covering theorems

The aim of this section is the following one: suppose to have a covering $\mathcal{F}$ made by closed balls of a set $A \subset \mathbb{R}^n$; we want to estimate the measure of $A$ using the measure of the covering. Since a mesure is $\sigma$-additive of countable disjoint mesurable sets, we would obtain a countable subfamily $\mathcal{G} \subset \mathcal{F}$ of disjoint sets that, in some sense, provide a covering of $A$. There are two principal ways to do this: construct $\mathcal{G}$ in such a way that $A$ is covered by an enlargment of balls in $\mathcal{G}$, or construct a finite number of disjoint countable subfamilies $\mathcal{G}_1, \dots, \mathcal{G}_k$, where $k$ depends only on the dimension $n$, whose union cover $A$. First way is proved in the Vitali covering Theorem (Theorem 2.6.1) but it required that the measure is (sub)-homogemeous. The other way is proved in the Besicovitch covering Theorem (Theorem 2.6.6): since we do not enlarge balls, we do not required any homogeneous property for the measure, but we can use it only if the set $A$ is the set of the centers of the balls in $\mathcal{F}$.

### 2.6.1 Vitali's covering Theorem

**Theorem 2.6.1.** ***(Vitali's covering theorem)*** *Let $\mathcal{F}$ Be any collection of nondegenerate closed balls in $\mathbb{R}^n$ with*

$$\sup\{\mathrm{diam}(B) \mid B \in \mathcal{F}\} < \infty$$

*Then there exists a countable subfamily $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} \widehat{B}$$

*where with $\widehat{B}$ we denote the closed ball with radius 5 times the radius of $B$.*

*Proof.* Let $D : \sup\{\,\mathrm{diam}(B) \mid B \in \mathcal{F}\,\}$, and set, for each $j$

$$\mathcal{F}_j := \left\{ B \in \mathcal{F} \mid \frac{D}{2^j} < \mathrm{diam}(B) \leq \frac{D}{2^{j-1}} \right\}$$

We define a sequence of subfamily $\mathcal{G}_j \subset \mathcal{F}_j$ as follows:

- let $\mathcal{G}_1$ be a maximal countable[1] collection of pairwise disjoint balls in $\mathcal{F}_1$
- let $\mathcal{G}_1, \dots, \mathcal{G}_{k-1}$ have been choosen, and let $\mathcal{G}_k$ be a maximal pairwise disjoint collection of

$$\left\{ B \in \mathcal{F}_k \mid B \cap B' = \emptyset \ \ \forall B' \in \bigcup_{i=1}^{k-1} \mathcal{G}_i \right\}$$

[1] It is possible because $\mathbb{R}^n$ is separable.

- define $\mathcal{G} := \bigcup_{k=1}^{\infty} \mathcal{G}_k$

We have that $\mathcal{G}$ is a collection of pairwise disjoint balls in $\mathcal{F}$. We need to prove that the enlargement of the ball in $\mathcal{G}$ are a covering of $\mathcal{F}$: let $B \in \mathcal{F}$; then there exists and index $j$ such that $B \in \mathcal{F}_j$. By the maximality of $\mathcal{G}_j$ there exists a ball $B' \in \bigcup_{i=1}^{j} \mathcal{G}_i$ such that $B \cap B' \neq \emptyset$. But $\operatorname{diam}(B') \geq \frac{D}{2^j}$ and $\operatorname{diam}(B) \leq \frac{D}{2^{j-1}}$. So we obtain that $\operatorname{diam}(B) \leq \operatorname{diam}(B')$, and thus $B \subset \widehat{B'}$. $\square$

**Definition 2.6.2.** *We say that a covering $\mathcal{F}$ of a set $A$ is a* ***fine*** *covering of $A$, if for each $x \in A$*

$$\inf\{\operatorname{diam}(B) \mid x \in B, B \in \mathcal{F}\} = 0$$

**Remark 2.6.3.** *We note that the hypothesy* $\sup\{\operatorname{diam}(B) \mid B \in \mathcal{F}\} < \infty$ *is necessary. In fact the thesis of the theorem above is not true for the family of balls $\mathcal{F} := (B_i(0))_{i \in \mathbb{N}}$.*

A thecnical consequence of this theorem is this the following

**Corollary 2.6.4.** *Let $\mathcal{F}$ be a fine covering of $A$ by closed balls. Then there exists a countable family of pairwise disjoint balls in $\mathcal{F}$ such that for each finite subset $\{B_1, \dots, B_m\} \subset \mathcal{F}$ we have*

$$A \setminus \bigcup_{i=1}^{m} B_i \subset \bigcup_{B \in \mathcal{G} \setminus \{B_1, \dots, B_m\}} B$$

*Proof.* Let $\mathcal{G}$ be the family obtained by the Vitali's covering theorem, and select $\{B_1, \dots, B_m\} \subset \mathcal{F}$. If $A \subset \bigcup_{i=1}^{m} B_i$ we have finish. Otherwise, let $x \in A \setminus \bigcup_{i=1}^{m} B_i$; since the balls are closed and $\mathcal{F}$ is a fine cover of $A$, there exists $B \in \mathcal{F}$ with $x \in B$ and $B \cap B_k = \emptyset$ for each $k = 1, \dots, m$. By the construction of the family $\mathcal{G}$ we see that there exists a ball $B'$ such that $B \cap B' \neq \emptyset$ and $B \subset \widehat{B'}$. Since $B \cap B' \neq \emptyset$, $B' \notin \{B_1, \dots, B_m\}$, and so we have done. $\square$

An important consequence of the Vitali's covering theorem, usefull for the result concerning the Lebesgue measure[2], is the following

**Corollary 2.6.5.** *Let $U \subset \mathbb{R}^n$ be an open set, and let $\delta > 0$. Then there exists a countable collection $\mathcal{G}$ of pairwise disjoint closed balls in $U$ such that* $\operatorname{diam}(B) \leq \delta$ *for each $B \in \mathcal{G}$ and*

$$\mathcal{L}^n\Big(U \setminus \bigcup_{B \in \mathcal{G}} B\Big) = 0$$

[2]In the following section we will derive a Radon measure $\nu$ with respect another Radon mesure $\mu$; if $\mu$ is the Lebesgue measure we can apply this Corollary instead of Corollary 2.6.8.

**Note:** it is important that the balls are **countable** and **disjoint** in order to apply the $\sigma$-additivity of a Borel measure, and that they are **in** $U$, in order to apply some property of balls that are in $U$.

*Proof.* Suppose first $\mathcal{L}^n(U) < \infty$, and fix $1 - \frac{1}{5^n} < \theta < 1$. We will define the family $\mathcal{G}$ by induction. Let $\mathcal{F}_1 := \{ B \mid B \in U,\ \operatorname{diam}(B) < \delta \}$; by Vitali's covering theorem we obtain a countable family of pairwise disjoint balls $\mathcal{G}_1 \subset \mathcal{F}_1$ such that

$$U \subset \bigcup_{B \in \mathcal{G}_1} \widehat{B}$$

where we reball that, since $U$ is open, $U = \bigcup_{B \in \mathcal{F}_1} B$.
Thus

$$\mathcal{L}^n(U) \leq \sum_{B \in \mathcal{G}_1} \mathcal{L}^n(\widehat{B}) = 5^n \sum_{B \in \mathcal{G}_1} \mathcal{L}^n(B) = 5^n \mathcal{L}^n\Big( \bigcup_{B \in \mathcal{G}_1} B \Big)$$

where in the last equality we have taken into account that the balls in $\mathcal{G}_1$ are pairwise disjoint.
Hence

$$\mathcal{L}^n\Big( \bigcup_{B \in \mathcal{G}_1} B \Big) \geq \frac{1}{5^n} \mathcal{L}^n(U)$$

Since $\bigcup_{B \in \mathcal{G}_1} B$ is measurable, we obtain that

$$\mathcal{L}^n\Big( U \setminus \bigcup_{B \in \mathcal{G}_1} B \Big) \leq \Big( 1 - \frac{1}{5^n} \Big) \mathcal{L}^n(U)$$

Since $\mathcal{L}^n(U) < \infty$ and $(U \setminus \bigcup_{k=1}^{i} B_k)_i$ is a decreasing sequence ($\mathcal{G}_1$ is countable) of measurable sets, we have that there exist disjoint balls $B_1, \dots, B_{M_1}$ in $\mathcal{G}_1$ such that

$$\mathcal{L}^n\Big( U \setminus \bigcup_{i=1}^{M_1} B_i \Big) \leq \theta \mathcal{L}^n(U)$$

Now let

$$U_2 := U \setminus \bigcup_{i=1}^{M_1} B_i$$

$$\mathcal{F}_2 := \{ B \mid B \in U_2, \operatorname{diam}(B) \leq \delta \}$$

Since $U_2$ is open, reasoning as above, we can find pairwise disjoint balls $B_{M_1+1}, \dots, B_{M_2}$ in $\mathcal{F}_2$ such that

$$\mathcal{L}^n\big( U \setminus \bigcup_{i=1}^{M_2} B_i \big) = \mathcal{L}^n\Big( U_2 \setminus \bigcup_{i=M_1+1}^{M_2} B_i \Big) \leq \theta \mathcal{L}^n(U_2) \leq \theta^2 \mathcal{L}^n(U)$$

Continue this process, we obtain for each $k$ pairwise disjoint balls $B_{M_{k-1}+1}, \dots, B_{M_k}$ in $\mathcal{F}_k$ such that

$$\mathcal{L}^n\Big(U \setminus \bigcup_{i=1}^{M_k} B_i\Big) \le \theta^k \mathcal{L}^n(U)$$

Since $\mathcal{L}^n(U) < \infty$ and $\theta < 1$, and so $\theta^k \to 0$, we obtain the desired result.
In the case $\mathcal{L}^n(U) = \infty$ we apply the above reasoning to the sets

$$U_m := \{\, x \in U \mid m < |x| < m+1 \,\}, \quad m \in \mathbb{N}$$

and since $\mathcal{L}^n(\partial B_m(0)) = 0$ for each $m \in \mathbb{N}$, we have done. □

### 2.6.2 Besicovitch's covering theorem

The fact that the Lebesgue measure is homogenous is foundamental for the validity of the above theorems. Now we want to obtain similar result for an arbitrary Radon measure, that not need to be homogenous. So, we need to find a new covering of the original one, without enlarging balls.

**Theorem 2.6.6** (**Besicovitch's covering theorem**). *There exists a integer $N_n$, depending only on $n$, with the following property:*

*let $\mathcal{F}$ be a family of nondegenerated closed balls in $\mathbb{R}^n$, and let $A$ be the set of the center of the balls in $\mathcal{F}$; suppose $A$ is bounded.*
*Then there exists $\mathcal{G}_1, \dots, \mathcal{G}_{N_n} \subset \mathcal{F}$ such that each $\mathcal{G}_i$ is a countable collection of pairwise disjoint balls in $\mathcal{F}$, and*

$$A \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B$$

*Proof.* First of all we note that if $\sup\{\, \mathrm{diam}(B) \mid B \in \mathcal{F} \,\} = \infty$, then we can easly prove the theorem just taking a ball $B \in \mathcal{F}$ such that $\mathrm{diam}(B) > \frac{\mathrm{diam}(A)}{2}$; this is possible because $A$ is bounded. So we can take as family $\mathcal{G}_1 := \{B\}$ and as the families $\mathcal{G}_2, \dots, \mathcal{G}_{N(n)}$ the empty family. These families clearly satisfied the thesis of the theorem.
So we can suppose $D := \sup\{\, \mathrm{diam}(B) \mid B \in \mathcal{F} \,\} < \infty$. We proced by steps:

**Step 1:** we start by defining a countable family of balls in $\mathcal{F}$ that we will use later to define the required families of balls.
Define inductively $B_i$ as follows:

- let $B_1 = B_{r_1}(a_1) \in \mathcal{F}$ such that $r_1 \ge \frac{3}{4}(\frac{D}{2})$
- for $j \ge 2$ let $A_j := A \setminus \bigcup_{i=1}^{j-1} B_i$; if $A_j = \emptyset$ we $J := j-1$ and we stop. If $A_j \neq \emptyset$, we choose $B_j = B_{r_j}(a_j)$ such that $a_j \in A_j$ and

$$r_j \ge \frac{3}{4} \sup\{\, r \mid B_r(a) \in \mathcal{F}, a \in A_j \,\}$$

If $A_j \neq \emptyset$ for each $j$, we set $J := \infty$.
Then the following facts hold:

- if $j > i$ then $r_j \leq \frac{4}{3} r_i$: in fact if $j > i$, then $a_j \in A_i$, and hence
$$r_i \geq \frac{3}{4} \sup\{ r \mid B_r(a) \in \mathcal{F}, a \in A_i \} \geq \frac{3}{4} r_j$$

- the balls $(B_{r_j/3}(a_j))_{j=1}^J$ are disjoint: if we take $j > i$, we have that $a_j \notin Bi$, and so
$$|a_i - a_j| > r_i = \frac{r_i}{3} + \frac{2}{3} r_i \geq \frac{r_i}{3} + \frac{2}{3}\frac{3}{4} r_j > \frac{r_i}{3} + \frac{r_j}{3}$$

- if $J = \infty$, then $r_j \to 0$: since $a_j \in A$ that is bounded and the balls $\{B_{r_j/3}(a_j)\}_{j=1}^J$ are disjoint, we must have that $r_j \to 0$

- $A \subset \bigcup_{i=1}^J B_i$: if $J < \infty$ it is trivial; otherwise, if $J = \infty$, let $a \in A$; then there exists $r > 0$ such that $B_r(a) \in \mathcal{F}$ and, for the claim above, the exists $r_j < \frac{3}{4} r$; but then $a \in \bigcup_{i=1}^{j-1} B_i$, otherwise we will have a contraddiction to the choise of $r_j$.

Now, fix $k > 1$, and define
$$I := \{ j \mid 1 \leq j \leq k, \ B_j \cap B_k \neq \emptyset \}$$
$$K := I \cap \{ j \mid r_j \leq 3 r_k \}$$

**Step 2:** we want to estimante the cardinality of $I$ for each $k > 1$.
We begin estimating the cardinality of $K$. Let $j \in K$: then $B_j \cap B_k \neq \emptyset$ and $r_j \leq 3 r_k$; let $x \in B_{r_j/3}(a_j)$, then
$$|x - a_k| \leq |x - a_j| + |a_j - a_k| \leq \frac{r_j}{3} + (r_j + r_k) \leq 5 r_k$$
so $B_{r_j/3}(a_j) \subset B_{5r_k}(a_k)$. Since the balls $(B_{r_j/3}(a_j))_{j=1}^J$ are disjoint, we have that
$$\begin{aligned} \alpha(n) 5^n r_k^n &= \mathcal{L}^n(B_{5r_k}(a_k)) \geq \sum_{j \in K} \mathcal{L}^n(B_{r_j/3}(a_j)) \\ &= \sum_{j \in K} \alpha(n) \Big( \frac{r_j}{3} \Big)^n \geq \sum_{j \in K} \alpha(n) \Big( \frac{r_k}{4} \Big)^n \\ &= \operatorname{Card}(K) \alpha(n) \frac{r_k^n}{4^n} \end{aligned}$$

So we have obtained that
$$\operatorname{Card}(K) \leq 20^n$$

Now we want to estimante the cardinality of $I \setminus K$. Let $i \neq j \in I \setminus K$: then $1 \leq i, j \leq k$ and

$$B_i \cap B_k \neq \emptyset\,, \quad r_i > 3r_k$$

$$B_j \cap B_k \neq \emptyset\,, \quad r_j > 3r_k$$

Without loss of generality, we can suppose $a_k = 0$; let $0 \leq \theta \leq \pi$ the angle between $a_i$ and $a_j$. We want to obtain a lower bound for $\theta$. We have the following facts:

- since $i, j < k$ we must have $a_k \notin B_i \cup B_j$, and so $r_i < |a_i|$ and $r_j < |a_j|$
- since $B_i \cap B_k \neq \emptyset$ and $B_j \cap B_k \neq \emptyset$ we have that $|a_i| < r_i + r_k$ and $|a_j| < r_j + r_k$

We can also suppose, without loss of generality, that $|a_i| \leq |a_j|$. In summary we have:

$$\begin{cases} 3r_k & < & r_i & < & |a_i| & \leq & r_i + r_k \\ 3r_k & < & r_i & < & |a_i| & \leq & r_i + r_k \end{cases}$$

We can suppose $i < j$; this imply $a_j \notin B_i$. Hence

$$\begin{aligned}
\cos\theta &= \frac{|a_i|^2 + |a_j|^2 - |a_i - a_j|^2}{2|a_i||a_j|} < \frac{(r_i + r_k)^2 + |a_j|^2 - r_i^2}{2r_i|a_i|} \\
&= \frac{2r_ir_j + r_k^2 + |a_j|^2}{2r_i|a_i|} = \frac{r_k}{|a_j|} + \frac{r_k^2}{2r_i|a_j|} + \frac{|a_j|}{2r_i} \\
&\leq \frac{r_k}{r_j} + \frac{r_k^2}{2r_ir_j} + \frac{r_j + r_k}{2r_i} \leq \frac{1}{3} + \frac{r_k^2}{2(3r_k)(3r_k)} + \frac{4r_k}{2(3r_k)} \\
&= \frac{13}{18} < 1
\end{aligned}$$

Hence:

$$\theta \ \geq \ \arccos\frac{13}{18} =: \theta_0$$

From the lower bound for $\theta$, we can derive an estimate for the cardinality of $I \setminus K$: let $r_0 > 0$ such that if $x \in \partial B_1(0)$, $y, z \in B_{r_0}(x)$ then the angle between $y$ and $z$ is less than $\theta_0$. Let $L_n$ such that $\partial B_1(0)$ can be covered by $L_n$ balls with center in $\partial B_1$ and radius $r_0$ but not by $L_n - 1$; this is possible because $\partial B_1(0)$ is compact. Then $\partial B_k$ can be covered by $L_n$ balls of radius $r_0r_k$ and center in $\partial B_k$. Now, if $i \neq j \in I \setminus K$, then the angle between $a_i - a_k$ and $a_j - a_k$ is more than $\theta_0$; so, by construction of $r_0$, the rays $a_i - a_k$ and $a_j - a_k$ cannot both go through the same ball on $\partial B_k$. So $\mathrm{Card}(I \setminus K) \leq L_n$.

In summary, setting $M_n := 20^n + L_n + 1$, we have that

$$\mathrm{Card}(I) < M_n$$

**Step 3:** Now we put the balls $(B_i)_{i=1}^J$ in rows, in a way that balls in the same row are disjoint. To do this we define the row index $Z(i)$ of the ball $B_i$ as follows:

- $Z(1) := 1$
- $Z(i+1) := \min\{\, j \mid B_{i+1} \bigcap B_k = \emptyset \;\; \forall k < i+1 \text{ such that } Z(k) = j \,\}$

From *Step 2* we have that $Z(i) < M_n$; so, defining the families

$$\mathcal{G}_j := \{\, B_i \mid Z(i) = j \,\}$$

for each $j = 1, \ldots, M_n$, we have that each family $\mathcal{G}_j$ consists of disjoint balls, and the families $\mathcal{G}_1, \ldots, \mathcal{G}_{M_n}$ cover $A$.

$\square$

**Remark 2.6.7.** *If in the previous theorem we have as hypothesis that $A$ general (not necessary bounded), but we suppose that* $\sup\{\, \mathrm{diam}(B) \mid B \in \mathcal{F} \,\} < \infty$ *then we can prove the same result. Reasoning as follows: for $l \geq 1$ we define*

$$A_l := A \cap \{\, x \in \mathbb{R}^n \mid 3D(l-1) \leq |x| < 3Dl \,\}$$

$$\mathcal{F}_l := \{\, B_r(a) \in \mathcal{F} \mid a \in A_l \,\}$$

*Then, for each $l \geq 1$, from* Step 3 *of the previous theorem there exists $\mathcal{G}_1^l, \ldots \mathcal{G}_{M_n}^l$ countable family of disjoint balls such that*

$$A_l \subset \bigcup_{i=1}^{M_n} \bigcup_{B \in \mathcal{G}_i^l} B$$

*So, if we define*

$$\mathcal{G}_j := \bigcup_{l=1}^{\infty} \mathcal{G}_j^{2l-1} \quad 1 \leq j \leq M_n$$

$$\mathcal{G}_{j+M_n} := \bigcup_{l=1}^{\infty} \mathcal{G}_j^{2l} \quad 1 \leq j \leq M_n$$

*and we set $L_n := 2M_n$, we have the desidered result.*

Now we present a result of the same spirit of Corollary 2.6.5.

**Corollary 2.6.8.** *Let $\mu$ be a Borel measure on $\mathbb{R}^n$, and $\mathcal{F}$ be any collection of nondegenerated closed balls. Let $A$ be the set of the center of the balls in $\mathcal{F}$; suppose $\mu(A) < \infty$ and for each $a \in A$* $\inf\{r \mid B_r(a) \in \mathcal{F}\} = 0$. *Then,* *for each open set $U \in \mathbb{R}^n$ there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that*

$$\bigcup_{B \in \mathcal{G}} B \subset U$$

*and*

$$\mu\Big( (A \cap U) \setminus \bigcup_{B \in \mathcal{G}} B \Big) = 0$$

*Proof.* Fix

$$1 - \frac{1}{N_n} < \theta < 1$$

We construct the family $\mathcal{G}$ inductively as follows:
let $\mathcal{F}_1 := \{ B \mid B \in \mathcal{F},\ \mathrm{diam}(B) \le 1,\ B \subset U \}$; this family is not empty, since $U$ is open, and $\mathcal{F}$ is a fine cover of $A$. by Theorem 2.6.6 there exist families $\mathcal{G}_1, \dots, \mathcal{G}_{N_n}$ of disjoint balls in $\mathcal{F}_1$ such that

$$A \cap U \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B$$

So:

$$\mu(A \cup U) \le \sum_{i=1}^{N_n} \mu\Big( (A \cap U)\ \cap \bigcup_{B \in \mathcal{G}_i} B \Big)$$

Then there exists an index $1 \le j \le N_n$ such that

$$\mu\Big( (A \cap U) \bigcap \bigcup_{B \in \mathcal{G}_j} B \Big) \ge \frac{1}{N_n} \mu(A \cap U)$$

Now, since $\mathcal{G}_j$ is countable and $\mu$ is a regular, there exists $M_1$ such that $B_1, \dots, B_{M_2} \in \mathcal{G}_j$ and

$$\mu\Big( (A \cap U) \bigcap \bigcup_{i=1}^{M_1} B_i \Big) \ge (1 - \theta) \mu(A \cap U)$$

Since $\bigcup_{i=1}^{M_1} B_1$ is $\mu$-measurable, we obtain that

$$\mu\Big( (A \cap U) \setminus \bigcup_{i=1}^{M_1} B_i \Big) \le \theta \mu(A \cup U)$$

Inductively we set, for $i \ge 1$

$$U_{i+1} := U \setminus \bigcup_{j=1}^{M_i} B_j$$

and

$$\mathcal{F}_{i+1} := \{ B \mid B \in \mathcal{F},\ \mathrm{diam}(B) \leq 1, B \subset U_{i+1} \}$$

Then $U_{i+1}$ is an open set, and $\mathcal{F}_{i+1}$ is a fine cover of $U_{i+1}$. Reasoning as above, we obtain disjoint balls $B_{M_i+1}, \ldots, B_{M_{i+1}} \in \mathcal{F}_{i+1}$ such that

$$\begin{aligned} \mu\Big((A \cap U) \setminus \bigcup_{j=1}^{M_{i+1}} B_j\Big) &= \mu\Big((A \cap U_{i+1}) \setminus \bigcup_{j=M_i+1}^{M_{i+1}} B_j\Big) \\ &\leq \theta\mu(A \cap U_{i+1}) \\ &\leq \theta^{i+1}\mu(A \cap U) \end{aligned}$$

Now, since $\mu(A \cap U) \leq \mu(A) < \infty$ and $\theta < 1$, we have the desidered result. □

## 2.7 Differentiation of Radon measures in $\mathbb{R}^n$

In this section we want to answer this question: do two measures that agree on balls agree everywhere? We will set the problem in $\mathbb{R}^n$, while the extension of the results of this and of the previous section will be made in chapter 4. In particular we will see that, if we take two Radon measures $\mu$, $\nu$ on $\mathbb{R}^n$ such that $\mu \ll \nu$, then we can express $\mu$ in terms of $\nu$ just integrating a function $D_\nu\mu$ with respect to $\nu$ (Theorem 2.7.4). The important fact is that the function $D_\nu\mu$ is defined as the derivate of $\mu$ with respect to $\nu$ (see Definition 2.7.1), and hence we can calculate it, not as in the case of the Radon-Nikodym Theorem (Theorems 2.5.12 and 2.5.13). In particular we can say that if two Radon measures $\mu$, $\nu$ such that $\mu \ll \nu$ agree on balls, than they agree on Borel sets.

**Definition 2.7.1.** *Let $\mu, \nu$ be two Radon measures on $\mathbb{R}^n$. For each point $x \in \mathbb{R}^n$ we define:*

$$\overline{D}_\nu\mu(x) := \begin{cases} \limsup\limits_{r \to 0} \dfrac{\mu(B_r(x))}{\nu(B_r(x))} & \text{,if } \nu(B_r(x)) > 0 \ \ \forall r > 0 \\ +\infty & \text{,if } \nu(B_r(x)) = 0 \text{ for some } r > 0 \end{cases}$$

$$\underline{D}_\nu\mu(x) := \begin{cases} \liminf\limits_{r \to 0} \dfrac{\mu(B_r(x))}{\nu(B_r(x))} & \text{,if } \nu(B_r(x)) > 0 \ \ \forall r > 0 \\ +\infty & \text{,if } \nu(B_r(x)) = 0 \text{ for some } r > 0 \end{cases}$$

*If $\overline{D}_\nu\mu(x) = \underline{D}_\nu\mu(x) < \infty$ then we say that $\mu$ is **differentiable** with respect to $\nu$, and denote by $D_\nu\mu(x)$ the common value of the limits.*

Now we want to understand when $D_\nu\mu$ exists and how we can recover $\mu$ by integrating $D_\nu\mu$.
The foundamental Lemma in this section is the following one

**Lemma 2.7.2.** *Let $\mu, \nu$ be two Radon mesure on $\mathbb{R}^n$, and let $0 < \alpha < \infty$. Define*

$$D^\infty(\mu,\nu) := \{\, x \in \mathbb{R}^n \mid \overline{D}_\nu\mu(x) = \infty \,\}, \quad D(\mu,\nu) := \mathbb{R}^n \setminus D^\infty(\mu,\nu)$$

*Then it hold*

1. $\nu(D^\infty(\mu,\nu)) = 0$
2. *for each $A \subset D(\mu,\nu)$, if $\nu(A) = 0$, then $\mu(A) = 0$*
3. *if $A \subset \{\, x \in \mathbb{R}^n \mid \underline{D}_\nu\mu(x) \le \alpha \,\}$, then $\mu(A) \le \alpha\nu(A)$*
4. *if $A \subset \{\, x \in \mathbb{R}^n \mid \overline{D}_\nu\mu(x) \ge \alpha \,\}$, then $\mu(A) \ge \alpha\nu(A)$*

*Proof.* First of all we note that if $\underline{D}_\nu\mu(x) = \infty$ then $\overline{D}_\nu\mu(x) = \infty$. Hence $D(\mu,\nu)$ is the set of points where both $\underline{D}_\nu\mu$ and $\overline{D}_\nu\mu$ are finite.
Let's prove 1: for $r > 0$ define

$$D_r^\infty := D^\infty(\mu,\nu) \cap B_r$$

Since $\mu$ is a Radon measure there exists an open set $U$ such that $D_r^\infty \subset U$ and $\mu(U) < \infty$. Now let $x \in D_r^\infty$, and for $h \in \mathbb{N}$ define

$$\mathcal{F}_x := \{\, B_r(x) \subset U \mid \nu(B_r(x)) > 0,\ \mu(B_r(x)) \ge h\nu(B_r(x)) \,\}$$

Define $\mathcal{F} := \bigcup_{x \in D_r^\infty} \mathcal{F}_x$; then $\mathcal{F}$ is a fine covering of closed balls of $D_r^\infty$. Hence for Corollary 2.6.8 there exists a countable disjoint subfamily $\mathcal{G} = \{B_i\}_i \subset \mathcal{F}$ such that[3]

$$\nu\Big( D_r^\infty \setminus \bigcup_{i=1}^{\infty} B_i \Big) = 0$$

Hence we obtain that

$$\nu(D_r^\infty) \le \nu\Big( \bigcup_{i=1}^{\infty} B_i \Big) = \sum_{i=1}^{\infty} \nu(B_i) \le \frac{1}{h} \sum_{i=1}^{\infty} \mu(B_i) = \frac{1}{h}\mu\Big( \bigcup_{i=1}^{\infty} B_i \Big) \le \frac{1}{h}\mu(U)$$

Since $\mu(U) < \infty$, letting $h \to \infty$ we obtain that $\nu(D_r^\infty) = 0$ for each $r > 0$, and hence $\nu(D^\infty)(\mu,\nu) = 0$.

Let's prove 2: for each $h \in \mathbb{N}$ set $A_h := A \cap \{\, x \in \mathbb{R}^n \mid \overline{D}_\nu\mu(x) < h \,\}$. Since $A = \bigcup_{h=1}^\infty A_h$ we have only to prove that $\mu(A_h) = 0$ for each $h$. So

[3]Note that $\nu(D_r^\infty) < \infty$, and that $U \cap D_r^\infty = D_r^\infty$.

fix $h$ and $\varepsilon > 0$; since $\nu(A_h) = 0$ and $\nu$ is a Radon measure, there exists an open set $U$ such that $A_h \subset U$ and $\nu(U) < \varepsilon$. Now for each $x \in A_h$ define

$$\mathcal{F}_x := \{\, B_r(x) \subset U \mid \nu(B_r(x)) > 0,\ \mu(B_r(x)) < h\nu(B_r(x)) \,\}$$

and $\mathcal{F} := \bigcup_{x \in A_h} \mathcal{F}_x$. Then $\mathcal{F}$ is a fine covering of $A_h$, and hence we can apply Theorem 2.6.6 to obtain countable disjoint subfamilies $\mathcal{G}_1, \dots, \mathcal{G}_{N_n} \subset \mathcal{F}$ such that

$$A_h \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B$$

Hence, if we write $\mathcal{G}_i = (B^i_j)_j$,

$$\begin{aligned} \mu(A_h) &\le \mu(U) \le \sum_{i=1}^{N_n} \sum_{j=1}^{\infty} \mu(B^i_j) \le \sum_{i=1}^{N_n} \sum_{j=1}^{\infty} h\nu(B^i_j) \\ &= \sum_{i=1}^{N_n} h\nu\Big( \bigcup_{B \in \mathcal{G}_i} B \Big) \le N_n h \nu(U) < N_n h \varepsilon \end{aligned}$$

and letting $\varepsilon \to 0$ we obtain the desired result.

Finally let's prove 3, because the proof of 4 is similar. Fix $\varepsilon > 0$, and let $U$ be an open set such that $U \supset A$; let

$$\mathcal{F} := \{\, B \mid B = B_r(a)\ a \in A,\ B \subset U,\ \mu(B) \le (\alpha + \varepsilon)\nu(B) \,\}$$

It is clear that $\mathcal{F}$ is a fine cover of $A$. So, by Corollary 2.6.8 we can find a countable family $\mathcal{G}$ of disjoint balls of $\mathcal{F}$ such that

$$\nu\Big( A \setminus \bigcup_{B \in \mathcal{G}} B \Big) = 0$$

Hence:

$$\mu(A) \le \sum_{B \in \mathcal{G}} \mu(B) \le (\alpha + \varepsilon) \sum_{B \in \mathcal{G}} \nu(B) \le (\alpha + \varepsilon)\nu(U)$$

Since $\varepsilon$ is arbitrary, we obtain

$$\mu(A) \le \alpha \nu(U)$$

for all open set $U \supset A$. Since $\nu$ is a regular measures, the estimate holds also for $A$. $\square$

**Theorem 2.7.3.** *Let $\mu, \nu$ be Radon measures on $\mathbb{R}^n$. Then $D_\nu \mu$ exists and it is finite for $\nu$-a.e. $x \in \mathbb{R}^n$. Moreover the function $x \mapsto D_\nu \mu(x)$ is $\nu$-measurable.*

*Proof.* Without loss of generality we can suppose $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$.
We want to prove that $D_\nu \mu$ exists $\nu$-a.e.. For each $a < b \in \mathbb{R}$ we define

$$R(a,b) := \{x \in \mathbb{R}^n \mid \underline{D}_\nu \mu(x) < a < b < \overline{D}_\nu \mu(x) < +\infty\}$$

Using again Lemma 2.7.2 we obtain that

$$b\nu(R(a,b)) \leq \mu(R(a,b)) \leq a\nu(R(a,b))$$

and since $\nu(R(a,b)) < \infty$ and $a < b$, we obtain that $\nu(R(a,b)) = 0$.
Now, since

$$\{x \in \mathbb{R}^n \mid \underline{D}_\nu \mu(x) < \overline{D}_\nu \mu(x)\} \; = \; \bigcup_{\substack{a<b \\ a,b\in\mathbb{Q}}} R(a,b)$$

we obtain that $D_\nu \mu$ exists and is finite $\nu$-a.e..
Now we prove that $x \mapsto D_\nu \mu(x)$ is $\nu$-measurable. Fix $x \in \mathbb{R}^n$, and let $(y_k)_k$ be a sequence of points converging to $x$ such that $B_r(y_k) \subset B_{2r}(x)$. Set $f_k := \chi_{B_r(y_k)}$ and $f := \chi_{B_r(x)}$; since the balls are closed, we have that $\limsup_k f_k \leq f$. Hence

$$\liminf_k (1 - f_k) \geq 1 - f \geq 0$$

and by the Fatou's Lemma

$$\mu(B_{2r}(x)) - \mu(B_r(x)) \leq \mu(B_{2r}(x)) - \limsup_k \mu(B_r(y_k))$$

that is the function $x \mapsto \mu(B_r(x))$ is upper semicontinous, and hence Borel measurable. A similar assertion holds for the function $x \mapsto \nu(B_r(x))$.
So, fixed $r > 0$, we have that the function

$$f_r(x) := \begin{cases} \dfrac{\nu(B_r(x))}{\nu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \\ +\infty & \text{if } \mu(B_r(x)) = 0 \end{cases}$$

is $\mu$-measurable. Since

$$D_\mu \nu = \lim_{r\to 0} f_r \qquad \mu - \text{a.e.}$$

we obtain that $D_\mu \nu$ is $\nu$-measurable. $\square$

Next theorem is the Foundamental Theorem of Calculus for Radon measures on $\mathbb{R}^n$, which states that if $\mu$ and $\nu$ are Radon measures on $\mathbb{R}^n$, than $\nu$ has densitiy with respect to $\mu$, and this density can be computed "differentiating" $\nu$ with respect to $\mu$.

**Theorem 2.7.4.** *Let $\mu, \nu$ be Radon measures on $\mathbb{R}^n$. Then*

$$\int_A D_\nu \mu \; d\nu \leq \mu(A)$$

*for all $\mu$-measurable $A \subset \mathbb{R}^n$. The equality holds if $\mu \ll \nu$.*

*Proof.* Let $A$ be $\mu$-measurable; since $\mu$ is regular, there exists a Borel set $B$ with $A \subset B$ and $\mu(B \setminus A) = 0$; thus $\nu(B \setminus A) = 0$, and hence $A$ is $\nu$-measurable, since $\nu$ is a Borel measure.
Now fix $1 < t < \infty$, and define, for each integer $m \in \mathbb{Z}$

$$A_m := A \cap \{x \in \mathbb{R}^n \mid t^m \leq D_\mu \nu(x) < t^{m+1}\}$$

Then, from the previous theorem, the sets $A_m$ are Borel sets. Moreover define

$$D_0(\mu, \nu) := \{x \in \mathbb{R}^n \mid D_\nu \mu(x) = 0\}$$

and

$$D^*(\mu, \nu) := \{x \in \mathbb{R}^n \mid \underline{D}_\nu \mu(x) < \overline{D}_\nu \mu(x)\}$$

Then, from Theorem 2.7.3 we have that

$$\nu(A \cap D^*(\mu, \nu)) = \nu(A \cap D^\infty(\mu, \nu)) = 0$$

Moreover

$$\int_{A \cap D_0(\mu,\nu)} D_\nu \mu(x) \, \mathrm{d}\nu(x) = 0$$

Hence, recalling the definition of the sets $A_m$ and point 4 of Lemma 2.7.2, we have that

$$\begin{aligned}
\int_A D_\nu \mu(x) \, \mathrm{d}\nu(x) &= \sum_{m=-\infty}^{\infty} \int_{A_m} D_\nu \mu(x) \, \mathrm{d}\nu(x) \leq \sum_{m=-\infty}^{\infty} t^{m+1} \nu(A_m) \\
&= t \sum_{m=-\infty}^{\infty} t^m \nu(A_m) \leq t \sum_{m=-\infty}^{\infty} \mu(A_m) \\
&= t\mu\Big(\bigcup_{m=-\infty}^{\infty} A_m\Big) \leq t\mu(A)
\end{aligned}$$

Hence, for all $t > 1$ we obtain that

$$\int_A D_\nu \mu(x) \, \mathrm{d}\nu(x) \leq t\mu(B)$$

Letting $t \to 1$ we have the first part of the theorem.
Now we prove the equality in the case $\mu \ll \nu$: in this case we have that $\mu(D^*(\mu,\nu)) = \mu(D^\infty(\mu,\nu)) = 0$. Moreoverwe have that $\mu(D_0(\mu,\nu)) = 0$: in fact, fixed $r, \varepsilon > 0$, for each $x \in D_0(\mu,\nu) \cap B_r$ it holds that $D_\nu\mu(x) \leq \varepsilon$; hence from Lemma 2.7.2 we have that

$$\mu(D_0(\mu,\nu) \cap B_r) \leq \nu(D_0(\mu,\nu) \cap B_r) \leq \varepsilon\nu(B_r) < \infty$$

Letting $\varepsilon \to 0$ we find, for each $r > 0$, that $\mu(D_0(\mu,\nu) \cap B_r) = 0$, and hence we conclude that $\mu(D_0(\mu,\nu)) = 0$. Then, recalling the definition of the sets $A_m$ and point 3 of Lemma 2.7.2 , it holds

$$\begin{aligned}
\mu(A) &= \mu\Big(\bigcup_{m=-\infty}^{\infty} A_m\Big) = \sum_{m=-\infty}^{\infty} \mu(A_m) \\
&\leq t \sum_{m=-\infty}^{\infty} t^m \nu(A_m) \leq t \sum_{m=-\infty}^{\infty} \int_{A_m} D_\nu\mu(x)\, \mathrm{d}\nu(x) \\
&= t \int_A D_\nu\mu(x)\, \mathrm{d}\nu(x)
\end{aligned}$$

Letting $t \to 1$ we obtain the desired result. □

Now we present two important consequences of the theorem above.

**Theorem 2.7.5.** ***(Lebesgue decomposition theorem)***
*Let $\mu, \nu$ be Radon measures on $\mathbb{R}^n$. Then we can write*

$$\nu = \nu_{ac} + \nu_s$$

*where $\nu_{ac}$ and $\nu_s$ are Radon measures on $\mathbb{R}^n$ such that*

$$\nu_{ac} \ll \mu \qquad \nu_s \perp \mu$$

*Furthermore*

$$D_\mu\nu_s = 0 \qquad D_\mu\nu = D_\mu\nu_{ac} \quad \mu - a.e.$$

*and hence*

$$\nu(A) = \int_A D_\mu\nu \; d\mu + \nu_s(A)$$

*for each Borel set $A \subset \mathbb{R}^n$.*
*Furthermore the measures $\nu_{ac}$ and $\nu_s$ are unique.*

*Proof.* We can suppose $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$.
Let

$$\mathcal{F} := \{A \subset \mathbb{R}^n \mid A \text{ di Borel }, \mu(\mathbb{R}^n \setminus A) = 0\}$$

Choose $(C_k)_k$ such that

$$\nu(C_k) = \inf_{A \in \mathcal{F}} \nu(A) + \frac{1}{k}$$

and define $C := \bigcap_{k=1}^{\infty} C_k$. Since

$$\mu(\mathbb{R}^n \setminus C) \leq \sum_{k=1}^{\infty} \mu(\mathbb{R}^n \setminus C_k) = 0$$

we have that $C \in \mathcal{F}$, and $\nu(C) = \inf_{A \in \mathcal{F}} \nu(A)$.
Now, if we define

$$\nu_{ac} := \nu \llcorner C$$

and

$$\nu_s := \nu(\mathbb{R}^n \setminus C)$$

from Theorem 2.1.11 we have that $\nu_{ac}, \nu_s$ are Radon measures.
Now, if we take $A \subset C$ such that $\mu(A) = 0$, we must have $\nu(A)$=0; otherwise we would have $C \setminus A \in \mathcal{F}$, and $\nu(C \setminus A) < \nu(C)$; absurd. Hence $\nu_{ac} \ll \mu$.
Moreover $\mu(\mathbb{R}^n \setminus C) = 0$, and hence $\nu_s \perp \mu$.
Now we want to prove the assertion concerning the densities: fix $\alpha > 0$ and set

$$D := \{x \in C \mid D_\mu \nu_s(x) \geq \alpha\}$$

By Lemma 2.7.2

$$\alpha \mu(D) \leq \nu_s(D) = 0$$

since $D \subset C$. Since $\nu_{ac} = 0$ on $\mathbb{R}^n \setminus C$, we obtain that $D_\mu \nu_s = 0$ $\mu$-a.e., and hence

$$D_\mu \nu_{ac} = D_\mu \nu \qquad \mu\text{-a.e.}$$

The proof of uniqueness is easy. □

The following result is a kind of generalization of the Mean Value Theorem for $L^1$ functions, and has a lot of important consequences.

**Theorem 2.7.6 (Lebesgue-Besicovitch differentiation Theorem).** *Let $\mu$ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n; \mu)$. Then*

$$\lim_{r \to 0} \fint_{B_r(x)} f \, d\mu = f(x)$$

*for $\mu$-a.e. $x \in \mathbb{R}^n$.*

*Proof.* Let's define two measures $\nu^+, \nu^-$ on the Borel sets $B \subset \mathbb{R}^n$ as follows:

$$\nu^+(B) := \int_B f^+ \, d\mu \qquad \nu^-(B) := \int_B f^- \, d\mu$$

Now we extend the measures to all the sets $A \subset \mathbb{R}^n$ as follows.

$$\nu^{\pm}(A) := \inf\{\nu^{\pm}(B) \mid B \subset A, B \text{ Borel }\}$$

By construction $\nu^{\pm}$ are Radon measures on $\mathbb{R}^n$ that are absolutly cointinous with respect to $\mu$. Hence, by Theorem 2.7.4 there exists $D_\mu\nu^+$ and $D_\mu\nu^-$ such that

$$\nu^+(A) = \int_A D_\mu\nu^+ \, d\mu \qquad \nu^-(A) = \int_A D_\mu\nu^- \, d\mu$$

for all $\mu$-measurable set $A \subset \mathbb{R}^n$. But then

$$D_\mu\nu^+ = f^+ \qquad D_\mu\nu^- = f^- \qquad \mu - \text{a.e.}$$

Then, by Theorem 2.7.3

$$\begin{aligned}
\lim_{r\to 0} \fint_{B_r(x)} f \, d\mu &= \lim_{r\to 0} \frac{1}{\mu(B_r(x))}[\nu^+(B_r(x)) - \nu^-(B_r(x))] \\
&= D_\mu\nu^+(x) - D_\mu\nu^-(x) \\
&= f^+(x) - f^-(x) = f(x)
\end{aligned}$$

for $\mu$-a.e. $x \in \mathbb{R}^n$. □

**Remark 2.7.7.** *In particular we have prove the following fact:*
*let $\mu$ and $\nu$ be two Radon measures on $\mathbb{R}^n$ such that $\nu \ll \mu$, let $f$ be the function obtain from the Radon-Nikodym Theorem (see Theorem 2.5.12), that is the function such that $\nu = f\mu$. Hence, from the definition of $D_\mu\nu$ and from the theorem above, we have that*

$$D_\mu\nu(x) = \lim_{r\to 0} \frac{(f\mu)(B_r(x))}{\mu(B_r(x))} = \lim_{r\to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, \mathrm{d}\mu = f(x) \qquad \mu - a.e.$$

*That is, the function $f$ obtain from the Radon-Nikodym Theorem coincide $\mu$-a.e. with the derivate of $\nu$ with respect to $\mu$.*

Moreover, from Theorem 2.7.6 we have the following

**Theorem 2.7.8.** *Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then*

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 1 \qquad \textit{for } \mathcal{L}^n - a.e.\ x \in E$$

*and*

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 0 \qquad \textit{for } \mathcal{L}^n - a.e.\ x \in \mathbb{R}^n \setminus E$$

Since we are working with measures, and sometimes measures can not see all the sets (i.e. there exist sets of measure 0), if we change a set of a set of measure 0, topologically we have different objects, but in measure the two sets are the same. So we need a definition of internal and external of a set that keep into account this fact.

**Definition 2.7.9.** *Let $E \subset \mathbb{R}^n$; we define the* ***measure theoretic interior*** *of $E$ as the set of points of density 1 for $E$, i.e. the set of points $x$ such that*

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))} = 1$$

*We define the* ***measure theoretic exterior*** *of $E$ as the set of points of density 0 for $E$, i.e. the set of points $x$ such that*

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))} = 0$$

**Note:** if $E$ is $\mathcal{L}^n$-measurable, from Theorem 2.7.8 we have that $\mathcal{L}^n$-a.e. point $x \in E$ is in the measure theoretic interior of $E$, and $\mathcal{L}^n$-a.e. point $x \in \mathbb{R}^n \setminus E$ is in the measure theoretic exterior of $E$.

As a Corollary of this result (actually the two results are equivalent!) we have the following

**Corollary 2.7.10 (Lebesgue's points Theorem).** *Let $\mu$ be a Radon measure on $\mathbb{R}^n$, $1 \le p < \infty$, and $f \in L^1(\mathbb{R}^n;\mu)$. Then*

$$\lim_{r\to 0} \fint_{B_r(x)} |f - f(x)|^p \, d\mu = 0$$

*for $\mu$-a.e. $x \in \mathbb{R}^n$.*
*A point for which this result holds, is called a* ***Lebesgue point*** *of $f$ with respect to $\mu$.*

*Proof.* Let $(r_i)_i$ be a dense subset of $\mathbb{R}^n$. If we fix an index $i$, by Theorem 2.7.6

$$\lim_{r\to 0} \fint_{B_r(x)} |f - r_i|^p \, d\mu = |f(x) - r_i|^p$$

for $\mu$-a.e. $x \in \mathbb{R}^n$. Then there exists a set $A \subset \mathbb{R}^n$ such that $\mu(A) = 0$ and

$$\lim_{r\to 0} \fint_{B_r(x)} |f - r_i|^p \, d\mu = |f(x) - r_i|^p$$

for all $i$ and $x \in \mathbb{R}^n \setminus A$.
Now, if we fix $x \in \mathbb{R}^n \setminus A$, and we choose $\varepsilon > 0$, we can find and index $i$

such that $|f(x) - r_i|^p < \frac{\varepsilon}{2^p}$. Then[4]

$$\begin{aligned}\limsup_{r\to 0} \fint_{B_r(x)} |f - f(x)|^p \, d\mu &\leq 2^{p-1}\Big[\limsup_{r\to 0} \fint_{B_r(x)} |f - r_i|^p \, d\mu \\ &\quad + \limsup_{r\to 0} \fint_{B_r(x)} |r_i - f(x)|^p\Big] \\ &= 2^{p-1}[|f(x) - r_i|^p + |f(x) - r_i|^p] < \varepsilon\end{aligned}$$

Hence, by the arbitrary of $\varepsilon$ we can conclude. $\square$

## 2.8 Riesz Representation Theorem

In this section we present an important theorem that links functional analysis and measure theory: the Riesz representation Theorem, that allow us to identify the dual space of $C_0(X;\mathbb{R}^p)$, where $X$ is a locally compact and separable metric space, with the space of finite vector valued Radon measures on $X$.

We start with some definitions.

**Definition 2.8.1.** *Let $X$ be topological space and $f : X \to \mathbb{R}^n$ be a continous function; we define the* ***support*** *of $f$ as*

$$\operatorname{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}$$

*Moreover we denote by $C_c(X;\mathbb{R}^n)$ the space of continous function $f : X \to \mathbb{R}^n$ with compact support. In the case $n = 1$ we write $C_c(X)$ instead of $C_c(X;\mathbb{R})$.*

If we define, for $f \in C_c(X;\mathbb{R}^n)$,

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}$$

we obtain that $\|\cdot\|_\infty$ is a norm on $C_c(X;\mathbb{R}^n)$. We denote by $C_0(X;\mathbb{R}^n)$ the closure of $C_c(X;\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_\infty$. We have that

$$f \in C_0(X;\mathbb{R}^n) \iff \forall \varepsilon > 0 \quad \exists K \subset X \text{ compact s.t. } |f(x)| < \varepsilon \quad \forall x \in X \setminus K$$

[4] We use the following inequality:

$$|a - b|^p \leq 2^{p-1}(|a - c|^p + |c - b|^p)$$

that can be proved noting that the funtion $f(x) := |a - x|^p + |b - x|^p$ achives its minimum in $x = \frac{a-b}{2}$.

**Definition 2.8.2.** *Let $X$ be a locally compact and separable metric space. We say that a function $\mu : X \to \mathbb{R}^p$ is a* ***Radon vector valued measure*** *if $\mu$ is a vector valued measure in each $U \Subset X$ defined on the $\sigma$-algebra of Borel sets of $U$. Moreover if also $\mu$ is a measure on $X$ we called $\mu$ a* ***finite vector valued Radon measure*** *on $X$.*

An important consequence of Lusin's Theorem (see Theorem 2.3.4), state in a way that is usefull for later, is the following one:

**Corollary 2.8.3.** *Let $X$ be a locally compact and separable metric space, and let $\mu$ be a finite Borel measure on $X$. Let $f : X \to \mathbb{R}$ be a $\mu$-measurable function. Then there exists a disjoint sequence $(K_i)_i$ of compact sets such that $\|v\|_\infty \le \|f\|_\infty$ and*

$$\mu\Big( X \setminus \bigcup_{i=1}^{\infty} K_i \Big) = 0$$

*and $f_{|K_i}$ is continous for each $i$. Equivalently we can say that there exists a sequence of functions $(f_i)_i \subset C_c(X)$ such that $f_i = f$ in $K_i$ and $\|f_i\|_\infty \le \|f\|_\infty$.*

**Note:** this theorem imply that if $X$ is $\sigma$-finite, then $C_c(X)$ is dense in $L^p(X, \mu)$ for each $1 \le p < \infty$.

**Notation:** Let $X$ be a topological space. We recall that we denote by $\mathcal{B}(X)$ the $\sigma$-agebra of the Borel sets of $X$. Moreover with the notation

$$K \prec f$$

we mean that $f \in C_c(X)$, $0 \le f \le 1$, $K$ compact and $f_{|K} \equiv 1$, and with the notation

$$f \prec V$$

we mean that $f \in C_c(X)$, $0 \le f \le 1$, $V$ open and $\mathrm{supp}(f) \subset V$.

Let $X$ be a locally compact Haurdorff space. Our aim is to identify the space of Radon vector valued measures on $X$ with the space of locally bounded linear functional on $C_c(X; \mathbb{R}^n)$.
Let $\mu : X \to \mathbb{R}^n$ be a Radon vector valued measure; we can define a linear operator $L_\mu$ on $C_c(X; \mathbb{R}^n)$ as

$$L_\mu(f) := \int_X f \,\mathrm{d}\mu = \sum_{i=1}^{n} \int_X f_i \,\mathrm{d}\mu_i$$

for $f = (f_1, \ldots, f_n) \in C_c(X; \mathbb{R}^n)$. Since $\mu$ is a Radon measure, $L_\mu$ is locally bounded: in fact let $K$ be a compact set

$$\begin{aligned}
\sup\{\, L_\mu(f) \mid f \prec K \,\} &= \sup\Big\{ \sum_{i=1}^{n} \int_X f_i \,\mathrm{d}\mu_i \mid f \prec K \Big\} \\
&= \sup\Big\{ \int_K \langle f, \sigma \rangle \,\mathrm{d}|\mu| \mid f \prec K \Big\} \\
&\leq |\mu|(K) < \infty
\end{aligned}$$

where $\mu = \sigma|\mu|$, and in the last step we have used the fact that $\mu$ is a measure on $K$, and hence $|\mu|(K) < \infty$. Moreover we note that we have equality in the last step if $X$ is $\sigma$-finite, thanks to the note after Corollary 2.8.3.
The other part of the identification is much harder, and it will be proved in the following theorems.

**Theorem 2.8.4** (**Riesz Representation Theorem - I form**). *Let $X$ be a locally compact Hausdorff space, and lt $L : C_c(X) \to \mathbb{R}$ be a positive linear functional, that is $L(f) \geq 0$ if $f \geq 0$. Then there exists a $\sigma$-algebra $\mathcal{M}$ on $X$ and a positive Radon measure on $\mathcal{M}$, such that*

$$L(f) = \int_X f \,\mathrm{d}\mu$$

*for each $f \in C_c(X)$.*

*Proof.* Note that $L$ is monotone: in fact if $f \leq g$, then

$$L(g) = L(f) + L(g - f) \geq L(f)$$

We start by proving the uniqueness of $\mu$. We racall that $\mu$ is a positive Radon measure on $X$ if $\mu$ is a Borel measure satisfying

1. $\mu(K) < \infty$ for each compact set $K \subset X$
2. $\mu(A) = \inf\{\, \mu(V) \mid V \text{ open }, \ V \supset A \,\}$ for all $A \in \mathcal{M}$
3. $\mu(V) = \sup\{\, \mu(K) \mid K \text{ compact }, \ K \subset V \,\}$ for each open set $V \subset X$

Hence $\mu$ is characterized by its value on compact sets.
So, let $\mu_1$ and $\mu_2$ positive Radon measures on $\mathcal{M}$ that satisfied the thesis of the theorem. Let $K$ be a compact set, and fix $\varepsilon > 0$.. Since $\mu_2$ is a Radon mesure, there exists an open set $V$ such that $K \subset V$ and

$$\mu_2(V) < \mu_2(K) + \varepsilon$$

Moreover, by the Urysohn's Lemma, there exists $f \in C_c(X)$ such that $K \prec f \prec V$. Hence

$$\begin{aligned} \mu_1(K) &= \int_X \chi_K \, \mathrm{d}\mu_1 \leq \int_X f \, \mathrm{d}\mu_1 = L(f) = \int_X f \, \mathrm{d}\mu_2 \\ &\leq \int_X \chi_V \, \mathrm{d}\mu_2 = \mu_2(V) < \mu_2(K) + \varepsilon \end{aligned}$$

Since $\varepsilon$ is arbitrary we conclude that $\mu_1(K) \leq \mu_2(K)$. Interchanging the role of $\mu_1$ and $\mu_2$ we obtain that $\mu_1$ and $\mu_2$ agree on the compact sets, and hence they are equal on $\mathcal{M}$.

Now we proced by constructing $\mu$ and $\mathcal{M}$. We define $\mu$ as follows

- if $V \subset X$ is an open set we define

$$\mu(V) := \sup\{\, L(f) \mid f \prec V \,\}$$

- for arbitrary $E \subset X$ we define

$$\mu(E) := \inf\{\, \mu(V) \mid E \subset V, V \text{ open } \}$$

First of all we note that, since $\mu$ is monotone on the open sets, then $\mu$ is well defined.
Define

- $\mathcal{M}_F$ as the class of the sets $E \subset X$ such that
  - $\mu(E) < \infty$
  - $\mu(E) = \sup\{\, \mu(K) \mid K \subset E, K \text{ compact } \}$
- $\mathcal{M}$ as the class of the sets $E \subset X$ such that $E \cap K \in \mathcal{M}_F$ for every compact set $K$

Now we will proced by steps.

*Step 1:* $\mu$ is an outer mesure on $X$.
In fact $\mu(\emptyset) = 0$ and, if $A \subset B$, $\mu(A) \leq \mu(B)$. To the $\sigma$-subadditivity, let $V_1, V_2$ be open sets: then $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$. Moreover let $g \prec V_1 \cup V_2$, and let $h_1, h_2$ such that $h_1 \prec V_1$, $h_2 \prec V_2$ and $h_1 + h_2 \equiv 1$ on $\mathrm{supp}(g)$. Hence

$$L(g) = L(h_1 g) + L(h_2 g) \leq \mu(V_1) + \mu(V_2)$$

Now we prove that $\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \sum_{i=1}^{\infty} \mu(E_i)$ for each $(E_i)_i \subset \mathcal{P}(X)$. If for some $i$ it holds $\mu(E_i) = \infty$, then the inequlity is trivial. Otherwise if for each $i$, $\mu(E_i) < \infty$, then fix $\varepsilon > 0$. Then for each $i$ there exists an open set $V_i$ such that $E_i \subset V_i$ and $\mu(V_i) \leq \mu(E_i) + \frac{\varepsilon}{2^i}$. Let $V := \bigcup_{i=1}^{\infty} V_i$, and let $f \prec V$. Since $\mathrm{supp}(f)$ is compact, there exist $V_{i_1}, \ldots, V_{i_n}$ such that $\mathrm{supp}(f) \subset V_{i_1} \cup \cdots \cup V_{i_n}$. Let $h_{i_1}, \ldots, h_{i_n}$ be a partition of unity subordinate to $V_{i_1}, \ldots, V_{i_n}$. Hence

$$L(f) = \sum_{j=1}^{n} L(h_{i_j} f) \leq \sum_{j=1}^{n} \mu(V_{i_j}) \leq \sum_{j=1}^{n} \left( \mu(E_{i_j}) + \frac{\varepsilon}{2^{i_j}} \right) \leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$$

Hence, for the arbitrary of $\varepsilon$ first, and of $f$ after, we obtain that $\mu$ is $\sigma$-subsdditive.

*Step 2:* $\mathcal{M}_F$ contains the compact sets.
Let $K$ be a compact set, $K \prec f$, and define the open set $V := \{ f > \frac{1}{2} \}$; then $K \subset V$. Let $g \prec V$; then $g \leq 2f$ and hence, since $L$ is monotone,

$$\mu(K) \leq \mu(V) = \sup\{ L(g) \mid g \prec V \} \leq L(2f) < \infty$$

Moreover it is obvious that a compact set satisfied the second condition that defined $\mathcal{M}_F$.

*Step 3:* every open set $V$ such that $\mu(V) < \infty$ belongs to $\mathcal{M}_F$.
Let $V$ open set with $\mu(V) < \infty$, and let $\alpha \in \mathbb{R}$ such that $\alpha < \mu(V)$. Let $f \prec V$ with $\alpha < L(f)$; this is possible thankss to the definition of $\mu$ on open sets. If we denote by $K := \mathrm{supp}(f)$ we have that for each open set $W$ with $K \subset W$, $f \prec W$, and hence $L(f) \leq \mu(W)$. Since $\mu(K) = \inf\{ \mu(W) \mid W \Supset K, W \text{ open } \}$, we obtain that $L(f) \leq \mu(K)$. Hence the compact set $K$ is such that $K \subset V$, $\alpha < \mu(K) < \mu(V)$. Since $\alpha$ is arbitrary we can conclude that $V \in \mathcal{M}_F$.

*Step 4:* Let $(E_i)_i \subset \mathcal{M}_F$ disjoint, and let $E := \bigcup_{i=1}^{\infty} E_i$. Then $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$. Moreover if $\mu(E) < \infty$, then $E \in \mathcal{M}_F$. We will prove it in three points:

1. let $K_1, K_2$ be disjoint compact sets; hence $\mu(K_1 \cup K_2) \geq \mu(K_1) + \mu(K_2)$. Since $X$ is a Hausdorff space, there exist disjoint open sets $V_1, V_2$ such that $K_1 \subset V_1$, $K_2 \subset V_2$. Moreover, if we fix $\varepsilon > 0$, there exists an open set $W$ such that $K_1 \cup K_2 \subset W$ and $\mu(W) \leq \mu(K_1 \cup K_2) + \varepsilon$. Finally there exist functions $f_1, f_2$ such that $f_i \prec W \cap V_i$ and $L(f_i) > \mu(W \cap V_i) - \varepsilon$, for $i = 1, 2$. Hence

$$\begin{aligned} \mu(K_1) + \mu(K_2) &\leq \mu(W \cap V_1) + \mu(W \cap V_2) \\ &< L(f_1) + L(f_2) + 2\varepsilon \\ &< \mu(W) + 2\varepsilon < \mu(K_1 \cup K_2) + 3\varepsilon \end{aligned}$$

2. if $\mu(E) = \infty$, then from the $\sigma$-subadditivity of $\mu$ the result follows. Otherwise, if $\mu(E) < \infty$, fix $\varepsilon > 0$; for each $E_i$ there exists a compact set $K_i \subset E_i$ such that $\mu(K_i) > \mu(E_i) - \frac{\varepsilon}{2^i}$. Hence, for each $n \in \mathbb{N}$

$$\mu(E) > \mu\Big(\bigcup_{i=1}^{n} K_i\Big) > \sum_{i=1}^{n} \mu(E_i) - \varepsilon$$

and hence

$$\mu(E) > \sum_{i=1}^{\infty} \mu(E_i) - \varepsilon$$

We conclude for the arbitrarity di $\varepsilon$.

3. if $\mu(E) < \infty$ then fix $\varepsilon > 0$; hence there exists $N > 0$ such that

$$\mu(E) \leq \sum_{i=1}^{N} \mu(E_i) + \varepsilon$$

From the previous point we obtain that

$$\mu(E) \leq \mu\Big(\bigcup_{i=1}^{N} K_i\Big) + 2\varepsilon$$

Since $K := \bigcup_{i=1}^{N} K_i$ is compact, for the arbitrarity of $\varepsilon$.

Hence we have obtained the desired result.

*Step 5:* Let $E \in \mathcal{M}_F$ and $\varepsilon > 0$. Then there exist a compact set $K$ and an open set $V$ such that $K \subset E \subset V$ and $\mu(V \setminus K) < \varepsilon$.
We known that there exist a compact set $K$ and an open set $V$ such that $K \subset E \subset V$ and

$$\mu(V) - \frac{\varepsilon}{2} < \mu(E) < \mu(K) + \frac{\varepsilon}{2}$$

Since $V \setminus K$ is open (we recall that a compact set in a Hausdorff space is closed!) and $\mu(V \setminus K) < \mu(V) < \infty$, from Step 3 we obtain that $V \setminus K \in \mathcal{M}_F$. hence, from the previous Step

$$\mu(K) + \mu(V - K) = \mu(V) < \mu(K) + \varepsilon$$

and since $\mu(K) < \infty$ we conclude.

*Step 6:* Let $A, B \in \mathcal{M}_F$. Hence $A \cap B, A \cup B, A \setminus B$ belong to $\mathcal{M}_F$.
Fix $\varepsilon > 0$; from Step 5 we have that there exist compact sets $K_1, K_2$ and open sets $V_1, V_2$ such that

$$K_1 \subset A \subset V_1, \quad \mu(V_1 \setminus K_1) < \varepsilon$$

$$K_2 \subset B \subset V_2, \quad \mu(V_2 \setminus K_2) < \varepsilon$$

Since

$$A \setminus B \subset V_1 \setminus K_2 \subset (V_1 - K_1) \cup (K_1 \setminus V_2) \cup (V_2 \setminus K_2)$$

we obtain that

$$\mu(A - B) \le \mu(K_1 - V_2) + 2\varepsilon$$

Since $K_1 \setminus V_2$ is a compact set contained in $A \setminus B$ we conclude that $A - B \in \mathcal{M}_F$.
Since $A \cup B = (A \setminus B) \cup B$, and $A \setminus B, B \in \mathcal{M}_F$ and $\mu(A \cup B) \le \mu(A) + \mu(B) < \infty$, from the previous Step we obtain that $A \cup B \in \mathcal{M}_F$. Same argument for $A \cap B = A \setminus (A \setminus B)$.

*Step 7:* $\mathcal{M}$ is a $\sigma$-algebra that contains the Borel sets of $X$.

1. $\mathcal{M}$ is closed for complementarity: let $A \in \mathcal{M}$ and let $K$ be a compact set. Then $A^c \cap K = K \setminus (A \cap K)$, and since $K$ and $A \cap K$ are in $\mathcal{M}_F$, then $A^c \in \mathcal{M}$.
2. $\mathcal{M}$ is closed under countable union: let $A := \bigcup_{i=1}^{\infty} A_i$, where $A_i \in \mathcal{M}$, and let $K$ be a compact set. Define $B_1 := A_1 \cap K$ and $B_n := (A_n \cap K) \setminus \bigcup_{i=1}^{n-1} B_i$ for $n \ge 2$. Hence $(B_n)_n$ is a sequence of disjoint sets in $\mathcal{M}_F$. Moreover $A \cap K = \bigcup_{i=1}^{\infty} B_n$, and hence $\mu(\bigcup_{i=1}^{\infty} B_n) = \mu(A \cap K) < \infty$. For Step 4 we obtain that $A \cap K \in \mathcal{M}_F$, and hence $A \in \mathcal{M}$.
3. If $C$ is a closed set, then $C \cap K$ is a compact set, and hence $C \cap K \in \mathcal{M}_F$, and hence $C \in \mathcal{M}$. Hence $\mathcal{M}$ contains all Borel sets. In particular $X \in \mathcal{M}$.

Hence we have obtained the desired result.

*Step 8:* $\mathcal{M}_F = \{ E \in \mathcal{M} \mid \mu(E) < \infty \}$.
Let $E \in \mathcal{M}_F$; then $E \cap K \in \mathcal{M}_F$ for each compact set $K$, and hence $E \in \mathcal{M}$. Now let $E \in \mathcal{M}$ with $\mu(E) < \infty$. Fix $\varepsilon > 0$; then, from the definition of $\mu$ we known that there exists an open set $V$ such that $E \subset V$ and $\mu(V) < \infty$, and hence, for Step 3, $V \in \mathcal{M}_F$. Moreover, by Step 5, taking $V$ itself as the open set, we find a compact set $K \subset V$ such that

$$\mu(V \setminus K) < \varepsilon$$

Since $E \cap K \in \mathcal{M}_F$, there exists a compact set $H \subset E \cap K$ such that

$$\mu(E \cap K) < \mu(H) + \varepsilon$$

Since $E \subset (E \cap K) \cup (V \setminus K)$, it follows that

$$\mu(E) \le \mu(E \cap K) + \mu(V \setminus K) < \mu(H) + 2\varepsilon < \infty$$

Hence $E \in \mathcal{M}_F$.

*Step 9:* $\mu$ is a measure on $\mathcal{M}$.
This easly follows from the previous Steps.

*Step 10:* $L(f) = \displaystyle\int_X f \,\mathrm{d}\mu$ for each $f \in C_c(X)$.
It is sufficient to prove that

$$L(f) \le \int_X f \,\mathrm{d}\mu \tag{2.1}$$

In fact, if (2.1) holds, then $L(-f) \le \int_X -f \,\mathrm{d}\mu$, and hence $L(f) \ge \int_X f \,\mathrm{d}\mu$. So we have to prove (2.1): fix $f \in C_c(X)$; since $f(X)$ is compact, there exists $a < b \in \mathbb{R}$ such that $f(X) \subset [a, b]$. Fix $\varepsilon > 0$ and choose pionts $y_0, \dots, y_n$ such that

$$y_0 < a < y_1 < \cdots < y_{n-1} < y_n = b$$

and $y_i - y_{i-1} < \varepsilon$. Denote by $K := \operatorname{supp}(f)$, and define for each $i = 1, \dots, n$

$$E_i := \{\, x \in K \mid y_{i-1} < f(x) \le y_i \,\}$$

Since $f$ is continous, $f$ is Borel measurable, and hence the sets $E_i$ are disjoint Borel sets, whose union is $K$. Since $\mu(K) < \infty$ there exists open sets $V_i$ such that $E_i \subset V_i$ and

$$\mu(V_i) < \mu(E_i) + \frac{\varepsilon}{n}$$

and $f(x) < y_i + \varepsilon$ for each $x \in V_i$. Let $(h_i)_{i=1}^n$ functions such that $h_i \prec V_i$ ($\Rightarrow L(h_i) \le \mu(V_i)$) and $\sum_{i=1}^n h_i \equiv 1$ on $K$. We have that $h_i f \le h_i(y_i + \varepsilon)$ and $y_i - \varepsilon < f(x)$ on $E_i$; moreover $\mu(K) \le L(\sum_{i=1}^n h_i)$. Hence we obtain that

$$\begin{aligned}
L(f) &= \sum_{i=1}^n L(h_i f) \le \sum_{i=1}^n (y_i + \varepsilon) L(h_i) \\
&= \sum_{i=1}^n (|a| + y_i + \varepsilon) L(h_i) - |a| \sum_{i=1}^n L(h_i) \\
&\le \sum_{i=1}^n (|a| + y_i + \varepsilon) \Big(\mu(E_i + \frac{\varepsilon}{n}\Big) - |a|\mu(K) \\
&= \sum_{i=1}^n (|a| + y_i + \varepsilon) \frac{\varepsilon}{n} + \sum_{i=1}^n (y_i - \varepsilon)\mu(E_i) + 2\varepsilon\mu(K) \\
&\le \varepsilon(|a| + b + \varepsilon + 2\mu(K)) + \int_X f \,\mathrm{d}\mu
\end{aligned}$$

Since $\varepsilon$ is arbitrary we conclude. $\square$

Now we present a particular version of the above theorem that it will be usefull for our aim.

**Theorem 2.8.5 (Riesz Representation Theorem - II form).** *Let $X$ be a locally compact and separable metric space, and let $L : C_c(X; \mathbb{R}^n) \to \mathbb{R}$ be linear and locally bounded, that is*

$$\sup\{\, L(f) \mid f \prec K \,\} < \infty$$

*for each compact set $K \subset X$. Then there exists a unique Radon vector valued measure $\mu = (\mu_1, \ldots, \mu_n)$ on $X$ such that*

$$L(f) = \sum_{i=1}^{n} \int_X f_i \,\mathrm{d}\mu_i$$

*for each $f \in C_c(X; \mathbb{R}^n)$. Moreover for each open set $A \subset X$ it holds*

$$|\mu|(A) = \sup \Big\{\, L(f) \mid f \in C_c(A; \mathbb{R}^n),\ |f| \leq 1 \,\Big\}$$

*Proof.* We start by proving the uniqueness: suppose $\mu_1, \mu_2$ are Radon vector valued measures satisfing the thesis of the theorem. Since

$$|\mu_1|(A) = \sup \Big\{\, L(f) \mid f \prec A \,\Big\} = |\mu_2|(A)$$

for each open set $A$, we obtain that $|\mu_1| = |\mu_2| =: \nu$. Now, writing $\mu_1 = \sigma_1 \nu$ and $\mu_2 = \sigma_2 \nu$, with $|\sigma_1(x)| = 1$ and $|\sigma_2(x)| = 1$ $|nu|$-a.e., we obtain that

$$\int_X \langle f, \sigma_1 - \sigma_2 \rangle \,\mathrm{d}\nu = 0$$

for each $f \in C_c(X : \mathbb{R}^n)$. Hence $\sigma_1 = \sigma_2$.

Now we prove the existence of this measure. Suppose first $n = 1$. We want to use the previous theorem, but since we do not known if $L$ is positive, we have to modify it: so we define the funcional $L^*$ on the space $\{\, f \in C_c(X) \mid f \geq 0 \,\}$ as

$$L^*(f) := \sup\{\, |L(g)| \mid g \in C_c(X),\ |g| \leq f \,\}$$

Hence

- $L^*(f) \in \mathbb{R}$: in fact since in the definition of $L^*(f)$ we work with functions $g$ such that $|g| \leq f$, we have that $\operatorname{supp}(g) \subset \operatorname{supp}(f)$; hence, since $\operatorname{supp}(f)$ is compact, for the homogeneity and the locally boundness of $L$ we conclude.

- $L^*$ is positive: if $f \geq 0$, take $g \equiv 0$; then $L^*(f) \geq |L(g)| = 0$.

- $L^*$ is linear: let $f_1, f_2 \in C_c(X)$ such that $f_1, f_2 \geq 0$. First we prove that $L^*(f_1 + f_2) \geq L^*(f_1) + L^*(f_2)$: let $g_1, g_2 \in C_c(X)$ such that $|g_i| \leq f_i$ for $i = 1, 2$; we can suppose that $g_1, g_2 \geq 0$. Then $|g_1 + g_2| \leq f_1 + f_2$, and hence

  $$L(g_1) + L(g_2) = L(g_1 + g_2) = |L(g_1 + g_2)| \leq L^*(f_1 + f_2)$$

  For the opposite inequality, let $g \in C_c(X)$ such that $g \leq f_1 + f_2$; define the funcions, for $i = 1, 2$

  $$g_i := \begin{cases} \dfrac{g}{f_1 + f_2} f_i & , f_1 + f_2 > 0 \\ 0 & f_1 + f_2 = 0 \end{cases}$$

  Then $g_i \in C_c(X)$, $g_i \leq f_i$ and $g_1 + g_2 = g$. Hence

  $$|L(g)| \leq |L(g_1)| + |L(g_2)| \leq L^*(f_1) + L^*(f_2)$$

Now we define the functional $\widetilde{L}$ on $C_c(X)$ as

$$\widetilde{L}(f) := L^*(f^+) - L^*(f^-)$$

Clearly $\widetilde{L}$ is linear and positive, and $\widetilde{L}(f) \in \mathbb{R}$ for each $f \in C_c(X)$. Hence for the previous theorem there exists a positive Radon measure $\nu$ on $X$ such that

$$\widetilde{L}(f) = \int_X f \, \mathrm{d}\nu$$

for each $f \in C_c(X)$. Moreover, for each open set $V$, it holds

$$\nu(V) = \sup\{\, L^*(f) \mid f \prec V \,\} = \sup\{\, L(f) \mid f \in C_c(V),\ |f| \leq 1 \,\}$$

Now we want represent $L$. Since

$$|L(f)| \leq \widetilde{L}(f) = \int_X f \, \mathrm{d}\nu \leq \|f\|_{L^1(X,\nu)}$$

we can extend $L$ to a functional $\overline{L} \in (L^1(X, \nu))' \equiv L^\infty(X, \nu)$. Hence there exists a function $\sigma \in L^\infty(X, \nu)$ such that

$$\overline{L}(f) = \int_X f\sigma \, \mathrm{d}\nu$$

for each $f \in L^1(X, \nu)$. In particular

$$L(f) = \int_X f\sigma \, \mathrm{d}\nu$$

for each $f \in C_c(X)$. Hence if we define the measure $\mu := \sigma\nu$ we have the desired representation of $L$.

For the case $n > 1$ we can reason component by component obtaining a function $\sigma = (\sigma_1, \dots, \sigma_n)$ such that $\sigma_i \in L^\infty(X, \nu)$ for each $i$ and

$$L(f) = \sum_{i=1}^{n} \int_X f_i \sigma_i \, \mathrm{d}\nu$$

for each $f \in C_c(X; \mathbb{R}^n)$. So we define the measure $\mu := \sigma\nu$.

Now we want to prove that $|\sigma(x)| = 1$, $\nu$-a.e.. Let $U \subset X$ with $\nu(U) < \infty$. Then, from Corollary 2.8.3 we obtain that there exists a sequence of functions $(f_k)_k \in C_c(X; \mathbb{R}^n)$ such that $|f_k| \leq 1$, $\operatorname{supp}(f_k) \subset U$ and $\langle f_k, \sigma \rangle \to |\sigma|$, $\nu$-a.e. on $U$. Then

$$\int_U |\sigma| \, \mathrm{d}\nu = \lim_{k\to\infty} \int_U \langle f_k, \sigma \rangle \, \mathrm{d}\nu = \lim_{k\to\infty} L(f_k) \leq \nu(U)$$

On the other hand if we take $f \in C_c(U; \mathbb{R}^n)$ with $|f| \leq 1$ we have that

$$\int_X \langle f, \sigma \rangle \, \mathrm{d}\nu \leq \int_U |\sigma| \, \mathrm{d}\nu$$

Hence $\nu(U) \leq \int_U |\sigma| \, \mathrm{d}\nu$. So we have obtained that $|\sigma(x)| = 1$, $\nu$-a.e. on every open set $U$ with $\nu(U) < \infty$. Since $X$ is a locally compact separable metric space, we can write $X$ as

$$X = \bigcup_{i=1}^{\infty} K_i$$

where the $K_i$'s are compact subsets. Moreover, since $\nu$ is a Radon measure on $X$, we have that $\nu(K_i) < \infty$ for each $i$; hence we obtain that, for each $i$, there exists an open set $U_i$ such that $K_i \subset U_i$ and $\nu(U_i) < \infty$. Applying the above result to each $U_i$ we obtain that $|\sigma(x)| = 1$, $\nu$-a.e. on $X$.
So we have obtain that $|\mu| = \nu$, and hence the desired result. □

## 2.9 Weak convergence and compactness of Radon measures

In this section we will introduced a notion of weak convergence for Radon measure, derived from the identification given by the Riesz Representation Theorem, and we will study the properties of this convergence.

**Definition 2.9.1.** *Let $X$ be a locally compact and separable metric space, and let $(\mu_k)_k$ be a sequence of vector valued Radon measures on $X$. We say that $\mu_k$* ***converge weakly*** *to the vector valued Radon measure $\mu$, or that $\mu_k$ is* ***weak* convergent*** *to $\mu$, written $\mu_k \rightharpoonup \mu$, if*

$$\lim_{k\to\infty} \int_X f \, \mathrm{d}\mu_k = \int_X f \, \mathrm{d}\mu$$

*for each $f \in C_c(X;\mathbb{R}^n)$.*

**Note**: we can endowed the space $C_c(X;\mathbb{R}^n)$ whith a topolgy, and consired the weak* convergence in the dual space of $C_c(X;\mathbb{R}^n)$, and transfert it to the space of vector valued Radon measures on $X$, thanks to the Riesz Representation Theorem.

First of all we prove two important results about the weak* convergence: lower semi-continouity and compactness.

**Theorem 2.9.2.** *Let $(\mu_k)_k, \mu$ be vector valued Radon measures on a locally compact and separable metric space $X$, and suppose that $\mu_k \rightharpoonup \mu$. Then for each open set $A$ it holds*

$$|\mu|(A) \leq \liminf_{k\to\infty} |\mu_k|(A)$$

*Proof.* Define the linear functionals on $C_c(X;\mathbb{R}^n)$

$$L_\mu(f) := \int_X f \, \mathrm{d}\mu, \qquad L_{\mu_k}(f) := \int_X f \, \mathrm{d}\mu_k$$

From the Riesz Representation Theorem (see Theorem 2.8.5) we have that, if $f \in C_c(X;\mathbb{R}^n)$, $|f| \leq 1$, $\mathrm{supp}(f) \subset A$

$$L_\mu(f) = \int_X f \, \mathrm{d}\mu = \lim_{k\to\infty} \int_X f \, \mathrm{d}\mu_k \leq \liminf_{k\to\infty} |\mu_k|(A)$$

and hence we obtain the desired result. $\square$

We also easily have compactness

**Theorem 2.9.3.** ***(De La Vallèe Poussin Theorem)***
*Let $X$ be a locally compact and separable metric space. Let $(\mu_k)_k$ be a sequence of vector valued Radon measures on $X$ such that*

$$\sup_k |\mu_k|(K) < \infty$$

*for each compact set $K \subset X$. Then there exists a vector valued Radon measure $\mu$ on $X$ and a subsequence $(\mu_{k_h})_h$ such that $\mu_{k_h} \rightharpoonup \mu$.*

*Proof.* Let $K \subset X$ be a compact set, and let $M := \sup_k |\mu_k|(K)$. Let $D := (f_h)_h$ be a contable dense subset of $C_c(X; \mathbb{R}^n)$. Since for each $h$ and $j$ we have that

$$\Big| \int_K f_h \, \mathrm{d}\mu_j \Big| \leq \|f_h\|_\infty M$$

we can find, using a diagonal process, a subsequence $(\mu_{h_j})_j$ and a sequence $(a_h)_h \subset \mathbb{R}^n$ such that

$$\int_K f_h \, \mathrm{d}\mu_{h_j} \stackrel{j \to \infty}{\longrightarrow} a_h$$

for each $h$. Hence we define the linear functional $L$ on $D$ as

$$L(f_h) := a_h$$

Since $|L(f_h)| \leq M\|f\|_\infty$ we can extend $L$ to a bounded linear functional $\overline{L}$ on $C_c(X; \mathbb{R}^n)$. From the Riesz Representation Theorem we have that $\overline{L}$ can be represent with a finite vector valued Radon measure $\mu$. Now we wanto to prove that $\mu_{h_j} \rightharpoonup \mu$. Let $f \in C_c(X; \mathbb{R}^n)$ and fix $\varepsilon > 0$; then there exists an integer $h$ such that $\|f_h - f\|_\infty < \frac{\varepsilon}{M}$. Next choose an integer $J$ such that for each $j > J$ it holds

$$\Big| \int_K f_h \, \mathrm{d}\mu_{h_j} - \int_K f_h \, \mathrm{d}\mu \Big| < \frac{\varepsilon}{2}$$

Hence, for $i > J$

$$\begin{aligned} \Big| \int_K f \, \mathrm{d}\mu_{h_j} - \int_K f \, \mathrm{d}\mu \Big| \ &\leq \ \Big| \int_K (f - f_h) \, \mathrm{d}\mu_{h_j} \Big| + \Big| \int_K (f - f_h) \, \mathrm{d}\mu \Big| \\ &+ \ \Big| \int_K f_h \, \mathrm{d}\mu_{h_j} - \int_K f_h \, \mathrm{d}\mu \Big| \\ &\leq \ 2M\|f - f_h\|_\infty + \varepsilon < 3\varepsilon \end{aligned}$$

For the arbitrary of $\varepsilon$ we conclude.

Since we can write $X$ as countable union of compact sets $(K_i)_i$, we can apply the above argument to each $K_i$, and hence using a diagonal argument to obtain the desired result. $\square$

Now we present some "measure kind" properties of the weak convergence of Radon measures, first in the case of non negative measures, and then for general vector valued measures.

**Theorem 2.9.4.** *Let $(\mu_k)_k, \mu$ be non negative Radon measures on a locally compact and separable metric space $X$. Supppose that $\mu_k$ converge weakly to $\mu$. Then for each compact set $K \subset X$*

$$\limsup_{k\to\infty} \mu_k(K) \le \mu(K)$$

*and for each open set $U \subset X$*

$$\liminf_{k\to\infty} \mu_k(U) \ge \mu(U)$$

*Proof.* Let $K \subset X$ be a compact set; fix $\varepsilon > 0$, and let $U \supset K$ be an open set. Choose $f \in C_c(X)$ such that $0 \le f \le 1$, $\mathrm{supp}(f) \subset U$ and $f \equiv 1$ on $K$. Then

$$\mu(U) \ge \int_X f \mathrm{d}\mu = \lim_{k\to\infty} \int_X f \mathrm{d}\mu_k \ge \limsup_{k\to\infty} \mu_k(K)$$

Since $\mu$ is a Radon measure, we can approssimate $\mu(K)$ from the outside with open sets. So

$$\mu(K) = \sup\{\, \mu(U) \mid U \supset K,\ U \text{ open } \} \ge \limsup_{k\to\infty} \mu_k(K)$$

The proof for the open sets is similar.

□

**Theorem 2.9.5.** *Let $X$ be a locally compact and separable metric space, and let $(\mu_k)_k$ be a sequence of vector valued Radon measures on $X$. Suppose that*

$$\mu_k \rightharpoonup \mu, \qquad |\mu_k| \rightharpoonup \sigma$$

*for some vector valued Radon measure $\mu$ and some non negative Radon measure $\sigma$, and that*

$$\sup_k |\mu_k|(X) < \infty$$

*Then $|\mu| \le \sigma$, and for each Borel set $B \Subset X$ such that $\sigma(\partial B) = 0$ it holds*

$$\lim_{k\to\infty} \mu_k(B) = \mu(B)$$

**Note:** if in the theorem above we have that the measures $\mu_k$ are non negative Radon measures, then $|\mu_h| = \mu_k$, and hence we can say that if $\mu_k \rightharpoonup \mu$, then for each Borel set $B \Subset X$ with $\mu(\partial B) = 0$ we have that $\lim_k \mu_k(B) = \mu(B)$.

*Proof.* We begin by proving that $|\mu| \le \sigma$: let $A \Subset X$, and define, for $t > 0$

$$A_t := \{\, x \in A \mid d(x, \partial A) > t \,\}$$

Let $f \in C_c(A)$ such that $\chi_{A_t} \le f \le \chi_A$. Hence, from Theorem 2.9.2

$$|\mu|(A_t) \le \liminf_{k\to\infty} |\mu_k|(A_t) \le \liminf_{k\to\infty} \int_X f \,\mathrm{d}|\mu_k| = \int_X f \,\mathrm{d}\sigma \le \sigma(A)$$

Now $A_t \uparrow A$, and since $\mu(\overline{A}) < \infty$ we have that $|\mu|(A_t) \to |\mu|(A)$; so we have obtained that $|\mu|(A) \le \sigma(A)$ for each $A \Subset X$. Since a locally compact and separable metric space can be write as a countable union of compact sets, we obtain that $|\mu| \le \sigma$.
For the second assertion: let $\mu_{k,i}$ be the $i^{th}$ component of the measure $\mu_k$, for $i = 1, \ldots, n$, and let $\mu_{k,i}^{\pm}$ be the positive and the negative part of $\mu_{k,i}$. Since $\mu_{k,i}^{\pm} \le |\mu_k| < M$ for some $M < \infty$, we can suppose that $\mu_{k,i}^{\pm} \rightharpoonup \nu_i^{\pm}$ for each $i = 1, \ldots, n$. Moreover, from $\mu_{k,i} = \mu_{k,i}^{+} - \mu_{k,i}^{-}$, passing to the limit we obtain that

$$\mu_i = \nu_i^{+} - \nu_i^{-}, \qquad \nu_i^{\pm} \le |\mu| \le \sigma$$

Now, let $B \Subset X$ be a Borel set such that $\sigma(\partial B) = 0$; then $\nu_i^{\pm}(\partial B) = 0$ for each $i = 1, \ldots, n$. Let $K$ be the closure of $B$ and $A$ be the internal of $B$. Since $E - A = \partial B$ we have that $\nu_i^{\pm}(K - A) = 0$, and hence $\nu_i^{\pm}(B) = \nu_i^{\pm}(A) = \nu_i^{\pm}(K)$. Hence from Theorem 2.9.4 we have for each $i = 1, \ldots, n$

$$\begin{aligned}
\limsup_{k\to\infty} \mu_{k,i}^{\pm}(B) &\le \limsup_{k\to\infty} \mu_{k,i}^{\pm}(K) \le \nu_i^{\pm}(K) \\
&= \nu_i^{\pm}(A) \le \liminf_{k\to\infty} \mu_{k,i}^{\pm}(A) \le \liminf_{k\to\infty} \mu_{k,i}^{\pm}(B)
\end{aligned}$$

Hence we obtain that $\mu_{k,i}^{\pm}(B) \to \nu_i^{\pm}(B)$ for each $i = 1, \ldots, n$. Thus

$$\begin{aligned}
\lim_{k\to\infty} \mu_{k,i}(B) &= \lim_{k\to\infty} (\mu_{k,i}^{+}(B) - \mu_{k,i}^{-}(B)) \\
&= \nu_i^{+}(B) - \nu_i^{-}(B) = \mu_i(B)
\end{aligned}$$

and hence $\mu_k(E) \to \mu(E)$. □

An important application of the theorem above is the following one: let $(A_t)_{t\in J}$ be an increasing family of relatively compact open sets labelled on an interval $J \subset \mathbb{R}$ such that $\overline{A}_s \subset A_t$ for $s < t$. Then we have that $\sigma(\partial A_t) = 0$ except for countable many $t \in J$, and hence $\mu_h(A_t) \to \mu(A_t)$ except for countable many $t \in J$. In fact let $B \Subset X$ and fix $\varepsilon > 0$; hence the set

$$\{\, t \in J \mid \sigma(\partial A_t) > \varepsilon,\ \overline{A}_t \Subset B \,\}$$

is finite, because the sets $\partial A_t$ are pairwise disjoint and $\sigma(B) < \infty$.

# Chapter 3

# Hausdorff measures

In this chapter we introduce the $s$-dimensional Hausdorff measures on a metric space $X$. This kind of measures are very useful in geometric measure theory, because they allow to define an intrinsic notion of $s$-dimenasional area. We will study the principal properties of this measures; in particular the notion of Hausdorff dimension of a set in a metric space (Definition 3.1.7), and densities properties for the Hausdorff measures (section 3.1.2). Then, in section 3.2, we will study the Hausdorff measures in $\mathbb{R}^n$ and their relation with the Lebesgue measure $\mathcal{L}^n$ (Theorem 3.2.6); in particular we prove the isodiametric inequality in $\mathbb{R}^n$ (Theorem 3.2.5).

## 3.1 Hausdorff measures in metric spaces

### 3.1.1 Definition and properties

We start by defining the $s$-dimensional Hausdorff measure in a generic metric space, using the so called "Carathéodory construction".

**Definition 3.1.1.** *Let $(X,d)$ be a metric space, and let $0 \le s < \infty$, $0 < \delta \le \infty$; we define the **pre-measure** $\mathcal{H}^s_\delta$ as follows:*

$$\mathcal{H}^s_\delta := \inf\Big\{\sum_{j=1}^{\infty}\alpha(s)\Big(\frac{\mathrm{diam}(C_j)}{2}\Big)^s \mid A \subset \bigcup_{j=1}^{\infty} C_j,\ \mathrm{diam}(C_j) \le \delta\Big\}$$

*for $A \subset X$, where*

$$\alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$$

*and $\Gamma(s) := \displaystyle\int_0^\infty e^{-x}x^{s-1}\,\mathrm{d}x$ is the Gamma function.*

The constant $\alpha(s)$ has been included in order to have, in $X = \mathbb{R}^n$, $\mathcal{L}^n = \mathcal{H}^n$. We recall that, if $n$ is an integer, then $\alpha(n) = \omega_n$.
Now we want to prove that $\mathcal{H}^s_\delta$ is an outer measure.

**Theorem 3.1.2.** *For each $0 \leq s < \infty$ and each $0 < \delta \leq \infty$, $\mathcal{H}^s_\delta$ is an outer measure.*

*Proof.* Fix $s$ and $\delta$. It is clear that $\mathcal{H}^s_\delta(\emptyset) = 0$. Now, let $(A_k)_k$; for each $k$ select $(C^k_j)_j$ such that $A_k \subset \bigcup_{j=1}^\infty C^k_j$ and $\mathrm{diam}(C^k_j) \leq \delta$. Then

$$\bigcup_{k=1}^\infty A_k \subset \bigcup_{k=1}^\infty \bigcup_{j=1}^\infty C^k_j$$

and hence, by the definition of the pre-measure $\mathcal{H}^s_\delta$

$$\mathcal{H}^s_\delta\Big(\bigcup_{kA_k}\Big) \leq \sum_{k=1}^\infty \sum_{j=1}^\infty \alpha(s)\Big(\frac{\mathrm{diam}(C^k_j)}{2}\Big)^s$$

Since the sets $(C^k_j)_j$ are arbitrary, we can take the infima over them, and hence obtain

$$\mathcal{H}^s_\delta\Big(\bigcup_{k=1}^\infty A_k\Big) \leq \sum_{k=1}^\infty \mathcal{H}^s_\delta(C^k_j)$$

$\square$

The pre-measure $\mathcal{H}^s_\delta$ is not $\sigma$-additive and not Borel. So we would have a measure with this properties. The idea to obtain this measure is to force the coverings that appear in the definition of the pre-measure to follow the local geometric nature of the set.

**Definition 3.1.3.** *We define the $s$-**dimensional Hausdorff measure** $\mathcal{H}^s$ on the subsets $A$ $X$ as follows*

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$$

We note that the definition above is a good definition: in fact if $\delta_1 < \delta_2$, then $\mathcal{H}^s_{\delta_1} \geq \mathcal{H}^s_{\delta_2}$, and hence the limit in the definition always exists.

**Theorem 3.1.4.** *$\mathcal{H}^s$ is a Borel regular measure, for each $0 \leq s < \infty$ and $\delta > 0$.*

*Proof.* It is clear that $\mathcal{H}^s(\emptyset) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(\emptyset) = 0$. Moreover

$$\mathcal{H}^s\Big(\bigcup_{k=1}^\infty A_k\Big) = \lim_{\delta \to 0} \mathcal{H}^s_\delta\Big(\bigcup_{k=1}^\infty A_k\Big) \leq \lim_{\delta \to 0} \sum_{k=1}^\infty \mathcal{H}^s_\delta(A_k) \leq \sum_{k=1}^\infty \mathcal{H}^s(A_k)$$

In order to prove that $\mathcal{H}^s$ is a Borel measure, we want to apply Theorem 2.1.7: so, let $A, B \subset X$ such that $d(A,B) > 0$; let

$$0 < \delta \leq \frac{d(A,B)}{3}$$

and $(C_k)_k$ such that $A \cup B \subset \bigcup_{k=1}^{\infty} C_k$, $\operatorname{diam}(C_k) \leq \delta$. We define

$$\mathcal{A} := \{k \mid A \cap C_k \neq \emptyset\}, \quad \mathcal{B} := \{k \mid B \cap C_k \neq \emptyset\}$$

Then, $A \subset \bigcup_{k \in \mathcal{A}} C_k$, $B \subset \bigcup_{k \in \mathcal{B}} C_k$, and, because of our choise of $\delta$, $\mathcal{A} \cap \mathcal{B} = \emptyset$. Hence

$$\begin{aligned} \alpha(s) \sum_{k=1}^{\infty} \Big(\frac{\operatorname{diam}(C_k)}{2}\Big)^s &\geq \alpha(s) \sum_{k \in \mathcal{A}} \Big(\frac{\operatorname{diam}(C_k)}{2}\Big)^s + \alpha(s) \sum_{k \in \mathcal{B}} \Big(\frac{\operatorname{diam}(C_k)}{2}\Big)^s \\ &\geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \end{aligned}$$

Finally, to prove the Borel regularity of $\mathcal{H}^s$, we note that $\operatorname{diam}(B) = \operatorname{diam}(\overline{B})$, and hence

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_{k=1}^{\infty} \alpha(s) \Big(\frac{\operatorname{diam}(C_k)}{2}\Big)^s \mid A \subset \bigcup_{k=1}^{\infty} C_k,\ \operatorname{diam}(C_k) \leq \delta,\ C_k \text{ closed } \}$$

Now, if $\mathcal{H}^s(A) < \infty$, we have $\mathcal{H}^s_\delta(A) < \infty$ for each $0 < \delta \leq \infty$; so we can find closed sets $B^j_k$ such that

$$\sum_{k=1}^{\infty} \alpha(s) \Big(\frac{\operatorname{diam}(B^j_k)}{2}\Big)^s \leq \mathcal{H}^s_\delta(A) + \frac{1}{j}$$

Now, setting

$$A_j := \bigcup_{k=1}^{\infty} B^j_k, \qquad B := \bigcap_{j=1}^{\infty} A_j$$

we have that $B$ is a Borel set; hence, since $\mathcal{H}^s_\delta(A_1) < \infty$, we obtain the desired result. $\square$

Now we present an property of the Hausdorff measures, useful to say when a set has measure 0.

**Theorem 3.1.5.** *If $A \subset X$ such that $\mathcal{H}^s_\delta(A) = 0$ for some $0 < \delta \leq \infty$, then $H^s(A) = 0$.*

*Proof.* Since $\mathcal{H}^s_\delta(A) = 0$ for some $0 < \delta \leq \infty$, then for each $\varepsilon > 0$ we can find sets $(C^\varepsilon_j)_j$ such that $A \subset \bigcup_{j=1}^{\infty}$, $\operatorname{diam}(C^\varepsilon_j) \leq \delta$, and

$$\sum_{j=1}^{\infty} \alpha(s) \Big(\frac{\operatorname{diam}(C^\varepsilon_j)}{2}\Big)^s \leq \varepsilon$$

Hence

$$\operatorname{diam}(C_j^\varepsilon) \leq 2\Big(\frac{\varepsilon}{\alpha(s)}\Big)^{\frac{1}{s}} =: \delta_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0$$

for each $j$. So we have obtained that the diameter of the sets $C_j^\varepsilon$ must go to 0 when $\varepsilon \to 0$. Hence

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \lim_{\varepsilon\to 0} \mathcal{H}^s_{\delta_\varepsilon}(A) = 0$$

$\square$

Next theorem links Hausdorff measures $\mathcal{H}^s$ when $s$ varies.

**Theorem 3.1.6.** *Let $A \subset X$, and $0 \leq s < t < \infty$. Then*

- *if $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$*
- *if $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$*

*Proof.* Let's prove the first assertion: let $A \subset X$ and $\delta > 0$ fixed; let $(C_j)_j$ such that $A \subset \bigcup_{j=1}^\infty C_j$, $\operatorname{diam}(C_j) \leq \delta$; since

$$\Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^t = \Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^s \Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^{t-s} \leq \Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^s \Big(\frac{\delta}{2}\Big)^{t-s}$$

We have that

$$\begin{aligned}
\mathcal{H}^t_\delta(A) &= \inf\Big\{\sum_{j=1}^\infty \alpha(t)\Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^t \mid A \subset \bigcup_{j=1}^\infty C_j,\ \operatorname{diam}(C_j) \leq \delta\Big\} \\
&\leq \frac{\alpha(t)}{\alpha(s)}\Big(\frac{\delta}{2}\Big)^{t-s} \inf\Big\{\sum_{j=1}^\infty \alpha(s)\Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^s \mid A \subset \bigcup_{j=1}^\infty C_j,\ \operatorname{diam}(C_j) \leq \delta\Big\} \\
&= \frac{\alpha(t)}{\alpha(s)}\Big(\frac{\delta}{2}\Big)^{t-s} \mathcal{H}^s(A) \overset{\delta\to 0}{\longrightarrow} 0
\end{aligned}$$

where in the last step, we have take into account that $t - s > 0$.
The second assertion is the dual of the first one. $\square$

The two properties of the above theorem suggest us how to define a notion of Hausdorff dimension of a set $A$: it will be the number $s$ for which $\mathcal{H}^s$ is the "correct" measure for measuring $A$.

**Definition 3.1.7.** *Let $A \subset X$; the* ***Hausdorff dimension*** *of $A$, $\mathcal{H}_{dim}(A)$ is defined as*

$$\mathcal{H}_{dim}(A) := \inf\{s \mid 0 \leq s \leq \infty,\ \mathcal{H}^s(A) = 0\}$$

We note that if $n = \mathcal{H}_{dim}(X)$, then for each $s > n$ $\mathcal{H}^s \equiv 0$.

We note that the construction of the Hausdorff measure, and in particular of the pre-measure, can be generalized as follows: let $(X, d)$ be a metric space, $\mathcal{F}$ a family of subsets of $X$ and $f := \mathcal{F} \to [0, \infty)$. Suppose that

- for each $\delta > 0$ there exists $(E_i)_i \subset \mathcal{F}$ such that $\mathrm{diam}(E_i) \leq \delta$ and $X = \bigcup_{i=1}^{\infty} E_i$
- for each $\delta > 0$ there exists $E \in \mathcal{F}$ such that $\mathrm{diam}(E) \leq \delta$ and $f(E) \leq \delta$

Then we can define the pre-measure

$$\psi_\delta(A) := \inf \Big\{ \sum_{i=1}^{\infty} f(C_i) \mid A \subset \bigcup_{i=1}^{\infty} C_i,\ \mathrm{diam}(C_i) \leq \delta,\ C_i \in \mathcal{F} \Big\}$$

and the measure

$$\psi(A) := \lim_{\delta \to 0} \psi_\delta(A)$$

for each $A \subset X$.
It turns out that $\psi_\delta$ is an outer measure, and that $\psi$ is a Borel regular outer measure. This way to construct a measure on a metric space is called **Carathèodory construction**.
Hence we can define the following measure

**Definition 3.1.8.** *Let $(X, d)$ be a metric space. For $0 \leq t < \infty$ define, for each $A \in X$, the* ***spherical Hausdorff measure*** *as*

$$\mathcal{S}^t(A) := \lim_{\delta \to 0} \mathcal{S}^t_\delta(A)$$

*where, for each $0 \leq \delta < \infty$*

$$\mathcal{S}^t_\delta(A) := \inf \Big\{ \sum_{i=1}^{\infty} \mathrm{diam}(C_i) \mid A \subset \bigcup_{i=1}^{\infty} C_i,\ \mathrm{diam}(C_i) \leq \delta,\ C_i \textit{ balls} \Big\}$$

It is easy to verify that, for each $A \in X$,

$$\frac{\mathcal{S}^t(A)}{2^t} \leq \mathcal{H}^t(A) \leq \mathcal{S}^t(A)$$

### 3.1.2 Densities

As we have seen in the previous chapter when we have prove the differentiation Theorem for Radon measures on $\mathbb{R}^n$, in order to understand when a measure $\mu$ can be represent in terms of another measure $\nu$ we have to look at
$$\frac{\mu(B_r(x))}{\nu(B_r(x))}$$
Since for the area formula if $S \subset \mathbb{R}^n$ is a $k$-dimensional surface, then $\mathcal{H}^k(S)$ coindices with the $k$-dimensional surface area, we have that
$$\mathcal{H}^k(B_r(x)) = \omega_k r^k$$
Hence, in order to understand when a measure $\mu$ can be represent in terms of the measure $\mathcal{H}^s$, we have to study
$$\frac{\mu(B_r(x))}{\omega_k r^k}$$
This fact suggests the following definition

**Definition 3.1.9.** *Let $(X, d)$ be a metric space, and let $\mu$ be a measure on $X$. Let $0 \leq k < \infty$ and $x \in X$; define the **upper $k$-dimensional density** of $\mu$ at $x$ as*
$$\overline{\Theta}(\mu, x) := \limsup_{r \to 0} \frac{\mu(B_r(x))}{\omega_k r^k}$$
*and the **lower $k$-dimensional density** of $\mu$ at $x$ as*
$$\underline{\Theta}(\mu, x) := \liminf_{r \to 0} \frac{\mu(B_r(x))}{\omega_k r^k}$$
*If $\overline{\Theta}(\mu, x) = \underline{\Theta}(\mu, x)$ then we called the common value the **$k$-dimensional density** of $\mu$ at $x$, and we denote it with $\Theta(\mu, x)$.*

In order to prove the foundamental result of this section, we need a kind of Vitali covering Theorem for the Hausdorff measures.

**Theorem 3.1.10.** *Suppose $(X, d)$ is a metric space, $E \subset X$, $k \geq 0$ and let $\mathcal{F}$ be a closed fine covering of $E$. Then there exists a countable disjoint subfamily $(V_i)_i \subset \mathcal{F}$ such that one of the following two conditions holds*

- $\displaystyle\sum_{i=1}^{\infty} (\mathrm{diam}(V_i))^k = +\infty$
- $\displaystyle\mathcal{H}^k\Big(E \setminus \bigcup_{i=1}^{\infty} V_i\Big) = 0$

*Proof.* Define $\mathcal{F}_0 := \mathcal{F}$ and choose $V_1 \in \mathcal{F}_0$ such that

$$\operatorname{diam}(V_1) > \frac{1}{2} \sup\{ \operatorname{diam}(V) \mid V \in \mathcal{F}_0 \}$$

Then inductively define, for $i \geq 1$

$$\mathcal{F}_i := \Big\{ V \in \mathcal{F} \mid V \cap \bigcup_{j=1}^{i} V_j = \emptyset \Big\}$$

If $\mathcal{F}_i = \emptyset$ then we stop. Else choose $V_{i+1} \in \mathcal{F}_i$ such that

$$\operatorname{diam}(V_{i+1}) > \frac{1}{2} \sup\{ \operatorname{diam}(V) \mid V \in \mathcal{F}_i \}$$

Clearly if the process is stopped, that is there is an integer $\bar{j}$ such that $\mathcal{F}_{\bar{j}} = \emptyset$, then it is obvious that

$$E \subset \bigcup_{i=1}^{\bar{j}} V_i$$

and hence the second conditions holds. Otherwise suppose that the process is not stopped and that $\sum_{i=1}^{\infty} (\operatorname{diam}(V_i))^k < +\infty$. For each $i$ select a point $x_i \in V_i$. Let $x \in E \setminus \bigcup_{i=1}^{s} V_i$ for some $s \geq 1$. Since the sets $V_i$ are closed and $\mathcal{F}$ is a fine covering of $E$, we have that there exists a set $V \in \mathcal{F}$ such that $x \in V$ and $V \cap \bigcup_{i=1}^{s} V_i = \emptyset$ and $\operatorname{diam}(V) < 2\operatorname{diam}(V_{s+1})$ (this is possible because of the way we have choosed $V_{s+1}$). Now we note that if $n > k$ and $V \cap \bigcup_{i=1}^{n} V_i = \emptyset$, then $\operatorname{diam}(V) < 2\operatorname{diam}(V_{n+1})$. Since the series of the diameters converges, then $\operatorname{diam}(V_n) \to 0$ for $n \to \infty$. Hence for $n > s$ sufficiently large we have that $V \cap V_n \neq \emptyset$. Let $n$ be the smallest integer with this property; then $\operatorname{diam}(V) < 2\operatorname{diam}(V_{n+1})$ and hence

$$d(x, x_n) \leq \operatorname{diam}(V) + \operatorname{diam}(V_n) < 3\operatorname{diam}(V_n)$$

So we have obtained that if $x \in E \setminus \bigcup_{i=1}^{s} V_i$ for some $s \geq 1$, then $x \in B_{3\operatorname{diam}(V_n)}(x_n)$ for some $n > s$. Hence, if we fix $\delta > 0$ and choose $s$ sufficiently large so that $6\operatorname{diam}(V_i) < \delta$ for each $i > s$ (this is possible because of the convergence of the series of the diameters), we have that

$$\begin{aligned}
\mathcal{H}^k_\delta\Big( B \setminus \bigcup_{i=1}^{s} V_i \Big) &\leq \mathcal{H}^s_\delta\Big( \bigcup_{i=s+1}^{\infty} B_{3\operatorname{diam}(V_i)}(x_i) \Big) \\
&\leq \sum_{i=s+1}^{\infty} \mathcal{H}^s_\delta\Big( B_{3\operatorname{diam}(V_i)}(x_i) \Big) \\
&\leq \frac{\omega_k}{2^k} \sum_{i=s+1}^{\infty} (6\operatorname{diam}(V_i))^k
\end{aligned}$$

where in the last step we have used $B_{3\mathrm{diam}(V_i)}(x_i)$ as a covering of itself. Since the series of the diameters converges, letting $k \to \infty$ we obtain the desired result. □

We have the following result

**Theorem 3.1.11.** *Let $\mu$ be a locally finite measure on a metric space $(X, d)$, and let $A$ be a Borel set of $X$. Then, for each $t \in (0, \infty)$ it hold*

$$\overline{\Theta}\mu, x \geq t \text{ for each } x \in A \;\Rightarrow\; \mu(A) \geq t\mathcal{S}^k(A) \geq t\mathcal{H}^k(A)$$

$$\underline{\Theta}\mu, x \leq t \text{ for each } x \in A \;\Rightarrow\; \mu(A) \leq t2^k\mathcal{H}^k(A)$$

*Proof.* Without loss of generaity we can suppose $t = 1$ and $A$ bounded, and clearly that $\mu(A) < \infty$.
Let's prove the first assertion: fix $0 < \delta < 1$, and let $U$ be an open bounded set such that $A \subset U$. Since $\mu(A) < \infty$ we can suppose that $\mu(U) < \infty$. Define the family

$$\mathcal{F} := \Big\{ B_r(x) \subset U \mid x \in A,\ \mathrm{diam}(B) < \delta,\ \mu(B_r(x)) \geq (1-\delta)\omega_k r^k \Big\}$$

Since the family $\mathcal{F}$ is a closed fine covering of $U$, from Theorem 3.1.10 we can find a countable disjoint family of closed balls $(B_i)_i \subset \mathcal{F}$ such that

$$\mathcal{H}^k\Big( U \setminus \bigcup_{i=1}^{\infty} B_i \Big) = 0$$

This because $\sum_{i=1}^{\infty}(\mathrm{diam}(B_i))^k \leq 2^k\mu(\bigcup_{i=1}^{\infty}(B_i)) \leq 2^k\mu(U) < \infty$. Hence

$$\mathcal{S}^k(A) \leq \sum_{i=1}^{\infty} \omega_k \mathrm{diam}(B_i)^k \leq \sum_{i=1}^{\infty} \frac{1}{1-\delta}\mu(B_i) \leq \frac{1}{1-\delta}\mu(U)$$

For the arbitrariness of $\delta$ we find out that $\mathcal{S}^k(A) \leq \mu(U)$, and hence, since $U$ is arbitrary, the desired result.
Now we prove the second assertion: let $\tau > 1$, and for $h \geq 1$ define the set

$$A_h := \Big\{ x \in A \mid \frac{\mu(B_r(x))}{\omega_k r^k} < \tau \; \forall r \in (0, \frac{1}{h}) \Big\}$$

Then $A = \bigcup_{h=1}^{\infty} A_h$ and $A_h$ is an increasing sequence; hence $\lim_{h\to\infty} \mu(A_h) = \mu(A)$. Let $(C_i)_i$ be a sequence of sets such that $\mathrm{diam}(C_i) < \frac{1}{h}$, $A_h \subset \bigcup_{i=1}^{\infty} C_i$, $\exists x_i \in A_h \cap C_i$ and

$$\sum_{i=1}^{\infty} \omega_k r_i^k < \mathcal{H}^k_{1/h}(A_h) + \frac{1}{h}$$

where $r_i := \frac{1}{2}\mathrm{diam}(C_i)$. Hence the sets $C_i' := B_{2r_i}(x_i)$ still cover $A_h$, and hence

$$\mu(A_h) \leq \sum_{i=1}^{\infty} \mu(C_i') \leq \tau \sum_{i=1}^{\infty} \omega_k (2r_i)^k < \tau 2^k \Big( \mathcal{H}^k(A) + \frac{1}{h} \Big)$$

Letting $h \to \infty$ and then $\tau \to 1$ we obtain the desired result. □

## 3.2 Hausdorff measures in $\mathbb{R}^n$

In this section we want to focus our attention on the metric space $\mathbb{R}^n$, proving some important properties relating the $s$-dimensional Hausdorff measures $\mathcal{H}^s$ and the Lebesgue measure $\mathcal{L}^n$.

### 3.2.1 Basic properties

First of all we want to study the behavior of the Hausdorff measures with respect to isometry, dilatations, and to study some first connections between Hausdorff measures and Lebesgue measure.

**Theorem 3.2.1.** *It hold:*

1. $\mathcal{H}^0$ *is the counting measure*

2. $\mathcal{H}^1 = \mathcal{L}^1$ *in* $\mathbb{R}^1$

3. $\mathcal{H}^s = 0$ *for all* $s > n$

4. $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ *for all* $\lambda > 0$ *and* $A \subset \mathbb{R}^n$

5. $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ *for every affine isometry* $L : \mathbb{R}^n \to \mathbb{R}^n$ *and* $A \subset \mathbb{R}^n$

*Proof.* 1 : since $\alpha(0) = 1$, it is clear that $\mathcal{H}^0(\{p\}) = 1$; since $\mathcal{H}^0$ is a Borel measure, points are measurable, and hence the thesis.
2 : let $A \subset \mathbb{R}^1$ and $\delta > 0$; then

$$\begin{aligned}
\mathcal{L}^1(A) &= \inf\Big\{\sum_{j=1}^{\infty} \operatorname{diam}(C_j) \mid A \subset \bigcup_{j=1}^{\infty} C_j\Big\} \\
&\le \inf\Big\{\sum_{j=1}^{\infty} \operatorname{diam}(C_j) \mid A \subset \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam}(C_j) \le \delta\Big\} \\
&= \mathcal{H}^1_\delta(A)
\end{aligned}$$

For the opposite inequality: let $A \subset \mathbb{R}^n$ sucht that $\mathcal{L}^1(A) < \infty$; fix $\varepsilon > 0$ and let $(C_j)_j$ such that $A \subset \cup_{j=1}^{\infty} C_j$ and

$$\mathcal{L}^1(A) + \varepsilon \ge \sum_{j=1}^{\infty} \operatorname{diam}(C_j)$$

For each $k \in \mathbb{Z}$ we define

$$I_k := [k\delta, (1 + k\delta)], \quad C_{j,k} := C_j \cap I_k$$

Then $\operatorname{diam}(C_j \cap I_k) \leq \delta$ for each $j, k$, and that $\operatorname{diam}(C_j) \geq \sum_{k=1}^{\infty} \operatorname{diam}(C_{j,k})$. Hence:

$$\sum_{j=1}^{\infty} \operatorname{diam}(C_j) \geq \sum_{k,j=1}^{\infty} \operatorname{diam}(C_{j,k}) \geq \mathcal{H}^1_\delta(A)$$

Since this inequality holds for each $\delta > 0$, it holds also for $\mathcal{H}^1$. Finally, since $\varepsilon$ is arbitrary, we can conclude.
3 : since $[0,1]^n$ is $\mathcal{H}^s$-measurable for each $s$, $\mathcal{H}^s$ is obviously translation invariant, and

$$\mathbb{R}^n = \bigcup_{z \in \mathbb{Z}} ([0,1]^n + z)$$

it is suffice to prove that $\mathcal{H}^s([0,1]^n) = 0$ if $s > n$. For this, let $\delta > 0$; the idea is to cover $[0,1]^n$ with cubes

$$\left[0, \frac{1}{N}\right]^n + \frac{q}{N}, \quad q \in \{0, \dots, N\}^n, \quad \frac{\sqrt{n}}{N} \leq \delta$$

In this way

$$\mathcal{H}^s_\delta([0,1]^n) \leq \sum_{i=1}^{N^n} \alpha(s) \Big(\frac{\sqrt{n}}{2N}\Big)^s = \frac{\alpha(s)(\sqrt{n})^s}{2^s} N^{n-s} \xrightarrow{\delta \to 0} 0$$

since $n - s < 0$ and $N > 1$.
4, 5 : easy:

$$\operatorname{diam}(\lambda A) = \lambda \operatorname{diam}(A), \qquad \operatorname{diam}(L(A)) = \operatorname{diam}(A)$$

for every $\lambda > 0$ and every affine transformation $L : \mathbb{R}^n \to \mathbb{R}^n$. □

**Remark 3.2.2.** *From the previous theorem it follows that, for each $A \subset \mathbb{R}^n$, $\mathcal{H}_{dim}(A) \leq n$. Moreover, it can be proved that, if $A$ is a k-dimensional submanifold, then $\mathcal{H}_{dim}(A) = k$. The converse is not true.*

### 3.2.2 Isodiametric inequality and $\mathcal{L}^n = \mathcal{H}^n$

Now we want to investigate the relation between $\mathcal{H}^n$ and $\mathcal{L}^n$. Motivated by the first two properties of Theorem 3.2.1, we might expect that $\mathcal{H}^n = \mathcal{L}^n$ for each $n \in \mathbb{N}$.
The inequality $\mathcal{H}^n \leq \mathcal{L}^n$ is proved using the fact that $\mathcal{H}^n_\delta \ll \mathcal{L}^n$ and Corollary 2.6.5. Instead, to prove the inequality $\mathcal{L}^n \leq \mathcal{H}^n$, the idea is the following: fix $\delta > 0$, if we take $A \subset \mathbb{R}^n$ and $(C_j)_j$ such that $\operatorname{diam}(C_j) \leq \delta$ and $A \subset \cup_{j=1}^\infty C_j$, then we must prove that

$$\mathcal{L}^n(A) \leq \sum_{j=1}^\infty \alpha(n)\Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^n$$

Using the monotony of $\mathcal{L}^n$, we obtain

$$\mathcal{L}^n(A) \leq \sum_{j=1}^\infty \mathcal{L}^n(C_j)$$

So we should try to prove that, for each set $C \subset \mathbb{R}^n$

$$\mathcal{L}^n(C) \leq \mathcal{L}^n\Big(B_{\frac{\operatorname{diam}(C)}{2}}\Big)$$

But $C \subset \mathbb{R}^n$ not need to be **in** $B_{\operatorname{diam}(C)/2}(x)$ for some $x \in \mathbb{R}^n$, so we can not apply directly the monotony of $\mathcal{L}^n$. This thecnical difficulty is resolved by the Steiner symmetrization.

**Notation:** Fix $a, b \in \mathbb{R}^n$ with $|a| = 1$. We define

$$L^a_b := \{b + ta \mid t \in \mathbb{R}\}$$

$$P_a := \{x \in \mathbb{R}^n \mid \langle x, a\rangle = 0\}$$

that are respectively the line through $b$ in direction $a$, and the orthogonal plane to $a$.

**Definition 3.2.3.** *Let $A \subset \mathbb{R}^n$; we define the **Steiner symmetrization** of $A$ with respect the plane $P_a$ as*

$$S_a(A) : \bigcup_{\substack{b \in P_a \\ A \cap L^a_b \neq \emptyset}} \Big\{b + ta \mid t \in \mathbb{R},\ |t| \leq \frac{1}{2}\mathcal{H}^1(A \cap L^a_b)\Big\}$$

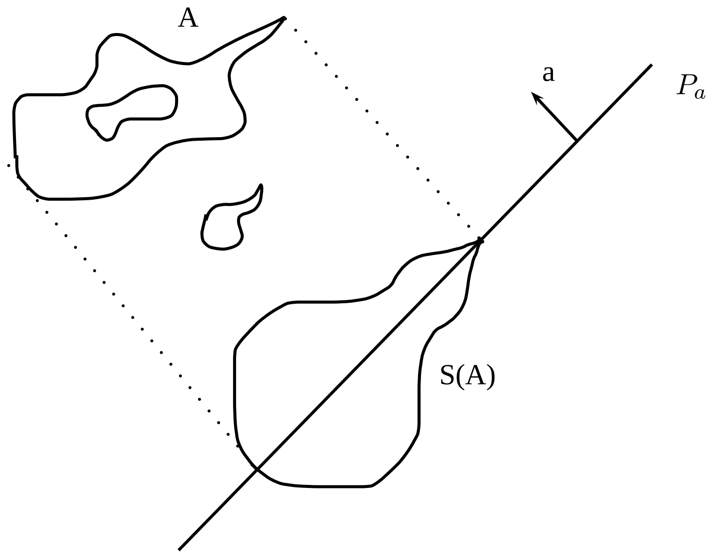



Figure 3.1: Steiner Symmetrization

Explain in words, the Steiner symmetrization works as follows: we “put ourseves ”in a point $b \in P_a$, and we look through the direction $a$; if we meet a section of the set $A$ with positive $\mathcal{H}^1 = \mathcal{L}^1$ measure, we construct a **line** in the direction of $a$, that is centered in $b$ and of lenght the lenght of the section. It is clear that the set $S_a(A)$ is symmetric with respect to $P_a$. But we also have two important properties, that are crucial for our purpose.

**Lemma 3.2.4.** *It hold:*

- $\mathrm{diam} S_a(A) \leq \mathrm{diam}(A)$
- *if $A$ is $\mathcal{L}^n$-measurable, so also $S_a(A)$ is; moreover $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$*

*Proof.* We prove $\mathrm{diam}(S_a(A)) \leq \mathrm{diam}(A)$. We may assume $\mathrm{diam}(A) < \infty$; then $A \subset B_R(0)$ for some $R > 0$, and hence $\mathcal{L}^1(L_b^a) \leq R$ for each $b \in P_a$; then $S_a(A) \subset B_R(0)$ and so $\mathrm{diam}(S_a(A)) < \infty$. Now fix $\varepsilon > 0$, and let $x, y \in P_a(A)$ suche that

$$\mathrm{diam}(S_a(A)) \leq |x - y| + \varepsilon$$

By definition of $S_a(A)$, there exist $b, c \in P_a$ such that

$$x = b + \langle x, a\rangle a, \qquad y = c + \langle y, a\rangle a$$

Then

$$|x - y|^2 \ = \ |(b - c) + (\langle x - y, a\rangle a)|^2 \ = \ |b - c|^2 + |\langle x - y, a\rangle|^2$$

where in the last step we have take into account that $P_a \perp \mathbb{R}a$, and $|a| = 1$. Now we want to estimate the last term: for this let

$$r := \sup\{t \mid b + ta \in A\}, \quad s := \inf\{t \mid b + ta \in A\}$$

$$v := \sup\{t \mid c + ta \in A\}, \quad t := \inf\{t \mid c + ta \in A\}$$

Then, if we suppose $r - t \geq v - s$

$$\begin{aligned} |\langle x, a\rangle - \langle y, a\rangle| &\leq |\langle x, a\rangle| + |\langle y, a\rangle| \leq \frac{1}{2}\mathcal{L}^1(L^a_b) + \frac{1}{2}\mathcal{L}^1(L^a_c) \\ &\leq \frac{r-s}{2} + \frac{v-t}{2} = \frac{r-t}{2} + \frac{v-s}{2} \leq r - t \end{aligned}$$

In this way we obtain that $b + ra, c + ta \in \overline{A}$, and hence

$$\begin{aligned} |\mathrm{diam}(S_a(A)) - \varepsilon|^2 &\leq |x - y|^2 \leq |b - c|^2 + |r - t|^2 \\ &= |(b + ra) - (c + ta)|^2 \leq \mathrm{diam}(\overline{A})^2 \\ &= \mathrm{diam}(A)^2 \end{aligned}$$

The second assertion follows directly from the Cavalieri's principle. □

Next theorem is of foundamental importance for two reason: first of all it will make us able to prove the inequality $\mathcal{L}^n \leq \mathcal{H}^n$; in second place it states that in the class of the sets of fixed diameter, those with maximal volume are the balls.

**Theorem 3.2.5 (Isodiametric inequality).** *For all sets $A \subset \mathbb{R}^n$ it holds*

$$\mathcal{L}^n(A) \leq \alpha(n)\Big(\frac{\mathrm{diam}(A)}{2}\Big)^n$$

*Proof.* The idea is this one: if we symmetrizing the set $A$ with respect all the principal direction, we obtain a set that is cointained in a ball with diameter less than those of $A$, and hence we can use the monodocity of the Lebesgue measure. So, let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, and define inductively

$$\overline{A}_1 := S_{e_1}(\overline{A})$$

and for $i = 2, \ldots, n$

$$\overline{A}_i := S_{e_1}(\overline{A}_{i-1})$$

We have take the closure of $A$ in order to work with $\mathcal{L}^n$-measurable sets, and hence $\mathcal{L}^n(\overline{A}) = \mathcal{L}^n(\overline{A}_i)$ for each $i$.
We now prove inductively on $i$ that $\overline{A}_i$ is symmetric with respect $P_{e_j}$ for each $j \leq i$. Let $S_j$ be the reflection through $P_{e_j}$.

- clearly $\overline{A}_1$ is symmetric with respect to $P_{e_1}$ for construction

- suppose $\overline{A}_k$ be symmetric with respect to $P_{e_1}, \dots, P_{e_k}$; for construction $\overline{A}_{k+1}$ is symmetric with rispect to $P_{e_{k+1}}$. Fix $b \in P_{e_{k+1}}$ and $1 \leq j < k+1$; since $S_j(A_k) = A_k$ we have that
$$\mathcal{L}^1(\overline{A}_k \cap L_b^{e_{k+1}}) = \mathcal{L}^1(\overline{A}_k \cap L_{S_j(b)}^{e_{k+1}})$$
and hence $S_j(A_{k+1}) = A_{k+1}$. So we have prove that, step by step, we mantain the symmetry obtain in the step before. Hence $\overline{A}_n$ is symmetric with respect to the origin. Then
$$\overline{A}_n \subset B_{\frac{\operatorname{diam}(\overline{A}_n)}{2}}(0)$$
and hence
$$\begin{aligned}\mathcal{L}^n(A) \ \leq \ \mathcal{L}^n(\overline{A}) \ &= \ \mathcal{L}^n(\overline{A}_n) \leq \alpha(n)\Big(\frac{\operatorname{diam}(\overline{A}_n)}{2}\Big)^n \\ &\leq \ \alpha(n)\Big(\frac{\operatorname{diam}(\overline{A})}{2}\Big)^n \leq \alpha(n)\Big(\frac{\operatorname{diam}(A)}{2}\Big)^n\end{aligned}$$

□

Now we are in position to prove that $\mathcal{L}^n = \mathcal{H}^n$.

**Theorem 3.2.6.** *$\mathcal{L}^n = \mathcal{H}^n$ in $\mathbb{R}^n$.*

*Proof.* First we prove the inequality $\mathcal{L}^n \leq \mathcal{H}^n$: let $A \subset \mathbb{R}^n$; fix $\delta > 0$ and let $(C_j)_j$ such that $\operatorname{diam}(C_j) \leq \delta$, $A \subset \cup_{j=1}^{\infty} C_j$; then, by the isodiametric inequality
$$\mathcal{L}^n(A) \leq \sum_{j=1}^{\infty} \mathcal{L}^n(C_j) \leq \sum_{j=1}^{\infty} \alpha(n)\Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^n$$
For the arbitrary of the sets $C_j$ we obtain $\mathcal{L}^n \leq \mathcal{H}^n_\delta$ for each $\delta > 0$, and hence $\mathcal{L}^n \leq \mathcal{H}^n$.
For the other inequality, first we need to prove that $\mathcal{H}^n_\delta \ll \mathcal{L}^n$: looking at $\mathcal{L}^n$ as a product measure, we have that, for each fixed $\delta > 0$, and for each $A \in \mathbb{R}^n$ that
$$\mathcal{L}^n(A) = \inf\Big\{ \sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i, \ \operatorname{diam}(Q_i) \leq \delta, \ Q_i \text{ cubes} \Big\}$$
and hence
$$\begin{aligned}\mathcal{H}^n_\delta(A) \ &\leq \ \inf\Big\{ \sum_{i=1}^{\infty} \alpha(n)\Big(\frac{\operatorname{diam}(Q_i)}{2}\Big)^n \mid A \subset \bigcup_{i=1}^{\infty} Q_i, \ \operatorname{diam}(Q_i) \leq \delta, \ Q_i \text{ cubes} \Big\} \\ &= \ \alpha(n)\Big(\frac{\sqrt{n}}{2}\Big)^n \inf\Big\{ \sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) \mid A \subset \bigcup_{i=1}^{\infty} Q_i, \ \operatorname{diam}(Q_i) \leq \delta, \ Q_i \text{ cubes} \Big\} \\ &= \ \alpha(n)\big(\frac{\sqrt{n}}{2}\big)^n \mathcal{L}^n(A)\end{aligned}$$

Since the result holds for each $\delta > 0$, it holds also for $\mathcal{H}^n$.
Now we can prove that $\mathcal{H}^n \le \mathcal{L}^n$: let $A \subset \mathbb{R}^n$, and fix $\delta > 0, \varepsilon > 0$; let $(Q_j)_j$ cubes such that $\operatorname{diam}(Q_j) \le \delta$, $A \subset \cup_{j=1}^{\infty} Q_j$ and

$$\sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) \le \mathcal{L}^n(A) + \varepsilon$$

By Corollary 2.6.5 we can find, for each $j$, a family of disjoint balls $(C_k^j)_k$ in $Q_j$ such that $\operatorname{diam}(C_k^j) \le \delta$ and

$$\mathcal{L}^n\Big( \overset{\circ}{Q}_j \setminus \bigcup_{k=1}^{\infty} C_k^j \Big) = 0$$

Keeping in mind that $\mathcal{H}^n_\delta \ll \mathcal{L}^n$, and that $\mathcal{H}^s_\delta$ is a Borel measure, we have that

$$\begin{aligned}
\mathcal{H}^n_\delta(A) \le \sum_{j=1}^{\infty} \mathcal{H}^n_\delta(Q_j) &= \sum_{j=1}^{\infty} \mathcal{H}^n_\delta(Q_j) \le \sum_{j=1}^{\infty} \mathcal{H}^n_\delta(\bigcup_{k=1}^{\infty} C_k^j) \\
&\le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mathcal{H}^s_\delta(C_k^j) \le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \alpha(n)\Big(\frac{\operatorname{diam}(C_k^j)}{2}\Big)^n \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mathcal{L}^n(C_k^j) = \sum_{j=1}^{\infty} \mathcal{L}^n(\bigcup_{k=1}^{\infty} C_k^j) \\
&= \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) \le \mathcal{L}^n(A) + \varepsilon
\end{aligned}$$

Since $\varepsilon$ is arbitrary we obtain that $\mathcal{H}^n_\delta \le \mathcal{L}^n$, and hence the desired result. $\square$

### 3.2.3 Densities

Since we have just proved that $\mathcal{L}^n = \mathcal{H}^n$, we know by Theorem 2.7.8 that if $E \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 1 \qquad \text{for } \mathcal{L}^n - \text{a.e. } x \in E$$

and

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 0 \qquad \text{for } \mathcal{L}^n - \text{a.e. } x \in \mathbb{R}^n \setminus E$$

Now we want to prove analogus density theorem for the lower dimensional Hausdorff measures $\mathcal{H}^s$ in $\mathbb{R}^n$.
For the points that are not in $E$ we have the following

**Theorem 3.2.7.** *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^s$-measurable, and $\mathcal{H}^s(E) < \infty$. Then*

$$\lim_{r\to 0} \frac{\mathcal{H}^s(E \cap B_r(x))}{\alpha(s) r^s} = 0$$

*for $\mathcal{H}^s$-a.e. $x \in \mathbb{R}^n \setminus E$.*

*Proof.* Let $t > 0$, and define

$$E_t := \Big\{ x \in \mathbb{R}^n \setminus E \mid \limsup_{r\to 0} \frac{\mathcal{H}^s(B_r(x) \cap E)}{\alpha(s) r^s} > t \Big\}$$

We will prove that $\mathcal{H}^s(E_t) = 0$ for all $t > 0$, from which follows the thesis. Fixed $\varepsilon > 0$, since $\mathcal{H}^s \llcorner E$ is a Radon measure, we can find a compact set $K \subset E$ such that

$$\mathcal{H}^s(E \setminus K) < \varepsilon$$

Then $U := \mathbb{R}^n \setminus E$ is an open set; set

$$\mathcal{F} := \Big\{ B_r(x) \subset U \mid r \leq \delta,\ \frac{\mathcal{H}^s(B(x, r \cap E))}{\alpha(s) r^s} > t \Big\}$$

then $\mathcal{F}$ is a covering of $U$. Thus, by Theorem 2.6.1, we can find a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B\in\mathcal{F}} \subset \bigcup_{B\in\mathcal{G}} \widehat{B}$$

Since $\operatorname{diam}(\widehat{B}) \leq 10\delta$, we have that

$$\begin{aligned}
\mathcal{H}^s_{10\delta}(E_t) &\leq \sum_{j=1}^{\infty} \alpha(s) \Big( \frac{\operatorname{diam}\widehat{B}_j}{2} \Big)^s = \alpha(s) 5^s \sum_{j=1}^{\infty} r_j^s \\
&< \frac{5^s}{t} \sum_{j=1}^{\infty} \mathcal{H}^s(B_j \cap E) = \frac{5^s}{t} \mathcal{H}^s\Big( \bigcup_{j=1}^{\infty} \cap E \Big) \leq \frac{5^s}{t} \mathcal{H}^s(E \setminus K) \\
&< \frac{5^s}{t} \varepsilon
\end{aligned}$$

For the arbitrary of $\varepsilon$ we obtain that $\mathcal{H}^s_{10\delta}(E_t) = 0$, and hence $\mathcal{H}^s(E_t) = 0$. □

But surprisingly, for the points in $E$, we have no informations on the $s$-dimensional density.

**Theorem 3.2.8.** *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^s$-measurable, and $\mathcal{H}^s(E) < \infty$. Then*

$$\frac{1}{2} \leq \limsup_{r\to 0} \frac{\mathcal{H}^s(E \cap B_r(x))}{\alpha(s) r^s} \leq 1$$

*for $\mathcal{H}^s$-a.e. $x \in E$.*

*Proof.* First we prove that

$$\limsup_{r\to 0} \frac{\mathcal{H}^s(E\cap B_r(x))}{\alpha(s)r^s} \;\le\; 1$$

Let $t > 1$, and define

$$A_t := \Big\{x \in E \mid \limsup_{r\to 0} \frac{\mathcal{H}^s(E\cap B_r(x))}{\alpha(s)r^s} \;\ge t\Big\}$$

Since $\mathcal{H}^s \llcorner E$ is a Radon measure, fixed $\epsilon > 0$ there exists an open set $U \supset A_t$ such that

$$(\mathcal{H}^s \llcorner E)(U) \le (\mathcal{H}^s \llcorner E)(A_t) + \epsilon$$

Fix $\delta > 0$, and define the family

$$\mathcal{F} := \Big\{\overline{B}(x,r) \in U \mid r \le \delta,\ \frac{\mathcal{H}^s(E\cap B_r(x))}{\alpha(s)r^s} \;\ge t\Big\}$$

Then $\mathcal{F}$ is a fine covering of $A_t$. By Corollary 2.6.4 there exists a countable family $\mathcal{G} := \{B_i\}_i$ of disjoint balls in $\mathcal{F}$ such that

$$A_t \subset \bigcup_{B\in\mathcal{G}} \widehat{B}$$

and for every $m \in \mathbb{N}$

$$A_t \setminus \bigcup_{i=1}^{m} \overline{B}_i \subset \bigcup_{i=m+1}^{\infty} \widehat{B}_i \tag{3.1}$$

Hence

$$\begin{aligned}
\mathcal{H}^s_{10\delta}(A_t) &\le \sum_{i=1}^{m} \alpha(s)\Big(\frac{\operatorname{diam}(B_j)}{2}\Big)^s + \sum_{i=m+1}^{\infty} \alpha(s)\Big(\frac{\operatorname{diam}(\widehat{B}_i)}{2}\Big)^s \\
&\le \sum_{i=1}^{m} \alpha(s) r_i^s + \sum_{i=m+1}^{\infty} \alpha(s)5^s r_i^s \\
&\le \frac{1}{t}\sum_{i=1}^{m} \mathcal{H}^s(E\cap \overline{B}_i) + \frac{5^s}{t}\sum_{i=m+1}^{\infty} \mathcal{H}^s(E\cap \overline{B}_i) \\
&= \frac{1}{t}\mathcal{H}^s\Big(E\cap \bigcup_{i=1}^{m} \overline{B}_i\Big) + \frac{5^s}{t}\mathcal{H}^s\Big(E\cap \bigcup_{i=m+1}^{\infty} \overline{B}_i\Big)
\end{aligned}$$

Now, letting $m \to \infty$ and recalling that $E \cap \bigcup_{i=m+1}^{\infty} \overline{B}_i \overset{m\to\infty}{\longrightarrow} \emptyset$, we obtain

$$\mathcal{H}^s_{10\delta}(A_t) \le \mathcal{H}^s(E\cap U) \le \frac{1}{t}(\mathcal{H}^s(A_t) + \epsilon)$$

Hence

$$\mathcal{H}^s(A_t) \leq \frac{1}{t}\mathcal{H}^s(A_t)$$

and since $t > 1$ we must have $\mathcal{H}^s(A_t) = 0$.
Finallly we note that we need a covering satisfying (3.1), because, with the simple Vitali's covering Theorem we would have had an estimate of the type $\mathcal{H}^s(A_t) \leq \frac{5^s}{t}\mathcal{H}^s(A_t)$; but $\frac{5^s}{t}$ is not greater than 1 for all $t$, and hence we couldn't have concluded that $\mathcal{H}^s(A_t) = 0$ for all $t$.

To prove the other inequality, set

$$A := \left\{x \in E \mid \limsup_{r\to 0} \frac{H^s(E \cap B_r(x))}{\alpha(s)r^s} < \frac{1}{2^s}\right\}$$

We will prove that $\mathcal{H}^s(A) = 0$. If we define, for each $k > 0$

$$B_k := \left\{x \in E \mid 2^s \frac{H^s(E \cap B_r(x))}{\alpha(s)r^s} < 1 - \frac{1}{k},\ \forall r \in \left(0, \frac{1}{k}\right]\right\}$$

we have that $A = \bigcup_{k=1}^{\infty}$; so we prove that $\mathcal{H}^s(B_k) = 0$ for all $k$.
Fix $\epsilon > 0$; then there exists $(E_j)_j$ with $r_j := \operatorname{diam}(E_j) < \frac{1}{k}$ for all $j$, and such that

$$\mathcal{H}^s(B_k) - \epsilon > \sum_{j=0}^{\infty} \frac{\alpha(s)}{2^s}{r_j}^s$$

Now let $x \in E_j \cap B_k \subset E$; from the definition of $B_k$ we have that

$$\frac{k}{k-1}\mathcal{H}^s(E \cap B(x_j, r_j)) \leq \frac{\alpha(s)r_j^s}{2^s}$$

Hence

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{\alpha(s)r_j^s}{2^s} &\geq \frac{k}{k-1}\sum_{j=0}^{\infty}\mathcal{H}^s(E \cap B(x_j, r_j)) \geq \frac{k}{k-1}\sum_{j=0}^{\infty}\mathcal{H}^s(E_j \cap B_k) \\ &\geq \frac{k}{k-1}\mathcal{H}^s\Big(B_k \cap \bigcup_{j=0}^{\infty} E_j\Big) = \frac{k}{k-1}\mathcal{H}^s(B_k) \end{aligned}$$

And hence

$$\mathcal{H}^s(B_k) - \epsilon > \frac{k}{k-1}\mathcal{H}^s(B_k$$

that is

$$\mathcal{H}^s(B_k) < (k+1)\epsilon$$

For the arbitrarity of $\epsilon$ we conclude. □

# Chapter 4

# Differentiation of Radon measures in metric spaces

The aim of this chapter is to extend the results of Section 2.6 and 2.7 to the setting of metric spaces. We will only state the principal results. In particular in Section 4.1 we extend the Vitali's covering Theorem and its corollaries to homogeneous spaces, i.e. a metric spaces endowed of a locally fintie measure $\mu$ that is sub-homogeneous. In Section 4.2 we extend the Besicovitch's covering Theorem and its corollaries to a special kind of metric spaces that generalized the property of $\mathbb{R}^n$ to have $n$ linearly indipendent directions. Thanks to these extensions of the covering theorems, we can prove differentiation theorems for Radon measures as the same spirit of those of Section 2.7 in these metric spaces.

## 4.1 Differentiation in homogeneous spaces

Vitali's covering Theorem provided a new cover from the original one enlarging the balls; we can use this covering theorem with the Lebesgue measure because Lebesgue measure is homogeneous, and hence we can controll the measure of the enlarged balls with the measure of the original balls. So, in order to extend Vitali's covering Theorem to more general metric spaces, we need an homogeneous measure on the space. Actually, since we have only to estimate the measure of the enlarged balls from above, we just need a sub-homogeneous measure. This idea is at the base of the following definitions.

**Definition 4.1.1.** *A metric space $(X, d)$ is called **doubling** if there exists a constant $C > 0$ such that every ball $B_r(\overline{x}) \subset X$ can be covered by at most*

*$C$ balls of radius $\frac{r}{2}$, i.e. there exists $x_1, \ldots, x_n \in X$ such that*

$$B_r(\overline{x}) \subset \bigcup_{i=1}^{C} B_{\frac{r}{2}}(x_i)$$

**Definition 4.1.2.** *Let $(X, d)$ be a metric space, and let $\mu$ be a measure on $X$. We say that $\mu$ is a* ***doubling measure****, and we call $(X, d, \mu)$ an* ***homogeneous metric space****, if*

- *$\mu(X) > 0$*
- *$\mu$ is locally finite*
- *there exists a constant $C_d \geq 1$ such that for each $x \in M$ and each $r > 0$*
$$\mu(B_{2r}(x)) \leq C_d \mu(B_r(x))$$

First of all we see that a doubling measure is sub-homogeneous

**Proposition 4.1.3.** *Let $(X, d, \mu)$ be an homogeneous metric space. Then for each $x \in X$ and $0 < r < R$ it holds*

$$\mu(B_R(x)) \leq C_d \Big(\frac{R}{r}\Big)^{\alpha} \mu(B_r(x))$$

*where $\alpha := \log_2 C_d$.*

The connection between the two notions given above is the following

**Lemma 4.1.4.** *Let $(X, d, \mu)$ be an homogeneous metric space. Then $(X, d)$ is doubling*

To extend the Vitali's covering theorem tohomogeneous spaces, we first need a definition.

**Definition 4.1.5.** *Let $(X, d)$ be a metric space, and let $\mu$ be a Radon measure on $X$. Let $\mathcal{F}$ be a cover of a set $A \subset X$ made by closed balls. We say that $\mathcal{F}$ cover $A$* ***in the sense of Vitali*** *if for each open set $V \subset X$ we can find a countable disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that*

$$\mu\Big((A \cap V) \setminus \bigcup_{C \in \mathcal{G}} C\Big) = 0$$

Then the following result holds

**Theorem 4.1.6.** *Let $(X, d, \mu)$ be an homogeneous metric space, and let $\mathcal{F}$ be a fine cover of a set $A \subset X$ made by closed balls. Then $\mathcal{F}$ cover $A$ in the sense of Vitali.*

Thanks to this covering theorem, that extend the Vitali's covering theorem in $\mathbb{R}^n$, we can prove analogous theorems as those in Section 2.7 for homogeneous spaces. In particular it hold:

**Lemma 4.1.7.** *Let $\mu, \nu$ be two Radon mesure on a metric space $(X, d)$, and suppose that $\nu$ is doubling. Let $0 < \alpha < \infty$. Define*

$$D^\infty(\mu, \nu) := \{ x \in X \mid \overline{D}_\nu \mu(x) = \infty \}, \quad D(\mu, \nu) := X \setminus D^\infty(\mu, \nu)$$

*Then it hold*

1. *$\nu(D^\infty(\mu, \nu)) = 0$*
2. *for each $A \subset D(\mu, \nu)$, if $\nu(A) = 0$, then $\mu(A) = 0$*
3. *if $A \subset \{ x \in X \mid \underline{D}_\nu \mu(x) \leq \alpha \}$, then $\mu(A) \leq \alpha \nu(A)$*
4. *if $A \subset \{ x \in X \mid \overline{D}_\nu \mu(x) \geq \alpha \}$, then $\mu(A) \geq \alpha \nu(A)$*

**Theorem 4.1.8.** *Let $\mu, \nu$ be Radon measures on $X$ and suppose that $\nu$ is doubling. Then $D_\nu \mu$ exists and it is finite for $\nu$-a.e. $x \in X$. Moreover the function $x \mapsto D_\nu \mu(x)$ is $\nu$-measurable.*

**Theorem 4.1.9.** *Let $\mu, \nu$ be Radon measures on $X$ and suppose that $\nu$ is doubling. Then*

$$\int_A D_\nu \mu \, d\nu \leq \mu(A)$$

*for all $\mu$-measurable $A \subset X$. The equality holds if $\mu \ll \nu$.*

## 4.2 Differentiation in metric spaces

In this section we want to extend the Besicovitch's covering theorem to metric spaces. Before doing this we have to underestand better which are the properties of $\mathbb{R}^n$ that make possible to have the thesis of the Besicovitch Theorem. The two important properties that are foundamental for the proof of the Besicovitch's covering Theorem are the following two.

**Lemma 4.2.1.** *Let $a, b \in \mathbb{R}^2$ such that $0 < |a|, |b| < |a - b|$. Then the angle between $a$ and $b$ are at least $\frac{\pi}{3}$, i.e.*

$$\left| \frac{a}{|a|} - \frac{b}{|b|} \right| \geq 1$$

*Proof.* We can suppose that $a = (|a|, 0)$. Write $b = (x_b, y_b)$. The condition $|b|^2 < |a-b|^2$ implies that $x_b \leq \frac{|a|}{2}$, while the condition $|a|^2 < |a-b|^2$ implies

$$\frac{y_b^2}{x_b^2} > 3$$

That is that angle between $a$ and $b$ are at least $\frac{\pi}{3}$, as desired. $\square$

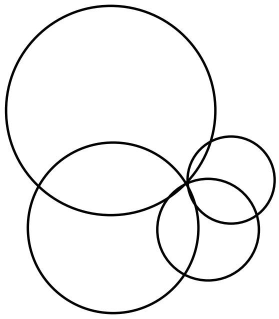


Figure 4.1: An example of balls as in Lemma 4.2.2

**Lemma 4.2.2.** *There exists a number $N = N(n)$ with the following property:*

*let $a_1, \dots, a_k \in \mathbb{R}^n$ and $r_1, \dots, r_k > 0$ be such that*

- $a_j \notin C_{r_i}(a_i)$ *if* $i \neq j$
- $\bigcap_{i=1}^k C_{r_i}(a_i) \neq \emptyset$

*Then $k < N$.*

*Proof.* Without loss of generality we can suppose that each $a_i$ is not the origin, and that

$$0 \in \bigcap_{i=1}^{k} C_{r_i}(a_i)$$

This condition implies that $|a_i| \leq r_i$, while the first condition implies that $r_i < |a_i - a_j|$ for each $i \neq j$. Hence we obtain that $|a_i| < |a_i - a_j|$ for each $i \neq j$. From the previous lemma we obtain that

$$\left| \frac{a_i}{|a_i|} - \frac{a_j}{|a_j|} \right| \geq 1$$

for each $i \neq j$. We can derive the existence of the number $N(n)$ as follows: consider the family of cover of $S^{n-1}$ made by closed balls $(C_{\frac{2}{3}}(y_i))_{i \in I}$ such that $y_i \in S^{n-1}$; since $S^{n-1}$ is compact from each of such cover $(C_{r_i}(y_i))_{i \in I}$ we can extract a finite cover $(C_{\frac{2}{3}}(y_{i_j}))_{j=1}^{n(I)}$. Let $N(n)$ be the minimum of this numbers, and select a covering $(C_{\frac{2}{3}}(y_{i_j}))_{j=1}^{N(n)}$. Now, if we take points $y_1, \dots y_k \in S^{n-1}$ such that $|y_i - y_j| \geq 1$ for each $i \neq j$, then we conclude that each of this point must be in a different ball of the covering $(C_{\frac{2}{3}}(y_{i_j}))_{j=1}^{n(I)}$. Hence $k < N(n)$ as desired. $\square$

This foundamental results are possible in $\mathbb{R}^n$ because we have $n$ linearly indipendend directions. The generalization of this property was made by Federer, and leads to the following

**Definition 4.2.3.** *Let $(X, d)$ be a metric space. We say that $d$ is $(\xi, \eta, \zeta)$-* ***directionally limited*** *in $A \subset X$ if $\xi > 0$, $0 < \eta \leq \frac{1}{3}$, $\zeta \in \mathbb{N}$, and the following property holds:*

*let $a \in A$ and $B \subset A \cap B_\xi(a) \setminus \{a\}$. If $\frac{d(x,c)}{d(a,c)} \geq \eta$ every time that $b, c \in B$ and $x \in X$ are such that $b \neq c$, $d(a, b) \geq d(a, c)$ and*

$$d(a, x) = d(a, c), \quad d(x, b) = d(a, b) - d(a, c)$$

*then* $\mathrm{Card}(B) \leq \zeta$.

We give an example of a such situation, in order to understand the terminology of the definition: let $(V, \|\cdot\|)$ be a normed linear space of finite dimension. We prove the the distance $d$ induced by the norm $\|\cdot\|$ is $(\xi, \eta, \zeta)$-directionally limited in the whole space $V$ for each $\eta > 0$ and for $\xi = +\infty$. Let $a, b, c \in V$, and define

$$x := a + \Big(\frac{\|c - a\|}{\|b - a\|}\Big)(b - a)$$

Then

$$\begin{aligned} d(x, c) &= \|x - c\| = \Big\| \frac{\|c - a\|}{\|b - a\|}(b - a) - \frac{\|c - a\|}{\|c - a\|}(c - a) \Big\| \\ &= d(a, c) \cdot d\Big(\frac{b - a}{\|b - a\|}, \frac{c - a}{\|c - a\|}\Big) \end{aligned}$$

Hence

$$\frac{d(x, c)}{d(a, c)} = d\Big(\frac{b - a}{\|b - a\|}, \frac{c - a}{\|c - a\|}\Big)$$

Now, since the vectors $\frac{b-a}{\|b-a\|}$ and $\frac{c-a}{\|c-a\|}$ belongs to the unit ball, that is compact because $V$ has finite dimension, we can find a number $\zeta$ for which the property of the definition above holds.

In this kind of metric spaces, the following two results holds

**Theorem 4.2.4 (Federer - Generalization of Besicovitch).** *Let $(X, d)$ be a metric space, and suppose that $d$ is $(\xi, \eta, \zeta)$-directionally limited in $A \subset X$. Let $0 < \rho < \frac{\xi}{2}$ and let $\mathcal{F} := \{C_r(a) \mid r < \rho\}$ be such that for each $a \in A$ there exists a ball $C_r(a) \in \mathcal{F}$. Then there exists $\mathcal{G}_1, \ldots, \mathcal{G}_{2\zeta+1} \subset \mathcal{F}$ countable disjoint families such that*

$$A \subset \bigcup_{i=1}^{2\zeta+1} \bigcup_{C \in \mathcal{G}_i} C$$

**Theorem 4.2.5.** *Let $(X, d)$ be a metric space and suppose that $d$ is $(\xi, \eta, \zeta)$-directionally limited in $A \subset X$. Let $\mathcal{F}$ be a fine cover of $A$ made of closed balls, such that for each $a \in A$ there exists a ball $C_r(a) \in \mathcal{F}$. Let $\mu$ be a Radon measure on $X$ such that $\mu(A) < \infty$. Then $\mathcal{F}$ cover $A$ in the sense of Vitali.*

These two theorems allows us to prove, for a metric space $(X, d)$ such that $d$ is $(\xi, \eta, \zeta)$-directionally limited in a subset $A \subset X$, the analougus of the theorems of Section 2.7. In particular it hold:

**Lemma 4.2.6.** *Let $(X, d)$ be a metric space that is $d$ is $(\xi, \eta, \zeta)$-directionally limited in a subset $A \subset X$. Let $\mu, \nu$ be two Radon mesure on $X$ such that $\mu(A), \nu(A) < \infty$. Let $0 < \alpha < \infty$. Define*

$$D^\infty(\mu, \nu) := \{\, x \in X \mid \overline{D}_\nu \mu(x) = \infty \,\}, \quad D(\mu, \nu) := X \setminus D^\infty(\mu, \nu)$$

*Then it hold*

1. $\nu(D^\infty(\mu, \nu)) = 0$
2. *for each $A \subset D(\mu, \nu)$, if $\nu(A) = 0$, then $\mu(A) = 0$*
3. *if $A \subset \{\, x \in X \mid \underline{D}_\nu \mu(x) \leq \alpha \,\}$, then $\mu(A) \leq \alpha \nu(A)$*
4. *if $A \subset \{\, x \in X \mid \overline{D}_\nu \mu(x) \geq \alpha \,\}$, then $\mu(A) \geq \alpha \nu(A)$*

**Theorem 4.2.7.** *Let $(X, d)$ be a metric space that is $d$ is $(\xi, \eta, \zeta)$-directionally limited in a subset $A \subset X$. Let $\mu, \nu$ be two Radon mesure on $X$ such that $\mu(A), \nu(A) < \infty$. Then $D_\nu \mu$ exists and it is finite for $\nu$-a.e. $x \in X$. Moreover the function $x \mapsto D_\nu \mu(x)$ is $\nu$-measurable.*

**Theorem 4.2.8.** *Let $(X, d)$ be a metric space that is $d$ is $(\xi, \eta, \zeta)$-directionally limited in a subset $A \subset X$. Let $\mu, \nu$ be two Radon mesure on $X$ such that $\mu(A), \nu(A) < \infty$. Then*

$$\int_A D_\nu \mu \, d\nu \leq \mu(A)$$

*for all $\mu$-measurable $A \subset X$. The equality holds if $\mu \ll \nu$.*

**Note**: in Section 12.4 we will give an example of space that is not $(\xi, \eta, \zeta)$-directionally limited. In particular in this space we can not apply the results of this section, and if we do not deal with doubling measures, we can not apply neither the results of the previous section.

# Chapter 5

# Sets of finite perimeter and $BV$ functions in $\mathbb{R}^n$

In this chapter we introduced the functions of bounded variations, and in particular the sets of finite perimeter. A function $f \in L^1(\mathbb{R}^n)$ is called of bounded variation if its distributional derivates are Radon measures on $\mathbb{R}^n$. We called a set $E \subset \mathbb{R}^n$ a set of finite perimeter if its characteristic function is of bounded variation, and we define its perimeter (or the $(n-1)$-dimensional area) as the total variation of the distributional gradient of its characteristic function. Sets of finite perimeter are the principal tool we will use to solve Bernstein probelm. In Section 5.1 we will prove some basic properties of this class of functions, while in Section 5.2 we will prove an approximation theorem for $BV$ functions (Theorem 5.2.1) and for their distributional derivates (Theorem 5.2.3). In Section 5.3 we will apply the direct method to prove the existence of minimal surfaces (Theorem 5.3.3) and, using the fact that we can approximate a bounded Caccippoli set whith smooth sets (Theorem 5.3.8), we will also prove the existence of a solution for another class of minimizing problems (Theorem 5.3.4). Finally in Section 5.4 we will prove a global and a local isoperimetric inequality (Theorem 5.4.2), that allow us to estimate the volume of a set using its perimeter.

## 5.1 Definitions and properties

To motivate the definition of this class we consider a minimal problem, the prescribed curvature problem, that allows us to find out the characterizing property of these particular sets. At the end of this chapter we will show (see Theorem 5.3.4) that the space of the sets of finite perimeter is good to apply the direct method to solve a weaker version of the prescribed curvature problem. Moreover we will prove (see Theorem 5.3.3) the existence of minimal surfaces and then, in the following chapters, we will focus on their properties and regularity. This study will allow us to solve the Bernstein Problem.

Let $E \subset \mathbb{R}^n$ be a set with $C^1$ boundary, and denote with $\sigma_{n-1}(\partial E)$ the $(n-1)$-dimensional area of $\partial E$. We consider the problem $(\mathcal{P})$:

$$\min\Big\{\sigma_{n-1}(\partial E) + \int_E f\,\mathrm{d}x \mid E \in \mathcal{R}\Big\}$$

where $f \in L^1(\mathbb{R}^n)$, and $\mathcal{R}$ is the class of the subsets of $\mathbb{R}^n$ having boundary of class $C^1$.
We note that we are obly to consider sets in the class $\mathcal{R}$ since, for arbitrary sets, we do not yet known how to define a notion of $\sigma_{n-1}(\partial E)$.
But it is note that, for sets in $\mathcal{R}$, the $(n-1)$-dimensional measure of $\partial E$ coincide with $\mathcal{H}^{n-1}(\partial E)$.

The problem $\mathcal{P}$ is called *prescibed curvature problem.* This terminology is motivated by the following fact: if $\Gamma \subset \mathbb{R}^n$ is a graph of a $C^2$ function $\phi$, we define the *average scalar curvature* $H$ of $\Gamma$

$$H(x) := -\sum_{i=1}^{\infty} \delta_i \nu_i(x)$$

where $\nu(x)$ is the normal vector to $\Gamma$ in $x$, and, if $g$ is a $C^1$ function defined in a neighborhood of $x \in \Gamma$, we denoted by $\nabla^\Gamma g(x) = (\delta_1 g(x), \ldots, \delta_n g(x))$ the projection of $\nabla g$ on the hyperplane tangential to $\Gamma$ in $x$. Since

$$\nu(x) = \frac{(-\nabla\phi(z), 1)}{\sqrt{1 + |\nabla\phi(z)|^2}}$$

we easly have

$$H(z, \phi(z)) = \operatorname{div}\Big(\frac{\nabla\phi(z)}{\sqrt{1 + |\nabla\phi(z)|^2}}\Big)$$

Now, let $E$ be a solution of the problem $(\mathcal{P})$; we can suppose that $f$ is continous in a open subset $A \subset \mathbb{R}^n$; we can also suppose that $A = D \times I$, where $D \subset \mathbb{R}^{n-1}$ and $I = (a, b)$, and that $A \cap \partial E$ is the graph of a $C^1$ function $\phi : D \to I$ such that $\inf\phi > a$ and $\sup\phi < b$; finally we suppose that $E \cap A$ is the subgraph of $\phi$ in $A$. Under this assumptions, it is easy to prove that

$$\operatorname{div}\Big(\frac{\nabla\phi(z)}{\sqrt{1 + |\nabla\phi(z)|^2}}\Big) = f(z, \phi(z))$$

in the sense of distributions.

The fact is that the class $\mathcal{R}$ is not good for searching our minimum, since in some cases[1] the minimizing set is not in $\mathcal{R}$. We want to find a "correct" space in which seraching our minimum, and to do this we reasoning as follows: let $(E_k)_k$ be a minimizing sequence for the problem $(\mathcal{P})$; we suppose[2] that $E_k$ converges locally to a set $E$. We want to find some properties of the set $E$, in order to build up our space. It is clear that, althought the sets $E_k$ have boundary of class $C^1$, the set $E$ need not to has boundary of class $C^1$. So we are searching for a property weakly than the $C^1$. Let $A \in \mathcal{R}$; define

$$F(A) := \sigma_{n-1}(A) + \int_A f \, \mathrm{d}x$$

and let

$$m := \inf \left\{ \sigma_{n-1}(\partial E) + \int_E f \, \mathrm{d}x \mid E \in \mathcal{R} \right\}$$

Then $m \in (-\infty, 0]$: in fact, if we choose $B_\varepsilon := B(0, \varepsilon)$, we obtain that

$$m \leq \sigma_{n-1}(\partial B_\varepsilon) + \int_{B_\varepsilon} f \, \mathrm{d}x = n\omega_n \varepsilon^{n-1} + \int_{E_\varepsilon} f \, \mathrm{d}x$$

and letting $\varepsilon \to 0^+$ we have that $m \leq 0$. Moreover, since

$$\sigma_{n-1}(\partial E_k) = F(E_k) - \int_{E_k} f \, \mathrm{d}x$$

we have that

$$\exists \lim_{k \to \infty} \sigma_{n-1}(\partial E_k) = m - \int_E f \, \mathrm{d}x$$

Now, since $\sigma_{n-1}(\partial E_k) \geq 0$, we have that

$$0 \leq m - \int_E f \, \mathrm{d}x \leq - \int_E f \, \mathrm{d}x$$

and hence

$$-\infty < \int_E f \, \mathrm{d}x \leq m \leq 0$$

and so $m \in (-\infty, 0]$. From this fact it follows that, for each $\varepsilon > 0$ there exists $\overline{k} \in \mathbb{N}$ such that $\forall k \geq \overline{k}$

$$\sigma_{n-1}(\partial E_k) < m - \int_E f \, \mathrm{d}x + \varepsilon \leq \|f\|_{L^1} + \varepsilon$$

Hence, using the Gauss-Green formula, we obtain

$$\int_{E_k} \mathrm{div}(\varphi) \, \mathrm{d}x = \int_{\partial E_k} \langle \varphi, \nu_k \rangle \, \mathrm{d}\sigma_{n-1} \leq \sigma_{n-1}(\partial E_h) \leq \|f\|_{L^1} + \varepsilon$$

[1] For example see [Amb97, page 6]

[2] We will see that this assumption is not restrictive, and that $E$ will be our minimum in a weak sense.

for each $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, where $\nu_k$ denote the outer normal to $\partial E_k$. Letting $k \to \infty$ we obtain

$$\int_E \operatorname{div}(\varphi)\,\mathrm{d}x \le \|f\|_{L^1}$$

for each $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$.

This property suggest the following definition

**Definition 5.1.1.** *Let $U \subset \mathbb{R}^n$ be an open set; we say that a measurable set $E \in \mathbb{R}^n$ has **finite perimeter** in $U$ if*

$$\sup\Big\{ \int_E \operatorname{div}(\varphi)\,\mathrm{d}x \mid \varphi \in C_c^1(U;\mathbb{R}^n),\ |\varphi| \le 1 \Big\} < \infty$$

*We call the number above the **perimeter of** $E$ **in** $U$, and denote it with $P(E,U)$. If a set $E$ has finite perimeter in each open bounded set, we say that $E$ is a **Caccioppoli** set.*

The above terminology is motivated by the following fact:
if $E \subset \mathbb{R}^n$ with $\chi_E \in L^1(\mathbb{R}^n)$, and with boundary of class $C^1$, using the Gauss-Green formula we have that

$$\int_E \operatorname{div}(\varphi)\,\mathrm{d}x = -\int_{\partial E} \langle \varphi, \nu \rangle\,\mathrm{d}\sigma_{n-1}$$

where $\nu$ is the outer normal to $\partial E$. Passing to the supremum we obtain that the $(n-1)$-dimensional measure of $\partial E$ coincide with what we call the perimeter of $E$. Since our definition required only the measurability of $E$ and the finiteness of the supremum above, we have in fact extended the definition of $(n-1)$-dimensional measure of $\partial E$ to a larger class of sets.

Now we want to extend the definition above to all $L^1$ functions, and not only to characteristic functions. To do this we observe that, if $\varphi \in C_c^1(U;\mathbb{R}^n)$, we have

$$\int_E \operatorname{div}(\varphi)\,\mathrm{d}x = \int_U \chi_E \operatorname{div}(\varphi)\,\mathrm{d}x$$

Hence the following definition is a natural generalization of Definition 5.1.1

**Definition 5.1.2.** *Let $U \subset \mathbb{R}^n$ be an open set; we say that $f \in L^1(\mathbb{R}^n)$ has **bounded variation** in $U$ if*

$$\sup\Big\{ \int_U f \operatorname{div}(\varphi)\,\mathrm{d}x \mid \varphi \in C_c^1(U;\mathbb{R}^n),\ |\varphi| \le 1 \Big\} < \infty$$

*We denote by $BV(U)$ the class of functions of bounded variation in $U$.*

**Remark 5.1.3.** *It is clear that E has finite perimeter in U* $\Leftrightarrow$ $\chi_E \in BV(U)$. *It is also clear that* $BV(U)$ *is a vector space.*

From the definition we easly have the following important

**Theorem 5.1.4 (Semicontinuity).** *Let* $U \subset \mathbb{R}^n$ *be an open set, and* $(f_k)_k \subset BV(U)$ *such that* $f_k \to f$ *in* $L^1_{loc}(U)$. *Then*

$$|Df|(U) \leq \liminf_{k\to\infty} |Df_k|(U)$$

*If* $\sup_k |Df_k|(U) < \infty$, *then* $f \in BV(U)$.

*Proof.* Let $\varphi \in C^1_c(U;\mathbb{R}^n)$, $|\varphi| \leq 1$; then

$$\int_U f \operatorname{div}(\varphi)\,\mathrm{d}x = \lim_{k\to\infty} \int_U f_k \operatorname{div}(\varphi)\,\mathrm{d}x \leq \liminf_{k\to\infty} |Df_k|(U)$$

Now, taking the supremum over all such $\varphi$ we obtain the desired result. $\square$

**Remark 5.1.5.** *We note that the theorem above does* ***NOT*** *says that the limit function f belongs to* $BV(U)$. *For example if we take the function*

$$u(x) := x\sin\left(\frac{1}{x}\right) \qquad x \in U := (0,\pi)$$

*and the functions*

$$u_j(x) := \begin{cases} 0 & , x \in \left(0, \frac{1}{j\pi}\right) \\ x\sin\left(\frac{1}{x}\right) & , x \in \left[\frac{1}{j\pi}, \pi\right) \end{cases}$$

*We have that the functions* $u_j \in BV(U)$, $u_j \to u$ *in* $L^1_{loc}(U)$, *but* $u \notin BV(U)$, *since (see Remark 5.1.10 for the first equality)*

$$|Du|(U) = \int_0^\pi \left|\sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)\right| \mathrm{d}x = \infty$$

*If we want to conclude, from the semicontinuity, that the function u is in* $BV(U)$ *we need to required that the functions* $u_j$ *have equibounded variation.*

**Remark 5.1.6.** *We show an example in which we have the strict inequality for the semicontinuity: let* $x \in U := (0,\pi)$ *and define* $f_j(x) := \frac{1}{j}\sin(jx)$ *and set* $f \equiv 0$. *Then* $(f_j)_j \subset C^1(U)$ *and* $f_j \to f$ *in* $L^1(U)$. *Moreover (see Remark 5.1.10 for the first equality)*

$$\int_0^\pi \mathrm{d}|Df_j| = \int_0^\pi |\cos(jx)|\,\mathrm{d}x = j\int_0^{\pi/j} |\cos(jx)|\,\mathrm{d}x = 2$$

*Hence* $(f_j)_j \subset BV(U)$ *and*

$$0 = |Df|(U) < \liminf_{j\to\infty} |Df_j|(U) = 2$$

We have called the functions in Definition 5.1.2 functions of bounded variation because of the following fact:

**Theorem 5.1.7 (Structure theorem for $BV$ functions).** *Let $f \in BV(U)$; then there exists a vector valued Radon measure $[Df]$ with values in $\mathbb{R}^n$ such that*

$$\int_U f \operatorname{div}(\varphi)\, \mathrm{d}x = -\int_U \varphi \cdot \mathrm{d}[Df] = -\int_U \langle \varphi, \sigma \rangle \, \mathrm{d}|Df|$$

*for all $\varphi \in C_c^1(U;\mathbb{R}^n)$, where $|Df|$ is the variation of the measure $[Df]$, and hence $\sigma$ is a $|Df|$-measurable function with $|\sigma(x)| = 1$ $|Df|$-a.e..*

*Proof.* We define the linear functional $L : C_c^1(U;\mathbb{R}^n) \to \mathbb{R}$ as

$$L(\varphi) := -\int_U f \operatorname{div}(\varphi)\, \mathrm{d}x$$

Since $f \in BV(U)$ we have that

$$C := \sup \Big\{ L(\varphi) \mid \varphi \in C_c^1(U;\mathbb{R}^n),\ |\varphi| \le 1 \Big\} < \infty$$

and hence

$$L(\varphi) \le \|\varphi\|_{L^\infty} C \tag{5.1}$$

Since $C_c^1(U;\mathbb{R}^n)$ is dense in $C_c(U;\mathbb{R}^n)$, we can uniquely extend the functional $L$ to a functional

$$\overline{L} : C_c(U;\mathbb{R}^n) \to \mathbb{R}$$

in this way: let $\varphi \in C_c(U;\mathbb{R}^n)$; since $\varphi$ has compact support in $U$, thanks to the smooth approximation made by the convolution we can find $(\varphi_k)_k \subset C_c^1(U;\mathbb{R}^n)$ such that $\varphi_k \to \varphi$ uniformly on $U$. By 5.1 we see that $(L(\varphi_k))_k$ is a Cauchy sequence in $\mathbb{R}$, and then we can define

$$\overline{L}(\varphi) := \lim_{k\to\infty} L(\varphi_k)$$

Using again (5.1) we note that this definition is indipendent of the choice of the sequence $(\varphi_k)_k$ converging to $\varphi$. Then, we have obtained a linear funtional $\overline{L} : C_c(U;\mathbb{R}^n) \to \mathbb{R}$ such that

$$\sup \Big\{ \overline{L}(\varphi) \mid \varphi \in C_c(U;\mathbb{R}^n),\ |\varphi| \le 1 \Big\} < \infty$$

So we can apply the Riesz Representation Theorem (Theorem 2.8.5) to obtain the desired result. ☐

**Notation:** if $E$ has finite perimeter in $U$, then we write $|\partial E|$ instead of $|D\chi_E|$, and $\nu_E$ instead of $-\sigma$.

**Remark 5.1.8.** *We reacall that, in the proof of the Riesz Representation Theorem, we defined the variation $|Df|$ of the measure $[Df]$ as*

$$|Df|(U) := \sup \left\{ \overline{L}(\varphi) \mid \varphi \in C_c(U;\mathbb{R}^n),\ |\varphi| \leq 1 \right\} < \infty$$

*In particular it holds*

$$P(E,U) = |\partial E|(U)$$

**Remark 5.1.9.** *Then the terminology* bounded variation *is referred to the fact that if $f \in BV(U)$, then the measure related to $f$ by the Riesz Representation Theorem has bounded variation. Moreover, by the identity*

$$\int_U f \operatorname{div}(\varphi)\, \mathrm{d}x = -\int_U \varphi \cdot \mathrm{d}[Df]$$

*we understand that the functions of bounded variation are the functions whose derivates, in the sense of distributions, are Radon measures. More precisely: let $f \in BV(U)$; if we define, for $i = 1,\ldots,n$*

$$\mu^i := \sigma^i |Df|$$

*by the Lebesgue Decomposition Theorem (Theorem 2.7.5) we can write*

$$\mu^i = \mu^i_{ac} + \mu^i_s$$

*where $\mu^i_{ac} \ll \mathcal{L}^n$ and $\mu^i_s \perp \mathcal{L}^n$. Hence, by the Radon-Nicodym Theorem (Theorem 2.5.12)*

$$\mu^i_{ac} = f_i \mathcal{L}^n$$

*for some $f_i \in L^1(U)$. Then, setting*

$$Df := (f_1,\ldots,f_n), \qquad [Df]_s := (\mu^1_s,\ldots,\mu^n_s)$$

*we can write*

$$[Df] = Df\, \mathcal{L}^n + [Df]_s$$

*Thus $f \in BV(U)$ belongs to $W^{1,p}(U)$ if and only if*

$$f \in L^p(U), \quad [Df]_s = 0, \quad Df \in L^p(U;\mathbb{R}^n)$$

*The main difference between Sobolev space and BV space is that in this last one we have a* singular *part of the Radon measure $[Df]$.*

**Remark 5.1.10.** *Now we present some important facts about BV functions and sets of finite perimeter, in order to understand them better.*

***Fact 1:*** *$W^{1,1}(U) \subset BV(U)$: in fact, if $f \in W^{1,1}(U)$, then $f \in L^1(U)$, and $Df \in L^1(U;\mathbb{R}^n)$; hence*

$$\int_U f \operatorname{div}(g)\,\mathrm{d}x = -\int_U \langle g, Df\rangle\,\mathrm{d}x$$

*Hence, passing to the supremum,*

$$\int_U \mathrm{d}|Df| = \int_U |Df|\,\mathrm{d}x$$

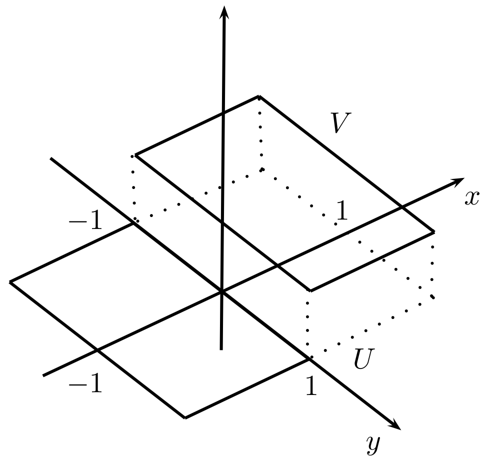
*and so*

$$\sigma := \begin{cases} \dfrac{Df}{|Df|} & Df \neq 0 \\ 0 & Df = 0 \end{cases}$$

*In particular in $f \in C^1(U)$ and $Df \in L^1(U;\mathbb{R}^n)$, then $f \in BV(U)$.*

***Fact 2:*** *The opposite inclusion does not hold. For example, let*

$$U := (-1,1)^2, \qquad V := (0,1)\times(-1,1)$$




*Then $V$ has finite perimeter in $U$, because, if $g \in C^1_c(U;\mathbb{R}^n), |g| \leq 1$, we have that*

$$\int_V \operatorname{div}(g)\,\mathrm{d}x = \int_V \Big(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2}\Big)\,\mathrm{d}x_1\mathrm{d}x_2$$

*Since $g$ has compact support in $U$, we have that*

$$\int_0^1 \mathrm{d}x_1 \int_{-1}^1 \frac{\partial g_2}{\partial x_2}\,\mathrm{d}x_2 = 0$$

*and hence, since* $|g| \le 1$,

$$\int_V \operatorname{div}(g)\,\mathrm{d}x = \int_{-1}^{1} \mathrm{d}x_2 \int_0^1 \frac{\partial g_1}{\partial x_2}\,\mathrm{d}x_1 = \int_{-1}^{1} -g_1(0,x_2)\,\mathrm{d}x_2 \le 2$$

*But* $\chi_E \notin W^{1,1}(U)$*: in fact, if for absurd* $\chi_E \in W^{1,1}(U)$*, then there exists* $f \in L^1(U)$ *such that*

$$\int_U \chi_E \frac{\partial g}{\partial x_1}\,\mathrm{d}x = -\int_U fg\,\mathrm{d}x$$

*for all* $g \in C_c^1(U)$*. Hence*

$$\int_U fg\,\mathrm{d}x = \int_{-1}^{1} \mathrm{d}x_2 \int_0^1 \frac{\partial g}{\partial x_1}\,\mathrm{d}x_1 = -\int_{-1}^{1} g(0,x_2)\,\mathrm{d}x_2 \tag{5.2}$$

*Then*

$$\|f\|_{L^1(U)} = \sup\Big\{ \int_U fg\,\mathrm{d}x \mid g \in C_c^1(U;\mathbb{R}^n),\ g_{|\{0\}\times(-1,1)} \equiv 0 \Big\} = 0$$

*Then* $f = 0$*, and hence, by (5.2) we obtain that*

$$\int_{-1}^{1} g(0,x_2)\,\mathrm{d}x_2 = 0$$

*for each* $g \in C_c^1(U)$*. Absurd.*

*The fact is that* $\chi_E$ *"jump"on a set of Lebesgue measure* $0$*, and hence with Sobolev functions, whose derivates are absolutely continous with respect to* $\mathcal{L}^n$ *we cannot measure this "jump". Hence we need the singular part of the Radon measure* $|\partial E|$ *to measure it.*

***Fact 3****: if* $E \subset \mathbb{R}^n$ *has* $C^1$ *boundary, and* $\mathcal{H}^{n-1}(E \cap U) < \infty$*, we have already seen that* $E$ *has finite perimeter in* $U$*, and*

$$|\partial E|(U) = \mathcal{H}^{n-1}(U \cap \partial E)$$

*and also*

$$\nu = \nu_E \quad \mathcal{H}^{n-1} - \text{a.e. on } \partial E \cap U$$

*where* $\nu$ *is the outer normal to* $\partial E$*.*

*But if* $E$ *has boudary not of class* $C^1$*, then* $|\partial E|(U)$ *and* $\mathcal{H}^{n-1}(U \cap \partial E)$ *can diagree violently. For example, let* $(q_j)_j$ *be an enumeration of* $\mathbb{Q}^2$*,* $B_j := B_{2^{-j}}(q_j)$*, and define*

$$E_k := \bigcup_{j=1}^{k} B_j\,, \qquad E := \bigcup_{j=1}^{\infty} B_j$$

*Since $\partial E_k$ is piecewise smooth we have that*

$$|\partial E_k|(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial E_k) \le \mathcal{H}^{n-1}\Big(\bigcup_{j=1}^{k} \partial B_j\Big) \le \frac{n\omega_{n-1}}{1-2^{-(n-1)}}$$

*Since $E_k \to E$, from the semicontinuity (see Theorem 5.1.4), we have that*

$$|\partial E| \le \liminf_{k\to\infty} |\partial E_k| < \infty$$

*But $\overline{E} = \mathbb{R}^2$, and*

$$\mathcal{L}^2(E) \le \sum_{j=1}^{\infty} \mathcal{L}^2(B_j) = \frac{4}{3}\pi$$

*Hence $\mathcal{L}^2(\partial E) = \infty$; since $\mathcal{L}^n = \mathcal{H}^n$ (see Theorem 3.1.6) we obtain that $\mathcal{H}^{n-1}(\partial E) = \infty$.*

*We note that this example also shows that $\mathcal{H}^{n-1}$ is **not** lower-semicontinous with respect the $L^1_{loc}$ convergence. This is the reason why we cannot use the measure $\mathcal{H}^{n-1}$ to solve problem $(\mathcal{P})$ with the direct method.*

***Fact 4**: if $U \subset U_1$, then $|\partial E|(U) \le |\partial E|(U_1)$, with equality holding if $E \Subset U$.*

***Fact 5**: $|\partial(E_1 \cup E_2)|(U) \le |\partial E_1|(U) + |\partial E_2|(U)$ , with equality holding when $d(E_1, E_2) > 0$.*

***Fact 6**: if $\mathcal{L}^n(E) = 0$, then $|\partial E|(\mathbb{R}^n) = 0$ ; in particular if $|E_1 \triangle E_2| = 0$, then $|\partial E_1|(\mathbb{R}^n) = |\partial E_2|(\mathbb{R}^n)$.*

***Fact 7**: it is important to note that $\operatorname{supp}|\partial E| \subset \partial E$: in fact, recalling the definition of $|\partial E|$ and of the support of a measure, we have that*

$$\operatorname{supp}|\partial E| = \mathbb{R}^n \setminus \bigcup \Big\{ A \text{ open } \mid \varphi \in C^1_c(A;\mathbb{R}^n) \Rightarrow \int_A \langle \varphi, \nu_E \rangle \, \mathrm{d}|\partial E| = 0 \Big\}$$

*Hence if $x \notin \partial E$, then there exists $r > 0$ such that $B_r(x) \subset \mathbb{R}^n \setminus E$; we have two cases:*

- *if $B_r(x) \subset \mathbb{R}^n \setminus E$, then $\chi_{E|_{B_r(x)}} \equiv 0$, and hence*

$$\int_{B_r(x)} \langle \varphi, \nu_E \rangle \, \mathrm{d}|\partial E| = \int_{B_r(x)} \chi_E \operatorname{div}(\varphi) \, \mathrm{d}x = 0$$

- *if* $B_r(x) \subset E$, *then* $\chi_{E|_{B_r(x)}} \equiv 1$, *and hence for every* $\varphi \in C^1_c(B_r(x);\mathbb{R}^n)$ *we have that*

$$\int_{B_r(x)} \langle \varphi, \nu_E \rangle \, \mathrm{d}|\partial E| = \int_{B_r(x)} \mathrm{div}(\varphi) \, \mathrm{d}x = \int_{\partial B_r(x)} \langle \varphi, \nu_{B_r(x)} \rangle \, \mathrm{d}\sigma_{n-1} = 0$$

*Hence* $|\partial E|$ *is a measure concentrated on* $\partial E$, *and then it holds*

$$\int_E \mathrm{div}(\varphi) \, \mathrm{d}x = \int_{\partial E} \varphi \cdot \nu_E \, \mathrm{d}|\partial E|$$

*a kind of Gauss-Green formula for sets of finite perimeter. We will see that this formula can be improved.*

*Moreover Caccioppoli sets are characterized by the property above: in fact if* $E \subset \mathbb{R}^n$ *is a Caccippoli set, then*

$$\int_E \mathrm{div}(\varphi) \, \mathrm{d}x = \int_{\partial E} \varphi \cdot \mathrm{d}[\partial E]$$

*for each* $\varphi \in C^1_c(U;\mathbb{R}^n)$ *and for each* $U$ *open bounded subset of* $\mathbb{R}^n$. *The converse is also true: let* $E$ *be a set such that there exists a vector valued Radon measure* $\omega$ *with locally finite total variation such that for each open bounded subset* $U$ *of* $\mathbb{R}^n$ *and for each* $\varphi \in C^1_c(U;\mathbb{R}^n)$ *it holds*

$$\int_E \mathrm{div}(\varphi) \, \mathrm{d}x = \int_{\mathbb{R}^n} \varphi \cdot \, \mathrm{d}\omega$$

*Hence, if* $|E| \leq 1$

$$\int_E \mathrm{div}(\varphi) \, \mathrm{d}x = \int_{\mathbb{R}^n} \varphi \cdot \, \mathrm{d}\omega \leq |\omega|(U) < \infty$$

*Thus* $|\partial E|(U) \leq |\omega|(U)\infty$, *and hence* $E$ *is a Caccioppoli set. Finally, tanks to the first part of this point, we have that*

$$\int_E \mathrm{div}(\varphi) \, \mathrm{d}x = \int_{\partial E} \varphi \cdot \, \mathrm{d}\omega$$

**BV as a Banach space:** we want to give to $BV(U)$ a Banach space structure. So we define, for $f \in BV(U)$

$$\|f\|_{BV(U)} := \|f\|_{L^1(U)} + |Df|(U)$$

It is clear that $\|\cdot\|_{BV(U)}$ is a norm. Moreover $(BV(U), \|\cdot\|_{BV(U)})$ is a Banach space: let $(f_k)_k$ be a Cauchy sequence; then, for every $\varepsilon > 0$ we can find $\overline{n} = \overline{n}(\varepsilon) \in \mathbb{N}$ such that, if $n, m \geq \overline{n}$ then

$$\|f_n - f_m\|_{L^1(U)} + |D(f_n - f_m)|(U) < \varepsilon$$

Hence $(f_k)_k$ is a Cauchy sequence in $L^1(U)$, and then we can find $f \in L^1(U)$ such that $f_k \to f$ in $L^1(U)$. Since $(f_k)_k$ is a Cauchy sequence in $BV(U)$, $|\partial f_k|(U)$ is bounded, and hence, by Theorem 5.1.4, $f \in BV(U)$. Now we prove that $f_k \to f$ in $BV(U)$: since we already have the $L^1$ convergence, we only need to prove that

$$|D(f_n - f)|(U) \to 0$$

To do this, we take $\varepsilon > 0$, and let $\overline{n}$ as above; then

$$|D(f_n - f_m)|(U) < \varepsilon$$

for each $n.m \geq \overline{n}$. Since $f_n - f_m \to f_n - f$ in $L^1(U)$, again from Theorem 5.1.4 we have that

$$|D(f_n - f)|(U) \leq \liminf_{m\to\infty} |D(f_n - f_m)|(U) < \varepsilon$$

And so, by the arbitrarity of $\varepsilon$ we can conclude.

## 5.2 Approximation

Now we present an important result of approximation of $BV$ functions due to Anzellotti e Giaquinta; this result allows us to transfer some properties of $C^\infty$ functions to $BV$ functions.

**Theorem 5.2.1** (**Anzellotti-Giaquinta**). *Let $f \in BV(U)$. Then there exists $(f_k)_k \subset BV(U) \cap C^\infty(U)$ such that*

$$f_k \to f \quad \textit{in } L^1(U)$$

$$|Df_k|(U) \to |Df|(U)$$

**Note:** we **do not** assert that $|D(f_k - f)|(U) \to 0$, since in this case, $(f_k)_k$ would be a Cauchy sequence in $W^{1,1}(U)$, and hence we would have that $f \in W^{1,1}(U)$. But we have seen that $BV(U) \not\subset W^{1,1}(U)$. For this reason most of the time we **do not** see $BV(U)$ as a Banach space, because $C^1(U)$ is not dense in $(BV(U), \|\cdot\|_{BV(U)})$.

*Proof.* Let $\varepsilon > 0$; then, since $|Df|(U) < \infty$, there exists an integer $m$ such that, if we define

$$U_k := \left\{ x \in U \mid d(x, \partial U) > \frac{1}{m+k} \right\}$$

we have

$$|Df|(U \setminus U_1) < \varepsilon$$

Now, set $U_0 := \emptyset$, and define for each $k > 1$

$$V_k := U_{k+1} \setminus \overline{U}_{k-1}$$

Let $(\xi_k)_k \in \mathbb{N}$ be a partition of unit subordinate to the covering $(V_k)$, that is

$$\xi_k \in C_c^\infty(V_k), \qquad 0, \leq \xi_k \leq 1, \qquad \sum_{k=1}^{\infty} \xi_k \equiv 1$$

Note that every $x \in U$ belongs at most to two sets $V_k$, and hence the summation above is finite for every $x \in U$. Let $\eta$ be a positive mollifier; then for each $k$ we can select $\varepsilon_k > 0$ such that

$$\operatorname{supp}(\eta_{\varepsilon_k} * (f\xi_k)) \subset V_k$$

$$\int_U |\eta_{\varepsilon_k} * (f\xi_k) - f\xi_k| \, \mathrm{d}x \leq \frac{\varepsilon}{2^k}$$

$$\int_U |\eta_{\varepsilon_k} * (fD\xi_k) - fD\xi_k| \, \mathrm{d}x \leq \frac{\varepsilon}{2^k}$$

We define

$$f_\varepsilon := \sum_{k=1}^{\infty} \eta_{\varepsilon_k} * (f\xi_k)$$

We have that $f_\varepsilon \in C^\infty(U)$ since in every point $x \in U$ there is a neighborhood of $x$ where it is the sum of two $C^\infty$ functions. We must prove that $f_{\varepsilon_k} \in BV(U)$ and that the the variations of the functions $f_k$ in $U$ converge to the variation of the function $f$ in $U$. Since $f_\varepsilon \to f$ in $L^1_{loc}(U)$, we have from Theorem 5.1.4 that

$$|Df|(U) \leq \liminf_{\varepsilon \to 0} |Df_{\varepsilon_k}|(U)$$

Then we need to prove that

$$\limsup_{\varepsilon \to 0} |Df_{\varepsilon_k}|(U) \leq |Df|(U)$$

To do this, we take $\varphi \in C_c^1(U;\mathbb{R}^n)$ with $|\varphi| \le 1$; then

$$
\begin{aligned}
\int_U f_\varepsilon \mathrm{div}(\varphi)\,\mathrm{d}x &= \sum_{i=1}^\infty \int_U \eta_{\varepsilon_k} * (f\xi_k)\mathrm{div}(\varphi)\,\mathrm{d}x \\
&= \sum_{i=1}^\infty \int_U \Big(\int_U \eta_{\varepsilon_k}(x-y)f(y)\xi_k(y)\mathrm{div}(\varphi)(x)\,\mathrm{d}y\Big)\,\mathrm{d}x && \text{(Fubini)} \\
&= \sum_{i=1}^\infty \int_U (\eta_{\varepsilon_k} * \mathrm{div}(\varphi))(y)\xi_k(y)f(y)\,\mathrm{d}y \\
&= \sum_{i=1}^\infty \int_U \mathrm{div}(\eta_{\varepsilon_k} * \varphi)(y)\xi_k(y)f(y)\,\mathrm{d}y && \text{(Leibnitz rule)} \\
&= \sum_{i=1}^\infty \int_U f(y)\mathrm{div}[\xi_k(\eta_{\varepsilon_k} * \varphi)]\,\mathrm{d}y - \sum_{i=1}^\infty \int_U f(y)\langle \nabla\xi_k, \eta_{\varepsilon_k} * \varphi\rangle\,\mathrm{d}x \\
&= \sum_{i=1}^\infty \int_U f(y)\mathrm{div}[\xi_k(\eta_{\varepsilon_k} * \varphi)]\,\mathrm{d}y - \sum_{i=1}^\infty \int_U f(y)\langle \varphi(y), \eta_{\varepsilon_k} * (f\nabla\xi_k)\rangle\,\mathrm{d}y \\
&= \sum_{i=1}^\infty \int_U f(y)\mathrm{div}[\xi_k(\eta_{\varepsilon_k *\varphi})]\,\mathrm{d}y - \sum_{i=1}^\infty \int_U f(y)\langle \varphi(y), \eta_{\varepsilon_k} * (f\nabla\xi_k) - f\nabla\xi_k\rangle\,\mathrm{d}y
\end{aligned}
$$

where in the last step we have take into account that $\sum_{k=1}^\infty \nabla\xi_k \equiv 0$ since $\sum_{k=1}^\infty \xi_k \equiv 1$. If we denote by $(I)$ the first integral, and by $(II)$ the second one, we have that

- for $(I)$: since $\xi_k(\eta_{\varepsilon_k} * \varphi) \in C_c^1(U;\mathbb{R}^n)$ and $|\xi_k(\eta_{\varepsilon_k} * \varphi)| \le 1$ we have that

$$
\begin{aligned}
& \sum_{i=1}^\infty \int_U f\mathrm{div}[\xi_k(\eta_{\varepsilon_k} * \varphi)]\,\mathrm{d}y \\
={}& \int_U f\mathrm{div}[\xi_1(\eta_{\varepsilon_k} * \varphi)]\,\mathrm{d}y + \sum_{i=2}^\infty \int_U f\mathrm{div}[\xi_k(\eta_{\varepsilon_k} * \varphi)]\,\mathrm{d}y \\
\le{}& |Df|(U) + \sum_{k=2}^\infty |Df|(V_k) \le |Df|(U) + 2|Df|(U - U_1) \\
\le{}& |Df|(U) + 2\varepsilon
\end{aligned}
$$

  where we have take into account that the intersection of more than two sets $V_k$ is empty.

- for $(II)$: since $|\varphi| \le 1$ we obtain that

$$|(II)| \le \sum_{k=1}^\infty \int_U |\eta_{\varepsilon_k} * (fD\xi_k) - fD\xi_k|\,\mathrm{d}x \le \varepsilon$$

Hence

$$\int_U f_\varepsilon \mathrm{div}\varphi \, \mathrm{d}x \le |Df|(U) + 3\varepsilon$$

uniformly in $\varphi$. Then, passing to the limit in $\varphi$, and for $\varepsilon \to 0$ we obtain that

$$\limsup_{\varepsilon \to 0} |Df_\varepsilon|(U) \le |Df|(U)$$

So we have obtained the desired result. □

As a consequence of the theorem above, we obtain two facts: the first one is important in the development of the trace of a $BV$ function (see Chapter 7), while the second one state that the functions given by the theorem above allow also to approximate weakly the distributional derivates.

**Corollary 5.2.2.** *Let $f$, $f_\varepsilon$ as in the theorem above. Then for each $\varepsilon > 0$, for each $N > 0$ and for each $x_0 \in \partial U$ we have*

$$\lim_{\rho \to 0} \frac{1}{\rho^N} \int_{B_\rho(x_0) \cap U} |f_\varepsilon - f| \, \mathrm{d}x = 0$$

*Proof.* Considering the construction made in the Theorem of Anzellotti-Giaquinta, if we take $\rho \le \frac{1}{m+2}$ we have that $B_\rho(x_0) \cap U_1 = \emptyset$. Now we want to see how many $V_k$'s intersect with $B_\rho(x_0)$:

$$V_k \subset V_{k+1} := \left\{ x \in U \mid d(x, \partial U) > \frac{1}{m+k+1} \right\}$$

and so

$$\frac{1}{m+k+1} > \rho > [\rho]^3$$

Then, if we take

$$k_0 := \left[\frac{1}{\rho}\right] - m - 1$$

we have that $B_\rho(x_0) \cap V_{k_0} = \emptyset$. Then

$$\begin{aligned} \int_{B_\rho(x_0) \cap U} |f_\varepsilon - f| \, \mathrm{d}x &\le \int_{B_\rho(x_0)} \sum_{k=k_0+1}^{\infty} |\eta_{\varepsilon_k} * (f\xi_k) - f\xi_k| \, \mathrm{d}x \\ &\le \sum_{k=k_0+1}^{\infty} \frac{\varepsilon}{2^k} = \frac{\varepsilon}{2^{k_0}} \end{aligned}$$

[3] For $r \in \mathbb{R}$ we define

$$[r] := \max\{n \in \mathbb{Z} \mid n \le r\}$$

and hence, recalling the definition of $k_0$,

$$\begin{aligned}\frac{1}{\rho^N}\int_{B_\rho(x_0)\cap U}|f_\varepsilon - f|\,\mathrm{d}x &\leq \frac{1}{\rho^N}\frac{\varepsilon}{2^{k_0}} < \overbrace{(k_0+m+1)}^{=:\nu}{}^N\frac{\varepsilon}{2^{k_0}}\\ &= \varepsilon\frac{\nu^N}{2^\nu}2^{m+2}\stackrel{\nu\to\infty}{\longrightarrow} 0\end{aligned}$$

□

**Theorem 5.2.3** (**Weak approximation of derivates**). *Let $(f_k)_k \subset C^\infty(U)\cap BV(U)$ be functions satisfing the thesis of the Theorem 5.2.1. Then, for each $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ we have that*

$$\lim_{k\to\infty}\int_U \varphi\cdot\,\mathrm{d}[Df_k] = \int_U \varphi\,\cdot\mathrm{d}[Df]$$

*Proof.* Let $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$; we note that if $\mathrm{supp}(\varphi)\subset U$ or $\mathrm{supp}\varphi \subset \mathbb{R}^n\setminus U$, then the result follows directly from the previous theorem. Fix $\varepsilon > 0$, and let $U_1 \Subset U$ as in the previous theorem; choose a cut-off function $\zeta \in C^\infty(U)$ such that

$$\begin{cases}\zeta \equiv 1 \text{ on } U_1, & \mathrm{supp}(\zeta)\subset U\\ 0\leq\zeta\leq 1\end{cases}$$

Then

$$\begin{aligned}\int_U\langle\varphi, Df_k\rangle\,\mathrm{d}x &= \int_U\langle\zeta\varphi Df_k\rangle\,\mathrm{d}x + \int_U\langle(1-\zeta)\varphi, Df_k\rangle\,\mathrm{d}x\\ &= -\int_U \mathrm{div}(\zeta\varphi)f_k\,\mathrm{d}x + \int_U\langle(1-\zeta)\varphi, Df_k\rangle\,\mathrm{d}x\end{aligned}$$

But

$$\begin{aligned}-\int_U\mathrm{div}(\zeta\varphi)f_k\,\mathrm{d}x &= \int_U\zeta\varphi\cdot\mathrm{d}[Df]\\ &= \int_U\varphi\cdot\mathrm{d}[Df] + \int_U(1-\zeta)\varphi\cdot\mathrm{d}[Df]\\ &\leq \int_U\varphi\cdot\mathrm{d}[Df] + \|\varphi\|_\infty|Df|(U\setminus U_1) \qquad (5.3)\\ &\leq \int_U\varphi\cdot\mathrm{d}[Df]+\varepsilon\end{aligned}$$

where in (5.3) we have take into account that $\mathrm{supp}(1-\zeta)\subset U\setminus U_1$ and that $|(1-\zeta)|_\infty \leq 1$.
Moreover, since $|Df_k|(U)\to|Df|(U)$, for $k$ big enought, we have that

$$\int_U\langle(1-\zeta)\varphi, Df_k\rangle\,\mathrm{d}x \leq \|\varphi\|_\infty|Df_k|(U\setminus U_1)\leq\|\varphi\|_\infty\varepsilon$$

So we have obtained that

$$\Big| \int_U \varphi \cdot \mathrm{d}[Df_k] \; - \; \int_U \varphi \cdot \mathrm{d}[Df] \Big| \leq 2\varepsilon \|\varphi\|_\infty$$

Letting $\varepsilon \to 0$ we obtain the desired result. $\square$

Now we present some results of the same spirit of Theorem 5.1.4.

**Theorem 5.2.4.** *Let $f, (f_j)_j \subset BV(U)$ such that $f_j \to f$ in $L^1_{loc}(U)$ and*

$$\lim_{j\to\infty} |Df_j|(U) = |Df|(U)$$

*Then for every $A \Subset U$*

$$|DF|(\overline{A} \cap U) \geq \limsup_{j\to\infty} |Df_j|(\overline{A} \cap U)$$

*In particular, if $|Df|(\partial A \cap U) = 0$ we have*

$$|Df|(A) = \lim_{j\to\infty} |Df_j|(A)$$

*Proof.* Define $B := U \setminus \overline{A}$; since $A$ and $B$ are an open sets, from the semi-continuity (see Theorem 5.1.4) it follows

$$|Df|(A) \leq \liminf_{j\to\infty} |Df_j|(A)$$

$$|Df|(B) \leq \liminf_{j\to\infty} |Df_j|(B)$$

From the other inequality

$$\begin{aligned}
|Df|(\overline{A} \cap U) + |Df|(B) &= |Df|(U) = \lim_{j\to\infty} |Df_j|(U) = \limsup_{j\to\infty} |Df_j|(U) \\
&\geq \limsup_{j\to\infty} |Df_j|(\overline{A} \cap U) + \liminf_{j\to\infty} |Df_j|(B) \\
&\geq \limsup_{j\to\infty} |Df_j|(\overline{A} \cap U) + |Df|(B)
\end{aligned}$$

Since $|Df|(B) < \infty$ we have the first assertion. The second one follows easly from the first one. $\square$

**Remark 5.2.5.** *In particular, if $f \in BV(B_R)$ we have for almost every $\rho < R$ that $|Df|(\partial B_\rho) = 0$; hence for almost every $\rho < R$ it holds*

$$\lim_{j\to\infty} |Df_j|(B_\rho) = |Df|(B_\rho)$$

**Proposition 5.2.6.** *Let $f \in BV(U)$ and $A \Subset U$ such that $|Df|(\partial A) = 0$. Then, if $f_\varepsilon := f * \eta_\varepsilon$,*

$$|Df|(A) = \lim_{\varepsilon \to 0} |Df_\varepsilon|(A)$$

*Proof.* Since $f_\varepsilon \to f$ in $L^1(U)$ we have that

$$|Df|(A) \leq \liminf_{\varepsilon \to 0} |Df_\varepsilon|(A)$$

For the opposite inequality take $\varphi \in C^1_c(U;\mathbb{R}^n)$ with $|\varphi| \leq 1$; then we have the following properties

$$\int_U f_\varepsilon \operatorname{div}(\varphi)\, \mathrm{d}x = \int_U f(\operatorname{div}(\varphi))_\varepsilon\, \mathrm{d}x = \int_U f \operatorname{div}(\varphi_\varepsilon)\, \mathrm{d}x$$

$$|\varphi| \leq 1 \Rightarrow |\varphi_\varepsilon| \leq 1$$

$$\operatorname{supp}(\varphi) \subset A \Rightarrow \operatorname{supp}(\varphi_\varepsilon) \subset A_\varepsilon := \{x \in \mathbb{R}^n \mid d(x, A) \leq \varepsilon\}$$

Hence

$$\int_U f_\varepsilon \operatorname{div}(\varphi)\, \mathrm{d}x \leq |Df|(A_\varepsilon)$$

Taking the supremum over all $\varphi$ we obtain

$$|Df_\varepsilon|(A) \leq |Df|(A_\varepsilon)$$

Hence

$$\limsup_{\varepsilon \to 0} |Df_\varepsilon|(A) \leq \lim_{\varepsilon \to 0} |Df|(A_\varepsilon) = |Df|(\overline{A})$$

where in the last step we have used the definition of the measure $|Df|$. Now, since $|Df|(\partial A) = 0$ we obtain the desired result. ☐

**Remark 5.2.7.** *If we take $f \in BV(\mathbb{R}^n)$ and $A = \mathbb{R}^n$ we obtain that*

$$|Df|(\mathbb{R}^n) = \lim_{\varepsilon \to 0} |Df_\varepsilon|(A)$$

*In particular, if $f = \chi_E$*

$$P(E) = \lim_{\varepsilon \to 0} |D(\chi_E)_\varepsilon|(\mathbb{R}^n)$$

*This is the original definition of perimeter of a set given by De Giorgi in [DG54]. Actually De Giorgi does not use our mollifiers, but the functions*

$$g_\varepsilon(y) := (\pi\varepsilon)^{-\frac{n}{2}} e^{-\frac{|y|^2}{\varepsilon}}$$

*This functions possess many of the properties of our mollifiers, and in particular it can be shown that*

$$\varepsilon \mapsto \int_U |Df_\varepsilon|\, \mathrm{d}x$$

*is a decreasing function. Hence De Giorgi defined the perimeter of a set E as*

$$P(E) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |D(\chi_E)_\varepsilon| \, \mathrm{d}x$$

*This definition coincides with our one.*

## 5.3 Existence of minimal surfaces

In this section we will prove that in the space of functions of bounded variation we can apply the direct method to solve two minimal problems: this is possible because the space $BV$ is relatively compact in $L^1$ (Theorems 5.3.2 and 5.3.5), while we have already prove the semicontinuity (Theorem 5.1.4). Having proved the existence theorems, the problem will be to prove the regularity of this minimal sets (see Chapter 9).

**Definition 5.3.1.** *We say that a Caccippoli set $E$ is a* ***minimal set*** *in $U$, or that $E$* ***has least area*** *in $U$, where $U$ is an open subset of $\mathbb{R}^n$, if for each $A \Subset U$ it holds*

$$|\partial E|(A) < \infty$$

*and*

$$|\partial E|(A) = \inf\{\, |\partial F|(A) \mid F \text{ Caccioppoli set }, E \triangle F \Subset A \,\}$$

Next result, together with Theorem 5.1.4, we will give us the existence of minimal surfaces.

**Theorem 5.3.2 (Compactness).** *Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Let $(f_k)_k \subset BV(U)$ such that*

$$\sup_k \|f_k\|_{BV(U)} < \infty$$

*Then there exists a subsequence $(f_{k_j})_j$ and a function $f \in BV(U)$ such that*

$$f_{j_k} \to f \quad \text{ in } BV(U)$$

*Proof.* For each $k$, let $g_k \in C^\infty(U)$ such that

$$\int_U |f_k - g_k| \, \mathrm{d}x < \frac{1}{k} \tag{5.4}$$

$$\int_U |Dg_k| \, \mathrm{d}x < \int_U \mathrm{d}|Df_k| + 1 \tag{5.5}$$

Such a functions $g_k$ exist by the Theorem 5.2.1. From the condition (5.5) we have that $(g_k)_k$ is bounded in $W^{1,1}(U)$. Hence, by the Rellich-Kondrachov Theorem there exists $(g_{k_j})_j$ and $f \in L^1(U)$ such that

$$g_{k_j} \to f \quad \text{ in } L^1(U)$$

Then, by (5.4) it follows that $f_{k_j} \to f \quad$ in $L^1(U)$, and hence, by Theorem 5.1.4, since $(f_k)_k$ has equibounded total variation, we obtain that $f \in BV(U)$. $\square$

Now we can easily prove the existence of minimal surfaces.

**Theorem 5.3.3** (**Existence of minimal surfaces**). *Let $U \subset \mathbb{R}^n$ be an open bounded set, and let $L \subset \mathbb{R}^n$ be a Caccioppoli set in $\mathbb{R}^n$. Then there exists $E \subset \mathbb{R}^n$ coinciding with $L$ outside $U$, and such that*

$$|\partial E|(\mathbb{R}^n) \leq |\partial F|(\mathbb{R}^n)$$

*for each set $F$ coinciding with $L$ outside $U$.*

*Proof.* Since $U$ is bounded, there exists $R > 0$ such that $U \subset B_R$; then

$$|\partial E| = |\partial E|(B_R) + |\partial E|(\mathbb{R}^n \setminus B_R)$$

Since $F = L$ outside $B_R$, we only need to prove that there exists $E \subset B_R$ coinciding with $L$ outside $U$, such that

$$|\partial E|(B_R) \leq |\partial F|(B_R)$$

for each $F \in B_R$ coinciding with $L$ outside $U$.

Let $(E_k)_k$ be a minimizing sequence; since $0 \leq |\partial F|(B_R)$, we have that $(|\partial E_k|(B_R))_k$ is uniformly bounded; moreover, since $B_R$ is bounded, also $\int_{B_R} |\chi_{E_k}| \, \mathrm{d}x$ is uniformly bounded. Then $(\chi_{E_k})_k$ is a bounded sequence in $BV(B_R)$; from the compactness theorem (Theorem 5.3.2) there exists a subsequence, still denoted by $(\chi_{E_k})_k$, and a function $f \in L^1(B_R)$, such that

$$\chi_{E_k} \to f \qquad \text{in } L^1(B_R)$$

Since $\chi_{E_k}(x) \to f(x)$ for $\mathcal{L}^n$-a.e. $x \in B_R$, we can suppose that $f$ is the characteristic function of a set $E$, coinciding with $L$ outside $U$. Finally, from the semicontinuity of the perimeter (see Theorem 5.1.4), we have that

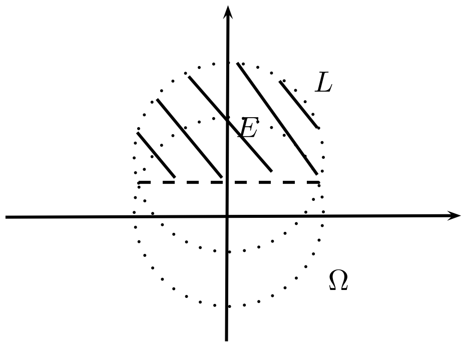
$$|\partial E|(B_R) \leq \liminf_{j \to \infty} |\partial E_{k_j}|(B_R)$$

and hence $E$ provides the required minimum. $\square$

**Note:** in some sense, the set $L$ determines the boundary values for $E$, that is, $E$ minimize the area among all surfaces with boundary $\partial L \cap \partial U$. For example, in $\mathbb{R}^2$ let

$$\Omega := B_2 \,, \quad L := \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y-1)^2 < 4\}$$

Then $E = \left\{(x, y) \in L \mid y > \frac{1}{2}\right\}$, as we can see in the figure below.




Now we want to prove that with the direct method in the $BV$ space, we can solve also a weaker form of problem $(\mathcal{P})$. So consider the problem $(\mathcal{P})^*$:

$$\min\Big\{ P(E,\mathbb{R}^n) + \int_E f(x)\,\mathrm{d}x \mid \; E \subset \mathbb{R}^n \text{ di Borel}\Big\}$$

where $f \in L^1(\mathbb{R}^n)$. We want to prove the following result

**Theorem 5.3.4.** *Problem $(\mathcal{P})^*$ has a solution. Moreover we have that*

$$\begin{aligned}&\inf\Big\{ \sigma_{n-1}(E) + \int_E f(x)\,\mathrm{d}x \mid E \in \mathcal{R} \Big\} = \\ &\min\Big\{ P(E,\mathbb{R}^n) + \int_E f(x)\,\mathrm{d}x \mid \; E \subset \mathbb{R}^n \text{ di Borel}\Big\}\end{aligned}$$

The second part of the above theorem is important because it states that the weak formulation of the problem $(\mathcal{P})$ does not decrease the value of the minimum. And this is not obvious.

To prove the theorem above we need to prove a more general compactness theorem than Theorem 5.3.2, and a theorem that allows us to approximate Caccioppoli sets with $C^\infty$ sets.

We start by proving the compactness theorem, that is of the same spirit of Theorem 5.3.2, but does not required condition on the boundary.

**Proposition 5.3.5.** *Let $\mathcal{F} \subset L^1_{loc}(U)$ be a family of functions such that*

$$\sup_{f\in\mathcal{F}} \|f\|_{BV(A)} < \infty$$

*for each $A \Subset U$. Then $\mathcal{F}$ is compact with respect to the $L^1_{loc}$ convergence.*

*Proof.* Since the convergence is metrizable, and the family $\mathcal{F}$ is clearly closed, we only need to show that every sequence $(f_j)_j \subset \mathcal{F}$ has a convergence subsequence. It is also sufficied to prove that if $K \subset U$ compact then $(f_j)_j$ has a convergence subsequence, because, in this case, we can fill up $U$ with an increasing sequence of compact sets, and hence use a diagonal process to obtain the desired result.

So let $K$ be a compact subset of $U$, and let $\delta := d(K, \partial U) > 0$; fix a convolution kernel $\eta$. For each $\varepsilon \in (0, \rho)$ let $f_j^\varepsilon := f_j * \eta_\varepsilon$. Then, for each fixed $\varepsilon$, the functions $f_j^\varepsilon$ satisfied the hypothesis of the Ascoli-Arzelá Theorem. In fact the functions $f_j^\varepsilon$ are continous and

$$\begin{aligned} |f_j^\varepsilon(y) - f_j^\varepsilon(x)| &= \Big| \int_0^1 \langle Df_j^\varepsilon, x - y \rangle \, \mathrm{d}t \Big| \leq |Df_j^\varepsilon| \cdot |x - y| \\ &\leq M \|u\|_{L^1(K)} |x - y| \leq L|x - y| \end{aligned}$$

So the family $(f_j^\varepsilon)_j$ is equi-Lipschitz, and hence the functions $f_j^\varepsilon$ are equi-uniformly continous. Hence we can find, with a diagonal process, a subsequence $(j_k)_k$ such that the sequence $(f_{j_k, \frac{1}{p}})_k$ converges uniformly in $K$ for each $p \geq 1$, and hence converge in $L^1(K)$, since $K$ is compact. Hence

$$\begin{aligned} \limsup_{k,k' \to \infty} \int_K |f_{h_k} - f_{h_{k'}}| \, \mathrm{d}x &\leq \limsup_{k,k' \to \infty} \Big[ \int_K |f_{h_k} - f_{h_k, \frac{1}{p}}| \, \mathrm{d}x + \int_K |f_{h_{k'}} - f_{h_{k'}, \frac{1}{p}}| \, \mathrm{d}x \\ &\quad + \int_K |f_{h_k, \frac{1}{p}} - f_{h_{k'}, \frac{1}{p}}| \, \mathrm{d}x \Big] \\ &\overset{(*)}{\leq} \frac{2C}{p} + \limsup_{k,k' \to \infty} \int_K |f_{h_k, \frac{1}{p}} - f_{h_{k'}, \frac{1}{p}}| \, \mathrm{d}x = \frac{2C}{p} \end{aligned}$$

where $C := \sup_k |Df_{h_k}|(\overline{K_\varepsilon}) < \infty$, because $\overline{K_\varepsilon} \Subset U$. For step $(*)$ we have used the results of the following lemma.
Since $p$ is arbitrary we can conclude that $(f_{h_k})_k$ is a Cauchy sequence in $L^1(K)$, and hence there exists a subsequence that converges in $L^1(K)$.
So we have obtained the desired result. □

**Lemma 5.3.6.** *Let $f \in BV_{loc}(U)$, $K \subset U$ be a compact set, and $\varepsilon < d(K, \partial U)$. Then*

$$\int_K |f - f_\varepsilon| \, \mathrm{d}x \leq \varepsilon |Df|(\{ x \in \mathbb{R}^n \mid d(x, K) < \varepsilon \})$$

*where $f_\varepsilon := f * \eta_\varepsilon$.*

*Proof.* We can suppose $f \in C^1(K_\varepsilon)$, where $K_\varepsilon := \{ x \in \mathbb{R}^n \mid d(x, K) < \varepsilon \}$. Let $y \in \mathbb{R}^n$ such that $|y| \leq 1$. Starting from the identity

$$f(x + \varepsilon y) - f(x) = \varepsilon \int_0^1 \frac{\partial f}{\partial y}(x + \varepsilon t y) \, \mathrm{d}t$$

we obtain

$$\begin{aligned}
&\int_K |f(x+\varepsilon y) - f(x)|\,\mathrm{d}x \le \varepsilon \int_K \mathrm{d}x \int_0^1 \Big|\frac{\partial f}{\partial y}(x+\varepsilon t y)\Big|\,\mathrm{d}t \\
= \;& \varepsilon \int_0^1 \mathrm{d}t \int_K \Big|\frac{\partial f}{\partial y}(x+\varepsilon t y)\Big|\,\mathrm{d}x = \varepsilon \int_0^1 \mathrm{d}t \int_{K-\varepsilon t y} \Big|\frac{\partial f}{\partial y}(x)\Big|\,\mathrm{d}x \\
\le \;& \varepsilon \int_0^1 \mathrm{d}t \int_{K_\varepsilon} \Big|\frac{\partial f}{\partial y}(x)\Big|\,\mathrm{d}x = \varepsilon \int_{K_\varepsilon} \Big|\frac{\partial f}{\partial y}(x)\Big|\,\mathrm{d}x \\
\le \;& \varepsilon |Df|(K_\varepsilon)
\end{aligned}$$

Hence, multipling by $\eta(y)$ and integrating over $\mathbb{R}^n$, we obtain

$$\int_K \mathrm{d}x \int_{\mathbb{R}^n} |f(x+\varepsilon y) - f(x)|\eta(y)\,\mathrm{d}y \le \varepsilon |Df|(K_\varepsilon)$$

But

$$\begin{aligned}
|f(x) - f_\varepsilon(x)| &= \Big|\int_{\mathbb{R}^n} [f(x+y) - f(x])\eta_\varepsilon(y)\,\mathrm{d}y\Big| = \Big|\int_{\mathbb{R}^n} f(x+\varepsilon y) - f(x)\eta(y)\,\mathrm{d}y\Big| \\
&\le \int_{\mathbb{R}^n} |f(x+\varepsilon y) - f(x)|\eta(y)\,\mathrm{d}y
\end{aligned}$$

and hence the conclusion. To prove the result for general $f \in BV_{loc}(U)$ the result follows by taking an approximating sequence $(f_j)_j \subset C^\infty(U) \cap BV(U)$ such as in Theorem 5.2.1. $\square$

Now we want to prove that we can approximate a Caccippoli set with $C^\infty$ sets. To do this we need the following

**Theorem 5.3.7** (**Coarea formula for** $BV$ **functions**). *Let $f \in L^1(U)$ and defined for $t \in \mathbb{R}$*

$$F_t := \{x \in U \mid f(x) > t\}$$

*Then*

$$|Df|(U) = \int_{-\infty}^{+\infty} |\partial F_t|(U)\,\mathrm{d}t$$

*In particular we obtain that if $f \in BV(U)$ $F_t$ has finite perimeter in $U$ for almost every $t$.*

*Proof.* Let $\varphi \in C_c^1(U;\mathbb{R}^n)$, $|\varphi| \le 1$; suppose $f \ge 0$; then

$$f(x) = \int_0^\infty \chi_{F_t}\,\mathrm{d}t$$

Hence

$$\begin{aligned}\int_U f\mathrm{div}(\varphi)\,\mathrm{d}x &= \int_U \mathrm{d}x \int_0^\infty \chi_{F_t}(x)\mathrm{div}(\varphi)\,\mathrm{d}t = \int_0^\infty \mathrm{d}t \int_U \mathrm{div}(\varphi)\,\mathrm{d}x \\ &= \int_0^\infty \mathrm{d}t \int_{F_t} \mathrm{div}(\varphi)\,\mathrm{d}x \le \int_0^\infty |\partial F_t|(U)\,\mathrm{d}t\end{aligned}$$

If $f \le 0$ we obtain that

$$f(x) = -\int_{-\infty}^0 (1 - \chi_{F_t}(x))\,\mathrm{d}t$$

and hence, with the same computation as above an recalling that $\int_U \mathrm{div}(\varphi)\, u dx = 0$, we have

$$\int_U f\mathrm{div}(\varphi)\,\mathrm{d}x \le \int_\infty^0 |\partial F_t|(U)\,\mathrm{d}t$$

Hence for arbitrary $f$ we have

$$\int_U f\mathrm{div}(\varphi)\,\mathrm{d} \le \int_{-\infty}^{+\infty} |\partial F_t|(U)\,\mathrm{d}t$$

Taking the supremum over all $\varphi$ we obtain

$$|Df|(U) \le \int_{-\infty}^{+\infty} |\partial F_t|(U)\,\mathrm{d}t$$

Hence we obtain that if the right-hand side is finite, then $f \in BV(U)$.

For the other inequality we can suppose $f \in BV(U)$, otherwise it is trivial. We proceed by steps.

*Step 1*: first we suppose that the formula holds for $f \in BV(U) \cap C^\infty(U)$; let$(f_k)_k \subset BV(U) \cap C^\infty(U)$ are the appoximation functions of $f$ given by the Anzellotti-Giaquianta Theorem. Hence, since $f_k \to f$ in $L^1(U)$ and

$$\int_U |f - f_k|\,\mathrm{d}x = \int_U \mathrm{d}t \int_{-\infty}^{+\infty} |\chi_{F_{kt}} - \chi_{F_t}|\,\mathrm{d}x$$

where $F_{kt} := \{x \in U \mid f_k(x) < t\}$, we obtain that there exists a subsequence, denoted again with $(f_k)_k$, such that

$$\chi_{F_{kt}} \to \chi_{F_t} \quad \text{in } L^1(U)$$

for almost all $t$. Hence

$$\begin{aligned}
|Df|(U) &= \lim_{k\to\infty}|Df_k|(U) = \lim_{k\to\infty}\int_{-\infty}^{+\infty}|\partial F_{kt}|(U)\,\mathrm{d}t \\
&= \liminf_{k\to\infty}\int_{-\infty}^{+\infty}|\partial F_{kt}|(U)\,\mathrm{d}t \geq \int_{-\infty}^{+\infty}\liminf_{k\to\infty}|\partial F_{kt}|(U)\,\mathrm{d}t \\
&\geq \int_{-\infty}^{+\infty}|\partial F_t|(U)\,\mathrm{d}t
\end{aligned}$$

and hence the desired result.

*Step 2*: now, if $f \in BV(U)\cap C^\infty(U)$, we can find a sequence $(f_j)_j$ of piecewise linear functions such that $f_j \to f$ a.e. and $|Df_j|(U) \to |Df|(U)$; if we suppose that the formula holds for this class of funcions, with the same calculation as above we prove the result for $f \in BV(U)\cap C^\infty(U)$.

*Step 3*: finally we prove the result for $f$ piecewise linear function. Write

$$U = \bigcup_{i=0}^{\infty} U_i \,\cup\, N$$

such that $U_i$ are disjoint open sets, $f(x) = \langle c_i, x\rangle + b_i$ if $x \in U_i$, where $c_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, and $\mathcal{H}^{n-1}(N) < \infty$. Then it holds

$$\int_{-\infty}^{+\infty}|\partial F_t|(U_i)\,\mathrm{d}t = \int_{U_i}|Df|\,\mathrm{d}x = c_i\mathcal{L}^n(U_i)$$

In fact, if $c_i = 0$ it is clear; if $c_i \neq 0$ let $\nu_i := \frac{c_i}{|c_i|}$; hence, since $F_t$ has piecewise smooth boundary, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty}|\partial F_t|(U_i)\,\mathrm{d}t &= \int_{-\infty}^{+\infty}\mathcal{H}^{n-1}(\{x\in U_i \mid \langle x, c_i\rangle + b_i = t\})\,\mathrm{d}t \\
&= \int_{-\infty}^{+\infty}\mathcal{H}^{n-1}(\{x\in U_i \mid \langle x, c_i\rangle = t\})\,\mathrm{d}t \\
&= |c_i|\int_{-\infty}^{+\infty}\mathcal{H}^{n-1}(\{x\in U_i \mid \langle \nu_i, x\rangle = t\})\,\mathrm{d}t \\
&= |c_i|\mathcal{L}^n(U_i)
\end{aligned}$$

where in the last step we have used the Fubini's Theorem. Now, since $\mathcal{H}^{n-1}(N) =< \infty$

$$\mathcal{H}^{n-1}(N \cap \{x \in U \mid f(x) = t\}) = 0$$

for almost all $t$, and hence

$$|\partial F_t|(U) = \mathcal{H}^{n-1}(N \cap \partial F_t) = 0$$

So

$$\int_{-\infty}^{+\infty} |\partial F_t|(U)\,\mathrm{d}t = \sum_{i=1}^{\infty} \int_{-\infty}^{+\infty} |\partial F_t|(U_i)\,\mathrm{d}t \le |Df|(U)$$

□

Now we can use this result to approximate Caccippoli sets with $C^\infty$ sets. We will use a lemma that we will prove after.

**Theorem 5.3.8.** *Let $E$ be a bounded Caccioppoli sets in $\mathbb{R}^n$. Then there exists a sequence $(E_j)_j$ of smooth sets such that*

$$E_j \to E$$

*and*

$$|\partial E_j|(\mathbb{R}^n) \to |\partial E|(\mathbb{R}^n)$$

*Proof.* From the Anzellotti-Giaquinta Theorem we known that $\chi_E$ can be approximated by a sequence of functions $f_\varepsilon := \chi_E * \eta_\varepsilon$. From the Coarea formula we have that

$$|Df_\varepsilon|(\mathbb{R}^n) = \int_0^1 |\partial E_{\varepsilon t}|(\mathbb{R}^n)\,\mathrm{d}t \tag{5.6}$$

where we have take into account that $0 \le f_\varepsilon \le 1$. But we known that

$$\lim_{\varepsilon \to 0} |Df_\varepsilon|(\mathbb{R}^n) = |Df|(\mathbb{R}^n) \tag{5.7}$$

From the next Lemma we known that if $\varepsilon_j \to 0$ for $j \to \infty$, then for each $0 < t < 1$

$$\chi_{E_{\varepsilon_j t}} \to \chi_E \quad \text{a.e. in } \mathbb{R}^n$$

Hence for the s.c.i. we obtain that

$$|\partial E|(\mathbb{R}^n) \le \liminf_{j \to \infty} |\partial E_{\varepsilon_j t}|(\mathbb{R}^n) \tag{5.8}$$

Hence

$$\begin{aligned} |\partial E|(\mathbb{R}^n) &\overset{(5.7)}{=} \lim_{j \to \infty} |Df_j|(\mathbb{R}^n) \overset{(5.3)}{=} \lim_{j \to \infty} \int_0^1 |\partial E_{\varepsilon_j t}|(\mathbb{R}^n)\,\mathrm{d}t \\ &\ge \int_0^1 \liminf_{j \to \infty} |\partial E_{\varepsilon_j t}|(\mathbb{R}^n)\,\mathrm{d}t \overset{(5.3)}{\ge} |\partial E|(\mathbb{R}^n) \end{aligned}$$

Hence for almost every $t \in (0,1)$ we have that

$$|\partial E|(\mathbb{R}^n) = \liminf_{j\to\infty} |\partial E_{\varepsilon_j t}|(\mathbb{R}^n)$$

Now, thanks to the Sard's Lemma[4], if we fix $j$ we can suppose that for almost every $t \in (0,1)$, $\partial E_{jt}$ is smooth. So there exists $t \in (0,1)$ such that, if we set $F_j := E_{\varepsilon_j t}$, $\partial F_j$ is smooth for each $j$. For such a $t$ it hold

- $\partial F_j$ smooth
- $F_j \to E$
- $|\partial E|(\mathbb{R}^n) = \liminf_{j\to\infty} |\partial F_j|(\mathbb{R}^n)$

Finally we can select a subsequence such that $|\partial E|(\mathbb{R}^n) = \lim_{j\to\infty} |\partial F_j|(\mathbb{R}^n)$, and hence obtained the desired result. □

**Lemma 5.3.9.** *Let $0 < t < 1$, and suppose $\varepsilon_j \to 0$ for $j \to \infty$; define*

$$E_{\varepsilon_j t} := \{x \in \mathbb{R}^n \mid f_{\varepsilon_j}(x) > t\}$$

*where $f_{\varepsilon j} := \eta_{\varepsilon j} * \chi_E$. Then*

$$\int_{\mathbb{R}^n} |\chi_{E_{\varepsilon_j t}} - \chi_E| \,\mathrm{d}x \le \frac{1}{\min\{t, 1-t\}} \int_{\mathbb{R}^n} |f_{\varepsilon_j} - \chi_E| \,\mathrm{d}x$$

*Proof.* By definition we have that

$$f_{\varepsilon_j} - \chi_E > t \quad \text{in } E_{\varepsilon_j t} \setminus E$$

$$\chi_E - f_{\varepsilon_j} \ge 1 - t \quad \text{in } E \setminus E_{\varepsilon_j t}$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}^n} |f_{\varepsilon_j} - \chi_E| \,\mathrm{d}x &\ge \int_{E_{\varepsilon_j t}\setminus E} |f_{\varepsilon_j} - \chi_E| \,\mathrm{d}x + \int_{E\setminus E_{jt}} |f_{\varepsilon_j} - \chi_E| \,\mathrm{d}x \\
&\ge t\mathcal{L}^n(E_{\varepsilon_j t} \setminus E) + (1-t)\mathcal{L}^n(E \setminus E_{\varepsilon_j t}) \\
&\ge \min\{t, 1-t\} \int_{\mathbb{R}^n} |\chi_{E_{\varepsilon_j t}} - \chi_E| \,\mathrm{d}x
\end{aligned}$$

□

Now we are in position to prove Theorem 5.3.4:

[4]**Theorem (Sard)**: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and let $C := \{x \in \mathbb{R}^n \mid \nabla f(x) = 0\}$. Then $\mathcal{L}^1(C) = 0$.

*Proof.* (of Theorem 5.3.4) The function

$$E \mapsto |\partial E|(\mathbb{R}^n) + \int_E f(x)\,\mathrm{d}x$$

is lower semi-continous with respect to the convergence $L^1_{loc}(\mathbb{R}^n)$ because the first term is lower semi-continous, while the second one is continous with respect to the convergence $L^1_{loc}(\mathbb{R}^n)$. The compactness follows by Theorem 5.3.5; in fact

$$|\partial E|(\mathbb{R}^n) + \int_E f(x)\,\mathrm{d}x \leq \max\{1, \|f\|_{L^1(\mathbb{R}^n)}\}\Big[|\partial E|(\mathbb{R}^n) + \int_{\mathbb{R}^n} \chi_E\,\mathrm{d}x\Big]$$

Hence if $(E_j)_j$ is a minimizing sequence of Borel sets for problem $(\mathcal{P})^*$ we have that

$$\sup\Big\{|\partial E_j|(\mathbb{R}^n) + \int_{\mathbb{R}^n} \chi_{E_j}\,\mathrm{d}x\Big\} < \infty$$

Hence if we apply Theorem 5.3.5 to the family $(\chi_{E_j})_j$ we obtain that this family is compact with respect the $L^1_{loc}$ convergence in $\mathbb{R}^n$.
So we can apply the direct method of the calculus of variation obtaining that there exists a minimum for the problem $(\mathcal{P})^*$.

For the equality of the two infima we clearly have that $\inf(\mathcal{P}) \geq \min(\mathcal{P})^*$; for the opposite inequality, from the above theorem we can approximate every ammisible set for the problem $(\mathcal{P})^*$ with sets ammissible for the problem $(\mathcal{P})$. □

Now we have to prove the regularity of the minimal sets, but first we have to understand better the structure of the sets of finite perimeter. We will do this in the following chapters.

## 5.4 Isoperimetric Inequalities

We conclude this chapter by presenting some inequalities relating the $\mathcal{L}^n$ measure of a set and its perimeter.

**Theorem 5.4.1 (Sobolev's and Poincaré's inequalities for $BV$).** *The following two facts hold:*

1. *There exists a constant $C_1 = C_1(n)$ such that*

$$\|f\|_{L^{1^*}(\mathbb{R}^n)} \leq C_1|Df|(\mathbb{R}^n)$$

*for all $f \in BV(\mathbb{R}^n)$.*

2. *There exists a constant $C_2 = C_2(n)$ such that*

$$\|f - (f)_{x,r}\|_{L^{1^*}(B_r(x))} \leq C_2 |Df|(B_r(x))$$

*for all $f \in BV(\mathbb{R}^n)$, where $(f)_{x,r} := \fint_{B_r(x)} f \,\mathrm{d}y$.*

*Proof.* (1) From the approximation Theorem 5.2.1 there exists $(f_k)_k \subset C^\infty(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$ such that

$$\|Df_k\|_{L^1} = \int_{\mathbb{R}^n} |Df_k| \,\mathrm{d}x \to \int_{\mathbb{R}^n} \mathrm{d}|Df| = |Df|(\mathbb{R}^n)$$

and

$$f_k \to f \quad \text{ in } L^1(\mathbb{R}^n)$$

and hence, possibly passing to a subsequence, $f_k \to f$ poinwise a.e.. From the Gagliardo-Nirenberg-Sobolev inequality we know that there exists a constant $C_1 = C_1(n)$ such that

$$\|f_k\|_{L^{1^*}(\mathbb{R}^n)} \leq C_1 \|Df_k\|_{L^1(\mathbb{R}^n)}$$

for each $k$. Hence

$$\liminf_{k\to\infty} \|f_k\|_{L^{1^*}(\mathbb{R}^n)} \leq C_1 \liminf_{k\to\infty} \|Df_k\|_{\mathbb{R}^n} = C_1 |Df|(\mathbb{R}^n)$$

Since

$$|f_k|^{1^*} = |f_k|^{\frac{n}{n-1}} \to |f|^{\frac{n}{n-1}} \quad \mathcal{L}^n - \text{a.e.}$$

by the Fatou's Lemma we have

$$\|f\|_{L^{1^*}} \leq \liminf_{k\to\infty} \|f_k\|_{L^{1^*}} \leq C_1 |Df|(\mathbb{R}^n)$$

(2) Again from Theorem 5.2.1 there exists $(f_k)_k \subset C^\infty(B_r(x)) \cap BV(B_r(x))$ such that

$$f_k \to f \quad \text{ in } L^1(B_r(x))$$

$$\int_{B_r(x)} |Df_k| \,\mathrm{d}x \to |Df|(B_r(x))$$

From the Poincaré inequality on balls, there exists $C_2 = C_2(n)$ such that

$$\|f_k - (f_k)_{x,r}\|_{L^{1^*}(B_r(x))} \leq C_2 \int_{B_r(x)} |Df_k| \,\mathrm{d}x$$

for each $k$. Hence

$$\liminf_{k\to\infty} \|f_k - (f_k)_{x,r}\|_{L^{1^*}(B_r(x))} \leq C_2 |Df|(B_r(x))$$

Since, possibly passing to a subsequence, $f_k \to f$ pointwise a.e., from the Fatou's Lemma follows that

$$\int_{B_r(x)} |f - (f)_{x,r}|^{1^*} \, dx \leq \liminf_{k \to \infty} \int_{B_r(x)} |f_k - (f_k)_{x,r}|^{1^*} \, dx$$

and hence the desired result. □

If we apply the previous theorem to characteristic functions of a set, we obtain the following

**Theorem 5.4.2 (Isoperimetric Inequality).** *Let $E \subset \mathbb{R}^n$ be a bounded set of finite perimeter. Let $C_1, C_2$ be the constants of the above theorem. Then*

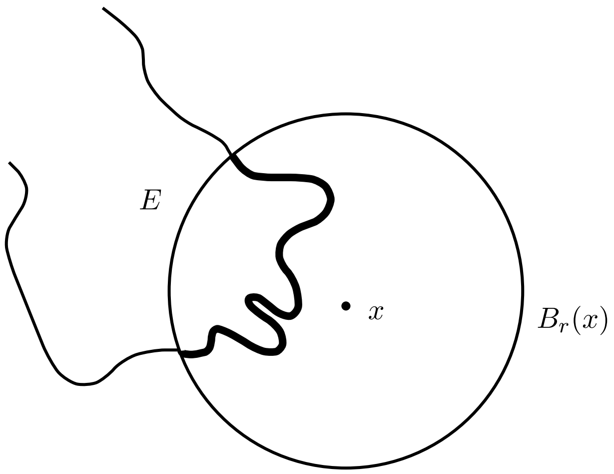
1. $\mathcal{L}^n(E)^{\frac{n-1}{n}} \leq C_1 |\partial E|(\mathbb{R}^n)$
2. *For each ball $B_r(x) \subset \mathbb{R}^n$*

$$\min\{\mathcal{L}^n(B_r(x) \cap E), \mathcal{L}^n(B_r(x) \setminus E)\}^{\frac{n-1}{n}} \leq 2C_2 |\partial E|(B_r(x))$$

**Note:** we would expect an estimate of the type

$$\mathcal{L}^n(B_r(x) \cap E)^{\frac{n-1}{n}} \leq c\Big[|\partial E|(B_r(x)) + \mathcal{H}^{n-1}(\partial B_r(x) \cap E)\Big]$$

So the estimate in the theorem is more accurate.




*Proof.* (1) Just apply point (1) of the previous theorem to $f = \chi_E$.

(2) We want to apply point (2) of the previous theorem to $f = \chi_{E \cap B_r(x)}$;, but first we have to check that $E \cap B_r(x)$ has finite perimeter in $\mathbb{R}^n$. Let $g_h$ be a smooth function such that

$$\operatorname{supp} g_h \subset B^{3h} := B_{6h}(x)$$

$$0 \leq g_h \leq 1, \qquad g_{|B} \equiv 1$$
$$|\nabla g_h| \leq h$$

Then $\chi_E g_h \to \chi_E \chi_B = \chi_{E\cap B}$ in $L^1(\mathbb{R}^n)$. Also $\chi_E g_h \in BV(\mathbb{R}^n)$: let $\varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$; then

$$\begin{aligned}
\int_{\mathbb{R}^n} \chi_E g_h \mathrm{div}(\varphi)\,\mathrm{d}x &= \int_{\mathbb{R}^n} \chi_E \mathrm{div}(g_h\varphi)\,\mathrm{d}x - \int_{\mathbb{R}^n} \chi_E \langle \nabla g_h, \varphi\rangle\,\mathrm{d}x \\
&\leq |\partial E|(\mathbb{R}^n) + \int_{\mathbb{R}^n} |\nabla g_h|\,\mathrm{d}x
\end{aligned}$$

But

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla g_h|\,\mathrm{d}x \;\leq\; h\mathcal{L}^n(B^{3h}\setminus B) &= h\alpha(n)\Big[(r+\frac{3}{h})^n - r^n\Big] && \text{(Lagrange)} \\
&= h\alpha(n)n(r+\xi)^{n-1}\Big(\frac{3}{h}\Big) && \Big(r<\xi<r+\frac{3}{h}\Big) \\
&\leq 3n\alpha(n)(r+3)^{n-1} =: c
\end{aligned}$$

Then by the semicontinuity we obtain that $\chi_E\chi_B \in BV(\mathbb{R}^n)$. So, aplling point (2) of the previous theorem, and writing $B$ for $B_r(x)$, we obtain that

$$\begin{aligned}
\|f-(f)_{x,r}\|^{1^*}_{L^{1^*}(B)} &= \int_B \Big|\chi_{E\cap B} - \frac{\mathcal{L}^n(E\cap B)}{\mathcal{L}^n(B)}\Big|^{1^*}\mathrm{d}x \\
&= \int_{B\setminus E}\Big|\frac{\mathcal{L}^n(E\cap B)}{\mathcal{L}^n(B)}\Big|^{1^*}\mathrm{d}x + \int_{B\cap E}\Big|1 - \frac{\mathcal{L}^n(E\cap B)}{\mathcal{L}^n(B)}\Big|^{1^*}\mathrm{d}x \\
&= \Big(\frac{\mathcal{L}^n(E\cap B)}{\mathcal{L}^n(B)}\Big)^{1^*}\mathcal{L}^n(B\setminus E) + \Big(\frac{\mathcal{L}^n(B\setminus E)}{\mathcal{L}^n(B)}\Big)^{1^*}\mathcal{L}^n(B\cap E) \\
&\leq \Big(\frac{\mathcal{L}^n(B\setminus E)}{\mathcal{L}^n(B)}\Big)^{1^*}\mathcal{L}^n(B\cap E)
\end{aligned}$$

Now, if we suppose $\mathcal{L}^n(B\cap E) \geq \mathcal{L}^n(B\setminus E)$ we have that

$$\begin{aligned}
\|f-(f)_{x,r}\|_{L^{1^*}(B)} &\geq \frac{\mathcal{L}^n(B\setminus E)}{\mathcal{L}^n(B)}\Big(\mathcal{L}^n(B\cap E)\Big)^{\frac{n-1}{n}} \\
&\geq \frac{1}{2}\Big(\mathcal{L}^n(B\cap E)\Big)^{\frac{n-1}{n}}
\end{aligned}$$

and hence

$$\mathcal{L}^n(B\cap E)^{\frac{n-1}{n}} \leq 2C_2|\partial E|(B)$$

□

# Chapter 6

# The Reduced boundary in $\mathbb{R}^n$

In this section we define a particular subset of the boundary of a set of finite perimeter $E$, the *reduced boundary* $\partial^* E$. This notion was introduced by De Giorgi and is the key concept of the *geometric* measure theory, that will play a foundamental role in proving the regularity of the boundary of minimizing sets. The principal result of this section is Theorem 6.3.2, that state that the reduced boundary is rettificable, i.e. $\partial^* E$ is, up to a set of zero $|\partial E|$-measure, a countable union of compact subsets of $C^1$ surfaces, and the vector $\nu_E$ assume the geometric role of the outer normal to these surfaces; moreover we will prove that the perimeter measure $|\partial E|$ is nothing else that the $\mathcal{H}^{n-1}$ measure restrict to the reduced boundary $\partial^* E$. The proof of this result uses a particular thecnique of the geometric measure theory, the *blow-up*: blowing up a set $E$ consist in exploding a set near a point of its boundary. We will prove in Theorem 6.2.1 that, in a point $x_0$ of the reduced boundary, a set $E$ of finite permiter has the same behaviour of an half space whose boundary can be consider as the *tangent plane* to the set $E$ in $x_0$. Finally, in Section 6.4 we will prove some useful properties of sets of finite perimeter we will use in the following chapaters.

## 6.1 Definition and properties

First of all we need a definition of boundary of a set that remains unchanged for sets that differ only by a set of measure zero, since we are working with equivalent classes of sets. We start with a lemma

**Lemma 6.1.1.** *Let $E \subset \mathbb{R}^n$ be a $\mathcal{L}^n$-measurable set. Then there exists a $\mathcal{L}^n$-measurable set $\widetilde{E} \subset \mathbb{R}^n$ equivalent to $E$ and such that*

$$0 < \mathcal{L}^n(\overline{E} \cap B_\rho(x)) < \omega_n \rho^n$$

*for all $x \in \partial\widetilde{E}$ and all $\rho > 0$.*

*Proof.* Define

$$E_0 := \{x \in \mathbb{R}^n \mid \text{ there exists } \rho > 0 \text{ with } \mathcal{L}^n(E \cap B_\rho(x)) = 0\}$$

$$E_1 := \{x \in \mathbb{R}^n \mid \text{ there exists } \rho > 0 \text{ with } \mathcal{L}^n(E \cap B_\rho(x)) = \omega_n \rho^n\}$$

- $E_0, E_1$ are open: let $x \in E_0$; then there exists $\rho > 0$ such that $\mathcal{L}^n(E \cap B_\rho(x)) = 0$; then, if $y \in B_\rho(x)$ and we define $r := \rho - |x - y|$ we have that $\mathcal{L}^n(E \cap B_r(y)) = 0$; hence $y \in E_0$.
  Let $x \in E_1$; then there exists $\rho > 0$ such that $\mathcal{L}^n(E \cap B_\rho(x)) = \omega_n \rho^n$, that is $\mathcal{L}^n(B_\rho(x) \setminus E) = 0$; then, if $y \in B_\rho(x)$ and we define $r := \rho - |x - y|$ we have that $\mathcal{L}^n(B_r(y) \setminus E) = 0$, and hence $\mathcal{L}^n(E \cap B_r(y)) = \omega_n r^n$; hence $y \in E_0$.

- $\mathcal{L}^n(E \cap E_0) = 0, \mathcal{L}^n(E_1 \setminus E) = 0$. For $E_0$: for each $x \in E_0$ let $\rho_x > 0$ such that $\mathcal{L}^n(E \cap B_{\rho_x}(x))) = 0$; since $\mathbb{R}^n$ is separable, we can find a countable family of points $(x_i)_i \subset E_0$ such that

  $$E_0 \subset \bigcup_{i=0}^{\infty} B_{\rho_{x_i}}(x_i)$$

  Then

  $$\mathcal{L}^n(E \cap E_0) \le \sum_{i=0}^{\infty} \mathcal{L}^n(E \cap B_{\rho_{x_i}}(x_i)) = 0$$

  For $E_1$, reasoning in the same way, we can find a countable family of points $(x_i)_i \subset E_1$ such that

  $$E_1 \subset \bigcup_{i=0}^{\infty} B_{\rho_{x_i}}(x_i)$$

  Then

  $$\mathcal{L}^n(E_1 \setminus E) \le \sum_{i=0}^{\infty} \mathcal{L}^n(B_{\rho_{x_i}}(x_i) \setminus E) = 0$$

Then, if we define $\widetilde{E} := (E \cup E_1) \setminus E_0$ we have that $\widetilde{E}$ is $\mathcal{L}^n$-measurable, $E$ and $\widetilde{E}$ are equivalent; moreover, since $E_0$ and $E_1$ are open, if $x \in \partial\widetilde{E}$, then $x \notin E_0 \cup E_1$. Hence we obtain the desired result. □

Then we can give the following definition

**Definition 6.1.2.** *Let $E$ be a $\mathcal{L}^n$-measurable set in $\mathbb{R}^n$, or better, an equaivalence class of sets. The* ***boundary*** *of $E$, still denoted with $\partial E$, is the set of points such that*

$$0 < \mathcal{L}^n(\overline{E} \cap B_\rho(x)) < \omega_n \rho^n \qquad \forall \rho > 0$$

*By the Lemma above, this is a good definition.*

**Note:** we note that, with this definition of boundary of a set, if $E$ is a set with finite permimeter in a open set $U$, then the support of the measure $|\partial E|$ coincides with $\partial E$.
We have seen in Examples 1 that the boundary of a set can have Lebesgue measure greater than 0.

**Definition 6.1.3.** *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. A point $x \in \mathbb{R}^n$ belongs to the reduced boundary of $E$, denoted by $\partial^* E$, if*

1. $|\partial E|(B_r(x)) > 0 \qquad \forall r > 0$
2. $\exists \lim_{r \to 0} \fint_{B_r(x)} \nu_E \, \mathrm{d}|\partial E| = \nu_E(x)$
3. $|\nu_E(x)| = 1$

Since $\mathrm{supp}|\partial E| \subset \partial E$, it is clear from condition (1) that $\partial^* E \subset \partial E$. Moreover, from Theorem 2.7.6 we have that

$$|\partial E|(\mathbb{R}^n \setminus \partial^* E) = |\partial E|(\partial E \setminus \partial^* E) = 0$$

So, in $|\partial E|$-measure, $\partial E$ and $\partial^* E$ are the same object.

**Example:** A simple example is when $\partial E$ is a $C^1$ hypersurface and $x \in \partial E$. We have already seen that in this case

$$\partial E = \nu \, \mathrm{d}\mathcal{H}^{n-1} \qquad \text{on } \partial E$$

where $\nu$ is the outer normal to $\partial E$. Since $\mathrm{supp}(|\partial E|) \subset \partial E$, we have that

$$\int_{B_r(x)} \mathrm{d}[\partial E] = \int_{\partial E \cap B_r(x)} \nu \, \mathrm{d}\mathcal{H}^{n-1}$$

Moreover

$$\int_{B_r(x)} \mathrm{d}|\partial E| = \mathcal{H}^{n-1}(\partial E \cap B_r(x))$$

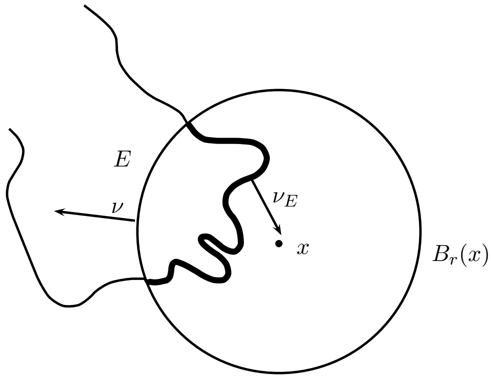
Since $\nu$ is continous on $\partial E$ we have that condition 2 of the definition above is satisfied in each point of $\partial E$. It is also clear that the other two conditions hold in every $x \in \partial E$. So, if $\partial E$ is an hypersurface, we have that $\partial^* E = \partial E$.

Now we present a divergence theorem, useful for prove some results, that we will refine later.

**Lemma 6.1.4.** *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, and let $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$ and for almost every $r > 0$ it holds:*

$$\int_{E \cap U_r(x)} \mathrm{div}(\varphi) \, \mathrm{d}x = \int_{B_r(x)} \langle \varphi, \nu_E \rangle \, \mathrm{d}|\partial E| + \int_{E \cap \partial U_r(x)} \langle \varphi, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}$$

*where $\nu$ is the outer normal to $\partial U_r(x)$.*




*Proof.* Let $h \in C^1(\mathbb{R}^n)$; then

$$\int_{\mathbb{R}^n} \langle (h\varphi), \nu_E \rangle \,\mathrm{d}|\partial E| = \int_E \mathrm{div}(h\varphi)\,\mathrm{d}y = \int_E h\mathrm{div}(\varphi)\,\mathrm{d}y + \int_E \langle \nabla h, \varphi \rangle \,\mathrm{d}y \tag{6.1}$$

Fix $\epsilon > 0$, we define the functions

$$g_\epsilon(t) := \begin{cases} 1 & , t \in [0, t] \\ 0 & , t \geq r + \epsilon \\ \dfrac{r + \epsilon - t}{\epsilon} & , r \leq t \leq r + \epsilon \end{cases}$$

and

$$h_\epsilon(y) := g_\epsilon(|x - y|)$$

Hence $h_\epsilon \in W^{1,1}(\mathbb{R}^n)$. So, taken an approximating by mollifier sequence $\eta_\sigma * h_\epsilon$ such that $\eta_\sigma * h_\epsilon \to h_\epsilon$ in $W^{1,1}(\mathbb{R}^n)$ and also uniformly[1]. Then (6.1) holds for every $\eta_\sigma * h_\epsilon$; letting $\sigma \to 0$ we obtain

$$\int_{\mathbb{R}^n} \langle (h_\epsilon\varphi), \nu_E \rangle \,\mathrm{d}|\partial E| = \int_E h_\epsilon\mathrm{div}(\varphi)\,\mathrm{d}y + \int_E \langle \nabla h_\epsilon, \varphi \rangle \,\mathrm{d}y$$

Now, letting $\epsilon \to 0$ we obtain

$$\int_{B_r(x)} \mathrm{div}(\varphi)\,\mathrm{d}y = \int_{B_r(x)} \langle \varphi, \nu_E \rangle \,\mathrm{d}|\partial E| + \lim_{\epsilon \to 0} \int_E \langle \varphi, \nabla h_\epsilon \rangle \,\mathrm{d}y \tag{6.2}$$

Since

$$\nabla h_\epsilon(y) = \begin{cases} 0 & , y \notin B_{r+\epsilon}(x) \setminus U_r(x) \\ -\dfrac{1}{\epsilon}\dfrac{y - x}{|y - x|} & , \text{otherwise} \end{cases}$$

if we define

$$F(r) := \int_{U_r(x)} \chi_E(y) \langle \varphi(y), \frac{y - x}{|y - x|} \rangle \,\mathrm{d}y = \int_{U_r(x)} \chi_E(y) \langle \varphi(y) \nu(y) \langle \,\mathrm{d}y$$

[1]This can be done since $h_\epsilon$ is continous and has compact support.

from the Coarea Formula (see [EG92], Chapter 3) we have that

$$F(r) = \int_0^r \Big( \int_{\partial U_\rho(x)} \chi_E \langle \varphi, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}(y) \Big) \, \mathrm{d}\rho$$

Hence, for almost every $r > 0$ we have that

$$\exists \;\; F'(r) = \int_{E \cap \partial U_r} \langle \varphi, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}$$

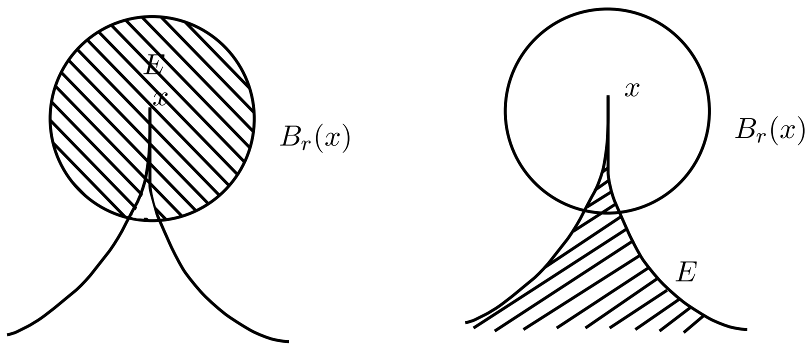
Since $F'(r)$ is the last term in (6.2), we have the desired result. □

Now we present some densities properties of a set of finite perimeter in his reduced boundary points.

**Theorem 6.1.5.** *Let $E \in \mathbb{R}^n$ be a set of finite perimeter, and $x \in \partial^* E$. Then there exists positive constants $A_1, A_2, A_3, A_4, A_5$ such that*

1. $\liminf_{r \to 0} \dfrac{\mathcal{L}^n(B_r(x) \cap E)}{r^n} > A_1$
2. $\liminf_{r \to 0} \dfrac{\mathcal{L}^n(B_r(x) \setminus E)}{r^n} > A_2$
3. $\liminf_{r \to 0} \dfrac{|\partial E|(B_r(x))}{r^{n-1}} > A_3$
4. $\limsup_{r \to 0} \dfrac{|\partial E|(B_r(x))}{r^{n-1}} \leq A_4$
5. $\limsup_{r \to 0} \dfrac{|\partial(E \cap B_r(x))|(\mathbb{R}^n)}{r^{n-1}} \leq A_5$

**Note**: conditions 1and 2 of the above theorem tell us that situations likes whose in figure below can not be possible



*Proof.* First of all we prove some implications for the inequalities above.

(1) $\Rightarrow$ (2) : let $\varphi \in C_c^1(E;\mathbb{R}^n)$, $|\varphi| \leq 1$; then, from the Gauss-Green Theorem

$$0 = \int_{\mathbb{R}^n} \operatorname{div}(\varphi)\,\mathrm{d}y = \int_E \operatorname{div}(\varphi)\,\mathrm{d}y + \int_{\mathbb{R}^n\setminus E} \operatorname{div}(\varphi)\,\mathrm{d}y$$

Hence we obtain that $E$ has finite perimeter in $\mathbb{R}^n \Leftrightarrow \mathbb{R}^n \setminus E$ has finite perimeter in $\mathbb{R}^n$. Moreover $|\partial E| = |\partial \mathbb{R}^n \setminus E|$, $\nu_E = -\nu_{\mathbb{R}^n\setminus E}$, and hence

$$\partial^* E = \partial^*(\mathbb{R}^n \setminus E)$$

(4) $\Rightarrow$ (5) : fix $R > r$; since $|\partial E \cap B_r(x)|(\mathbb{R}^n) = |\partial E \cap B_r(x)|(B_R(x))$, from Remark 7.3.6 we have that

$$|\partial E \cap B_r(x)|(B_R(x)) = |\partial E|(B_r(x)) + \mathcal{H}^{n-1}(E \cap \partial B_r(x))$$

Since

$$\frac{\mathcal{H}^{n-1}(E \cap \partial B_r(x))}{r^{n-1}} \leq \frac{\mathcal{H}^{n-1}(\partial B_r(x))}{r^{n-1}} = n\omega_n$$

then, if (4) holds, passing to the superior limit we obtain that

$$|\partial E \cap B_r(x)|(B_R(x)) \leq A_4 + n\omega_n =: A_5$$

To prove (4) : let $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ such that $\varphi_{|_{B_r(x)}} \equiv \nu_E(x)$; then, from Theorem 6.1.4 we have that

$$0 = \int_{B_r(x)} \operatorname{div}(\varphi)\,\mathrm{d}x = \langle\, \nu_E(x), \int_{B_r(x)} \nu_E\,\mathrm{d}|\partial E|\,\rangle + \int_{E\cap\partial B_r(x)} \langle \nu_E(x), \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}$$

Averaging with respect $|\partial E|(B_r(x))$

$$0 = \langle \nu_E(x), \fint_{B_r(x)} \nu_E\,\mathrm{d}|\partial E|\rangle + \frac{1}{|\partial E|(B_r(x))}\int_{E\cap\partial B_r(x)} \langle \nu_E(x), \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}$$

Since $x \in \partial^* E$ the first integral goes to $\nu_E(x)$, and hence

$$\lim_{r\to 0} \frac{\left|\int_{E\cap\partial B_r(x)} \langle \nu_E(x), \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}\right|}{|\partial E|(B_r(x))} = 1$$

Then, for $r$ sufficiently small

$$\frac{1}{2} \leq \frac{\left|\int_{E\cap\partial B_r(x)} \langle \nu_E(x), \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}\right|}{|\partial E|(B_r(x))} \leq \frac{\mathcal{H}^{n-1}(E \cap \partial B_r(x))}{|\partial E|(B_r(x))}$$

Hence

$$\frac{|\partial E|(B_r(x))}{r^{n-1}} \leq \frac{2\mathcal{H}^{n-1}(E \cap \partial B_r(x))}{r^{n-1}} = 2n\omega_n =: A_4$$

To prove (1) : define the function

$$g(r) := \mathcal{L}^n(E \cap B_r(x)) = \int_0^r \mathcal{H}^{n-1}(E \cap \partial B_\rho(x))\,\mathrm{d}\rho$$

where the equality above is by the Coarea formula (see [EG92], Chapter 3). Hence, for almost every $r$

$$\exists \quad g'(r) = \mathcal{H}^{n-1}(E \cap \partial B_r(x))$$

Then

$$\begin{aligned} g(r)^{\frac{n-1}{n}} &\leq C|\partial(E \cap B_r(x))|(\mathbb{R}^n) && \text{(isodiametric ineq.)} \\ &\leq C\Big[|\partial E|(B_r(x)) + \mathcal{H}^{n-1}(E \cap \partial B_r(x))\Big] && \text{(see (4) } \Rightarrow \text{ (5))} \\ &\leq 3C\mathcal{H}^{n-1}(E \cap \partial B_r(x)) = 3Cg'(r) \end{aligned}$$

where in the last step we have used a inequality proved in the previous point. Hence

$$g(r)^{\frac{1}{n}-1} g'(r) \geq \frac{1}{3C}$$

and integrating from 0 to $r$ we obtain

$$g(r) \geq \frac{r^n}{(3Cn)^n}$$

To prove (3) : From the local isoperimetric inequality (Theorem 5.4.2) we have that there exists a constant $C$ such that

$$\frac{|\partial E|(B_r(x))}{r^{n-1}} \geq C \min\left\{\frac{\mathcal{L}^n(E \cap B_r(x))}{r^n}, \frac{\mathcal{L}^n(E \setminus B_r(x))}{r^n}\right\}^{\frac{n-1}{n}}$$

Then, using point (1) and (2), for $r$ sufficiently small

$$\frac{|\partial E|(B_r(x))}{r^{n-1}} \geq C \min\left\{\frac{A_1}{2}, \frac{A_2}{2}\right\} =: A_3$$

□

## 6.2 Blow-up

In this section we will study better the reduced boundary using the blow-up, an useful thecnique in the study of the geometrical properties of a set.

Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, and let $x \in \partial^* E$. Define, for $r > 0$,

$$g_r(y) := x + \frac{y-x}{r}$$

and set

$$E_r := g_r(E)$$

We have a "change of variable formula": let $R > 0$ fixed, and consider $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$; then

$$\begin{aligned}\int_{B_R(x)\cap E_r} \operatorname{div}(\varphi(x))\,\mathrm{d}x &= \frac{1}{r^n}\int_{B_{rR}(x)\cap E} \operatorname{div}(\varphi(g_r(x)))\,\mathrm{d}x \\ &= \frac{1}{r^{n-1}}\int_{B_{rR}(x)\cap E} \operatorname{div}(\varphi \circ g_r)(x)\,\mathrm{d}x\end{aligned}$$

Since $\varphi \in C_c^1(B_R(x)) \Leftrightarrow \varphi \circ g_r \in C_c^1(B_{rR}(x))$ we have the following two equalities

$$\int_{B_R(x)} \mathrm{d}[D\chi_{E_r}] = \frac{1}{r^{n-1}}\int_{B_{rR}(x)} \mathrm{d}[D\chi_E]$$

$$\int_{B_R(x)} \mathrm{d}|\partial E_r| = \frac{1}{r^{n-1}}\int_{B_{rR}(x)} \mathrm{d}|\partial E|$$

We will use a lot this formulae.

Now, the idea we want to prove is that the unit vector $\nu_E(x)$ define a "normal"to $\partial E$ in $x$; more precisely, let

$$H(x) := \Big\{y \in \mathbb{R}^n \mid \langle y - x, \nu_E(x)\rangle = 0\Big\}$$

$$H^+(x) := \Big\{y \in \mathbb{R}^n \mid \langle y - x, \nu_E(x)\rangle\ > 0\Big\}$$

$$H^-(x) := \Big\{y \in \mathbb{R}^n \mid \langle y - x, \nu_E(x)\rangle\ < 0\Big\}$$

The result is the following

**Theorem 6.2.1.** *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, $x \in \partial^* E$. Then*

$$E_r \to H^-(x)$$

*Proof.* We can suppose $x = 0$ and $\nu_E(0) = -e_1$. Let $(r_j)_j \to 0$, and set $E_j := E_{r_j}$. We want to apply the Compactness Theorem (see Theorem 5.3.2); to do this we need to work in a open bounded set with lipschitz boundary. Since $0 \in \partial^* E$ and $\nu_E(0) = e_n$ from the formulae above we have that

$$\lim_{r\to 0} \frac{\displaystyle\int_{B_R(0)} \mathrm{d}D_1\chi_{E_r}}{|\partial E_r|(B_R(0))} = -1 \tag{6.3}$$

and

$$\lim_{r\to 0} \frac{\displaystyle\int_{B_R(0)} \mathrm{d}D_i\chi_{E_r}}{|\partial E_r|(B_R(0))} = 0 \qquad i = 2, \dots, n \tag{6.4}$$

From point (4) of Theorem 6.1.5 we obtain that

$$\limsup_{r\to 0} \int_{B_R(0)} \mathrm{d}|\partial E_r| \le \infty$$

Moreover

$$\|\chi_{E_r}\|_{L^1(B_R(0))} \le \mathcal{L}^n(B_R(0)) < \infty$$

Hence

$$\|\chi_{E_r}\|_{BV(B_R(0))} < \infty$$

So we can apply the Compactness Theorem obtaining a subsequence, still denoted by $(E_{r_j})_j$, and a function $f_R \in L^1(B_R(0))$ such that

$$E_{r_j} \to f_R \quad \text{in } B_R(0)$$

We can supppose that $f_R$ is the characteristic function of a set in $B_R(0)$. Repeating the same reasoning to every $R > 0$, and using a diagonal argument, we obtain that there exists a subsequence, still denoted with $(r_j)_j$, and a set $C \subset \mathbb{R}^n$ such that $E_{r_j} \to C$ in $\mathbb{R}^n$. By semicontinuity we also have that $C$ as finite peimeter in every bounded set. Moreover, by Theorem 2.9.5 we have that

$$\lim_{j\to\infty} \int_{B_r(0)} \mathrm{d}[D\chi_{E_j}] = \int_{B_r(0)} \mathrm{d}[D\chi_C]$$

for almost every $r$ (in particular for those $r$ such that $|\partial C|(\partial B(r)(0)) = 0$). Hence, recalling (6.3) and (6.4) we have that

$$\lim_{j\to\infty} \int_{B_r(0)} \mathrm{d}|\partial E_j| = \lim_{j\to\infty} \mathrm{d}D_1\chi_{E_j} = -\int_{B_r(0)} \mathrm{d}D_1\chi_C$$

Thus, by semicontinuity

$$\int_{B_r(0)} \mathrm{d}|\partial C| \leq -\int_{B_r(0)} \mathrm{d}D_1\chi_C$$

and hence, since $D_1\chi_C \leq |\partial C|$,

$$\int_{B_r(0)} \mathrm{d}|\partial C| = -\int_{B_r(0)} \mathrm{d}D_1\chi_C$$

If we differentiate $D_1\chi_C$ with respect to $|\partial C|$, from the identity above we obtain that

$$D_1\chi_C = -|\partial C|$$

Hence, since

$$(D_1\chi_C, \dots, D_n\chi_C) = \nu_C|\partial C|$$

and $|\nu_C| = 1$ we obtain that

$$D_i\chi_C = 0 \qquad i = 2, \dots n$$

Hence $\nu_C = -e_1$, $|\partial C|$-a.e.; so, if we take an approximating sequence of smooth functions $f_\epsilon := \eta_\epsilon * \chi_C$, and consider $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ we have that

$$\int_{\mathbb{R}^n} \langle \varphi, Df_\epsilon \rangle \,\mathrm{d}x = \int_C \mathrm{div}(\eta_\epsilon * \varphi) \,\mathrm{d}x = \int_C \langle (\eta_\varepsilon * \varphi), \nu_C \rangle \,\mathrm{d}|\partial C| = -\int_{\mathbb{R}^n} (\eta_\epsilon * \varphi_1) \,\mathrm{d}|\partial C|$$

So all the functions $f_\epsilon$ depend only on $x_1$, and they are decreasing functions. Then there exists $\gamma \in \mathbb{R}$ such that

$$C = \{x \in \mathbb{R}^n \mid x_1 \leq \gamma\}$$

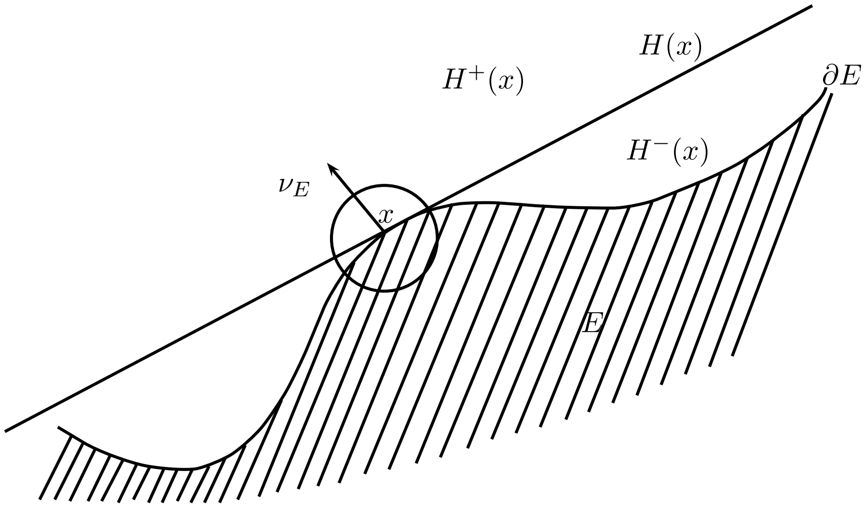
We want to show that $\gamma = 0$. Suppose $\gamma < 0$; since $E_j \to C$

$$0 = \mathcal{L}^n(C \cap B_{|\gamma|}(0)) = \lim_{j\to\infty} \mathcal{L}^n(E_j \cap B_{|\gamma|}(0)) = \lim_{j\to\infty} \frac{1}{r_j^n} \mathcal{L}^n(E \cap B_{r_j|\gamma|}(0))$$

A contraddiction to (1) of Theorem 6.1.5. On the other hand, if $\gamma > 0$ we have that

$$1 = \mathcal{L}^n(C \cap B_{|\gamma|}(0)) = \lim_{j\to\infty} \mathcal{L}^n(E_j \cap B_{|\gamma|}(0)) = \lim_{j\to\infty} \frac{1}{r_j^n} \mathcal{L}^n(E \cap B_{r_j|\gamma|}(0))$$

Hence we have a contraddiction to (2) of Theorem 6.1.5. Then $\gamma = 0$ and the desired result is proved. $\square$




So we can say that $C$ is a "tangent"plane to $\partial E$ in 0. More precisely

**Theorem 6.2.2.** *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, and $x \in \partial^* E$. Then*

1. $\lim\limits_{r \to 0} \dfrac{\mathcal{L}^n(B_r(x) \cap E \cap H^+(x))}{\omega_n r^n} = 0$

2. $\lim\limits_{r \to 0} \dfrac{(\mathcal{L}^n(B_r(x) \setminus E) \cap H^-(x))}{\omega_n r^n} = 0$

3. $\lim\limits_{r \to 0} \dfrac{\mathcal{L}^n(B_r(x) \cap E \cap H^-(x))}{\omega_n r^n} = \dfrac{1}{2}$

4. $\lim\limits_{r \to 0} \dfrac{|\partial E|(B_r(x))}{r^{n-1}} = \omega_{n-1}$

*Proof.* We can suppose $x = 0$. For (1) : since $\chi_{E_r} \to \chi_{H^-(0)}$ we have that

$$\begin{aligned} \mathcal{L}^n(B_r(0) \cap E \cap H^+(0)) &= r^n \mathcal{L}^n(B_1(0) \cap E_r \cap H^+(0)) \\ &= r^n \int_{B_1(0) \cap H^+(0)} \chi_{E_r} \, \mathrm{d}y \stackrel{r \to 0}{\longrightarrow} = 0 \end{aligned}$$

The proof of (2) is similar as (1). For (3) : we have that

$$\begin{aligned} \frac{\mathcal{L}^n(B_r(x) \cap E \cap H^-(x))}{\omega_n r^n} &= \frac{\mathcal{L}^n(B_r(x) \cap H^-(x))}{\omega_n r^n} - \frac{\mathcal{L}^n((B_r(x) \setminus E) \cap H^-(x))}{\omega_n r^n} \\ &\stackrel{r \to 0}{\longrightarrow} \frac{1}{2} \end{aligned}$$

where in the last step we have take into account that $H^-(x)$ is an half-space.

For (4) : if we take a $L > 0$ such that $|\partial C|(\partial B_L(0)) = 0$, that is for almost every $L > 0$, from Theorem 6.2.1 we have that

$$\lim_{r\to 0} |\partial E_r|(B_L(0)) = |\partial H^-(0)|(B_L(0)) = \mathcal{H}^{n-1}(H^-(0) \cap B_L(0)) = \omega_{n-1}L^{n-1}$$

Hence

$$\frac{|\partial E|(B_{rL}(0))}{\omega_{n-1}(rL)^{n-1}} = \frac{|\partial E_r|(B_L(0))}{\omega_{n-1}L^{n-1}} \to 1$$

□

## 6.3 Regularity of the reduced boundary

Now we can prove the foundamental result, due to De Giorgi, concerning the regularity of the reduce boundary. First we need the following

**Lemma 6.3.1.** *There exists a constant $C = C(n)$ such that*

$$\mathcal{H}^{n-1}(B) \leq C|\partial E|(B)$$

*for each $B \subset \partial^* E$.*

*Proof.* Fix $\epsilon > 0$; since $|\partial E|$ is a Radon measure there exists an open set $A \supset B$ such that

$$|\partial E|(A) \leq |\partial E|(B) + \epsilon$$

From point (3) of Theorem 6.1.5 there exists $A_3 > 0$ such that for every $x \in \partial^* E$

$$\liminf_{r\to 0} \frac{|\partial E|(B_r(x))}{r^{n-1}} > A_3$$

Then, for $r$ sufficiently small, and for a fixed $k \in (0, A_3)$

$$\frac{|\partial E|(B_r(x))}{r^{n-1}} \geq k$$

for each $x \in \partial^* E$. Then, if we define

$$\mathcal{F} := \left\{ B_r(x) \subset A \mid x \in B,\ \rho > 10r,\ |\partial E| \geq kr^{n-1} \right\}$$

we have that $\mathcal{F}$ is a fine covering of $B$. Then, from Theorem 2.6.5, there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$B \subset \bigcup_{i=0}^{\infty} B_{5r_j}(x_j)$$

Hence

$$
\begin{aligned}
\mathcal{H}^{n-1}_\rho(B) &\leq \sum_{i=0}^{\infty} \omega_{n-1}\Big(5^{n-1} r_j^{n-1}\Big)^{n-1} \leq \frac{\overbrace{\omega_{n-1}5^{n-1}}^{=:C}}{k} \sum_{i=0}^{\infty} |\partial E|(B_{r_j}(x_j)) \\
&= C|\partial E|\Big(\bigcup_{i=0}^{\infty} B_{r_j}(x_j)\Big) \leq C|\partial E|(U) \leq C(\,|\partial E|(B) + \epsilon\,)
\end{aligned}
$$

Since, first $\epsilon$ and then $\rho$, are arbitrary, we can conclude. □

Next theorem allow us to say that a set of finite perimeter has " measure theoretic $C^1$ boundary".

**Theorem 6.3.2** (**Structure theorem for sets of finite perimeter - De Giorgi**). *Le $E \subset \mathbb{R}^n$ be a Caccioppoli set. Then*

1. *It holds*
$$\partial^* E = \Big(\bigcup_{i=0}^{\infty} K_i\Big) \cup N$$
*where $|\partial E|(N) = 0$ and $K_h$ is a compact subset of an hypersurface $S_i$ of class $C^1$.*

2. *$\nu_{E|_{K_i}}$ is perpendicular to $S_i$*

3. *$|\partial E| = \mathcal{H}^{n-1} \llcorner \partial^* E$*

4. *$\overline{\partial^* E} = \partial E$*

*Proof.* Consider (1) and (3) of Theorem 6.2.2:

$$(1) \quad \lim_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \cap E \cap H^+(x))}{\omega_n r^n} = 0$$

$$(3) \quad \lim_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \cap E \cap H^-(x))}{\omega_n r^n} = \frac{1}{2}$$

For each $i$, by Egoroff's Theorem (see Theorem 2.3.6) we can find disjoint $|\partial E|$-measurable sets $(F_i)_i$ such that

$$|\partial E|\Big(\partial^* E \setminus \bigcup_{i=0}^{\infty} F_i\Big) = 0$$

and the convergences in (1) and (2) are uniform.

Now fix an $i$; by Lusin's Theorem (see Theorem 2.3.4) we can find a countable family of disjoint compact sets $(G_h^i)_h \subset F_i$ such that

$$|\partial E|\Big(F_i \setminus \bigcup_{h=0}^{\infty} G_h^i\Big) = 0$$

and

$$\nu_{E|_{G_h^i}} \text{ is continous}$$

Denoted by $(k_i)_i$ the family $(G_h^i)_{i,h}$, we define

$$N := \Big(\partial^* E \setminus \bigcup_{i=0}^{\infty} F_i\Big) \cup \Big(\bigcup_{i=0}^{\infty} \Big(F_i \setminus \bigcup_{h=0}^{\infty} G_h^i\Big)\Big)$$

we have that $|\partial E|(N) = 0$ and the convergences in (1) and (3) are uniform in every $K_i$, and $\nu_{E|_{K_i}}$ is continous.

Now we want to prove that each $K_i$ is contained in a $C^1$ hypersurface $S_i$. To do this we want to apply the Whitney Extension Theorem (see [EG92] Section 6.5) to: function $f \equiv 0$ on $K_i$, $d = \nu_{E|_{K_i}}$. To do this we have to prove that

$$\rho(\delta) := \sup\Big\{\frac{|\langle \nu_E(x), y-x\rangle|}{|y-x|} \mid x, y \in K_i,\ 0 < |y-x| < \delta\Big\} \xrightarrow{\delta \to 0} 0$$

Fix $0 < \varepsilon < 1$; since the convergence in (1) and (3) are uniform in $K_i$, there exists $r_\varepsilon > 0$ such that for each $\xi \in K_i$ and each $r \in (0, r_\varepsilon)$

$$\frac{\mathcal{L}^n(B_r(x) \cap E \cap H^+(x))}{\omega_n r^n} \leq \frac{\varepsilon^n}{2^{n+2}}$$

and

$$\frac{\mathcal{L}^n(B_r(x) \cap E \cap H^-(x))}{\omega_n r^n} \geq \frac{1}{2} - \frac{\varepsilon^n}{2^{n+2}}$$

We state that if $\delta < \frac{r_\varepsilon}{2}$ then $\rho(\delta) \leq \varepsilon$. Suppose not; then there exists $x, y \in K_i$ such that $0 < |y-x| \leq \delta$ and

$$\frac{|\langle \nu_E(x), y-x\rangle|}{|y-x|} > \varepsilon$$

Suppose $\langle \nu_E(x), y-x\rangle > 0$; then, for $z \in \mathbb{R}^n$

$$\begin{aligned}\langle \nu_E(x), z-x\rangle &= \langle \nu_E(x), y-x\rangle + \langle \nu_E(x), z-y\rangle \geq \varepsilon|y-x| - |\langle \nu_E(x), z-y\rangle| \\ &\geq \varepsilon|y-x| - |z-y|\end{aligned}$$

Hence $B_{\varepsilon|y-x|}(y) \subset H^+(x)$. Moreover $B_{\varepsilon|y-x|}(y) \subset B_{|y-x|}(x,)$: in fact

$$|z-x| \leq |z-y| + |y-x| \leq (\varepsilon+1)|y-x| < 2|y-x|$$

where in the last step we have take into account that $\varepsilon < 1$. Hence

$$B_{\varepsilon|y-x|}(y) \subset H^+(x) \cap B_{2|y-x|}(x) \tag{6.5}$$

Since $|y-x| \leq \delta < \frac{r_\varepsilon}{2}$

$$\mathcal{L}^n(E \cap B_{2|y-x|}(x) \cap H^+(x)) \leq \frac{\varepsilon^n}{2^n}\omega_n|y-x|^n$$

and

$$\mathcal{L}^n(E \cap B_{\varepsilon|y-x|}(y)) \geq \mathcal{L}^n(E \cap B_{\varepsilon|y-x|}(y) \cap H^-(y)) \geq \Big(\frac{1}{2} - \frac{\varepsilon^n}{2^{n+2}}\Big)\omega_n(\,\varepsilon|y-x|\,)^n$$

Hence by inclusion (6.5) we obtain that

$$\Big(\frac{1}{2} - \frac{\varepsilon^n}{2^{n+2}}\Big)\omega_n(\,\varepsilon|y-x|\,)^n \leq \frac{\varepsilon^n}{2^n}\omega_n|y-x|^n$$

that hyelds $\varepsilon \geq 2$. Absurd. The case $\langle \nu_E(x), y-x\rangle < 0$ is similar.

So we can apply the Whitney Extension Theorem and obtain that there exists $\widetilde{f} \in C^1(\mathbb{R}^n)$ such that $\widetilde{f}_{|K_i} \equiv 0$ and $(D\widetilde{f})_{|K_i} = \nu_{E|K_i}$. so, if we define

$$S_i := \Big\{x \in \mathbb{R}^n \mid \widetilde{f}(x) = 0,\ |D\widetilde{f}(x)| \geq \frac{1}{2}\Big\}$$

we obtain that $S_i$ is an hypersurface, $K_i \subset S_i$ and $\nu_{E|K_i}$ is perpendicular to $S_i$.

Now we prove (3): since $|\partial E|$ and $\mathcal{H}^{n-1}$ are regular, we can prove (3) only for Borel sets. So let $B \subset \mathbb{R}^n$ be a Borel set; by the previous Lemma, $\mathcal{H}^{n-1}(N) = 0$; so

$$\mathcal{H}^{n-1}(B \cap \partial^* E) = \mathcal{H}^{n-1}\Big(\bigcup_{i=0}^{\infty} B \cap K_i\Big) = \sum_{i=0}^{\infty}\mathcal{H}^{n-1}(B \cap K_i)$$

Let $\gamma_i := \mathcal{H}^{n-1} \llcorner K_i$. From the Area Formula (see [EG92], Chapter 3) we have that

$$\lim_{r\to 0}\frac{\gamma_i(B_r(x))}{\omega_{n-1}r^{n-1}} = 1$$

Thus, from (4) of Theorem 6.2.2, we obtain that

$$\lim_{r\to 0}\frac{\gamma(B_r(x))}{|\partial E|(B_r(x))} = 1$$

Since $|\partial E|$ and $\gamma$ are Radon measures, from the Differentiation Theorem for Radon measures (see Theorem 2.7.4) we obtain that

$$|\partial E| = \gamma$$

Hence

$$\begin{aligned}\mathcal{H}^{n-1}(B \cap \partial^* E) &= \sum_{i=0}^{\infty} \mathcal{H}^{n-1}(B \cap K_i) = \sum_{i=0}^{\infty} |\partial E|(B \cap K_i) \\ &= |\partial E|(B \cap \partial^* E) = |\partial E|(B)\end{aligned}$$

To prove (4) let $A$ be an open set such that $A \cap \partial^* E = \emptyset$; then from point (3) we obtain that

$$|\partial E|(A) = 0$$

Hence $\chi_E$ is constant in $A$; since $\mathrm{supp}(|\partial E|) \subset \partial E$ we obtain that $A \cap \partial E = \emptyset$. □

## 6.4 Some applications

In this section we will use the results of the previous section to study the behaviour of the union and the intersection of Caccioppoli sets.

**Lemma 6.4.1.** *Let $E, F$ be Caccioppoli sets in $\mathbb{R}^n$. Then, for any open set $A \subset \mathbb{R}^n$ it holds*

$$|\partial(E \cup E)|(A) + |\partial(E \cap F)|(A) \leq |\partial E|(A) + |\partial F|(A)$$

*Proof.* Supppose $f, g$ are smooth functions such that $0 \leq f, g \leq 1$, and let

$$\varphi := f + g - fg, \quad \psi := fg$$

Then

$$|D\varphi| = |Df + Dg + fDg + gDf| \leq (1-f)|Df| + (1-g)|Df|$$

$$|D\psi| \leq f|Dg| + g|Df|$$

Hence

$$|D\varphi|(A) + |D\psi|(A) \leq |Df|(A) + |Dg|(A)$$

Now note that if $|\partial E|(A)$ or $|\partial F|(A)$ is not finite, then the theorem is clearly true. If both $E$ and $F$ have finite perimeter in $A$, let $(f_j)_j$ and $(g_j)_j$ be respectively the approximating smooth functions of $\chi_E$ and $\chi_F$ given by the Anzellotti-Giaquinta Theorem. In particular we have that

$$|Df_j|(A) \to |\partial E|(A), \quad |Dg_j|(A) \to |\partial F|(A)$$

Moreover, if we define $\varphi_j := f_j + g_j - f_j g_j$ and $\psi_j := f_j g_j$, we have that

$$\varphi_j \to \chi_{E\cup F}, \quad \psi_j \to \chi_{E\cap F}$$

Hence, from the semi-continuity Theorem, we have that

$$\begin{aligned} |\partial(E \cup E)|(A) + |\partial(E \cap F)|(A) &\leq \liminf_{j\to\infty}(|D\varphi_j|(A) + |D\psi_j|(A)) \\ &\leq \liminf_{j\to\infty}(|Df_j|(A) + |Dg_j|(A)) \\ &= |\partial E|(A) + |\partial F|(A) \end{aligned}$$

□

**Remark 6.4.2.** *From this lemma we have an important consequence: if $E$ and $F$ have least perimeter in $A$, and if $E\triangle F \Subset A$, then both $E \cap F$ and $E \cup F$ have least perimeter in $A$. In fact, since we can write*

$$E \cup F = F \cup (E \setminus F), \quad (E \cap F) = E \setminus (E \setminus F)$$

*from the minimality of $E$ and $F$ we get*

$$|\partial F|(A) \leq |\partial(E \cup F)|(A)$$

*and*

$$|\partial E|(A) \leq |\partial(E \cap F)|(A)$$

*Hence, from the lemma above we obtain that*

$$|\partial(E \cup E)|(A) + |\partial(E \cap F)|(A) = |\partial E|(A) + |\partial F|(A)$$

*and so*

$$|\partial F|(A) = |\partial(E \cup F)|(A)$$

$$|\partial E|(A) = |\partial(E \cap F)|(A)$$

*In particular*

$$|\partial E|(A) = |\partial F|(A) = |\partial(E \cup F)|(A) = |\partial(E \cap F)|(A)$$

**Lemma 6.4.3.** *Let $E := E_1 \cup E_2$ and suppose that $\mathcal{H}^{n-1}(\overline{E_1} \cap \overline{E_2}) = 0$. Then for any open set $A$ we have*

$$|\partial E|(A) = |\partial E_1|(A) + |\partial E_2|(A)$$

*Moreover if $E$ has least perimeter in $A$, then the same is true for $E_1$ and $E_2$.*

*Proof.* Since the reduced boundary of a Caccioppoli set is $\mathcal{H}^{n-1}$-measurable, we have that

$$\begin{aligned}
|\partial E_1|(A) + |\partial E_2|(A) &= \mathcal{H}^{n-1}(\partial^* E_1 \cap A) + \mathcal{H}^{n-1}(\partial^* E_2 \cap A) \\
&= \mathcal{H}^{n-1}((\partial^* E_1 \cup \partial^* E_2) \cap A) - \mathcal{H}^{n-1}((\partial^* E_1 \cap \partial^* E_2) \cap A) \\
&= \mathcal{H}^{n-1}([(\partial^* E_1 \cup \partial^* E_2) \setminus \partial^* E] \cap A) \\
&\quad + \mathcal{H}^{n-1}([(\partial^* E_1 \cup \partial^* E_2) \cap \partial^* E] \cap A) \\
&\quad - \mathcal{H}^{n-1}((\partial^* E_1 \cap \partial^* E_2) \cap A) \\
&\leq \mathcal{H}^{n-1}([(\partial^* E_1 \cup \partial^* E_2) \setminus \partial^* E]) + \mathcal{H}^{n-1}(\partial^* E \cap A) \\
&\quad - \mathcal{H}^{n-1}((\partial^* E_1 \cap \partial^* E_2))
\end{aligned}$$

Now, since

$$\partial^* E_1 \cap \partial^* E_2 \subset \partial E_1 \cap \partial E_2 \subset \overline{E_1} \cap \overline{E_2}$$

and

$$\partial^* E_1 \cup \partial^* E_2 \setminus \partial^* E \subset (\partial^* E_1 \cap \overline{E_2}) \cup (\partial^*(E_2) \cap \overline{E_1}) \cup (\partial^* E_1 \cap \partial^* E_2) \subset \overline{E_1} \cap \overline{E_2}$$

we have that

$$\mathcal{H}^{n-1}(\partial^* E_1 \cap \partial^* E_2) = 0$$

and

$$\mathcal{H}^{n-1}(\partial^* E_1 \cup \partial^* E_2 \setminus \partial^* E) = 0$$

Hence we get

$$|\partial E_1|(A) + |\partial E_2|(A) \leq |\partial E|(A)$$

The opposite inequality clearly holds.

Now, suppose $E$ has least perimeter in an open set $A$; let $F$ be a Caccioppoli set such that $F = E$ outside a compact set $K \subset A$. Then

$$|\partial F|(A) + |\partial E_2|(A) \geq |\partial(F \cup E_2)|(A) \geq |\partial E|(A) = |\partial E_1|(A) + |\partial E_2|(A)$$

Hence $E_1$ is a minimal set in $A$. With the same method we can prove the minimality of $E_2$ in $A$. □

# Chapter 7

# Traces and extensions in $\mathbb{R}^n$

From the definition of $BV$ functions, if $f \in BV(U)$, and we take $\varphi \in C_c^1(U\ \mathbb{R}^n)$, we can write

$$\int_U f \operatorname{div}(\varphi)\,\mathrm{d}x = -\int_U \varphi \cdot \mathrm{d}[Df]$$

But if $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ we cannot write a similar formula. Inspired from the fact that if $f \in BV(U) \cap C^\infty(U)$, and $U$ is bounded and have Lipschitz boundary we can write (see Theorem 7.0.4)

$$\int_U f \operatorname{div}(\varphi)\,\mathrm{d}x = -\int_U \langle \varphi, Df\rangle\,\mathrm{d}x + \int_{\partial U} f\langle \varphi, \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}$$

where $\nu$ is the outer normal to $\partial U$, we want to extend the above formula to all $BV$ functions. To do this we need to talk about the value of $f \in BV(U)$ on $\partial U$, even if $\mathcal{L}^n(\partial U) = 0$, and so we need to define the trace of a $BV$ function of the boundary of a set. This is the aim of this chapter. Moreover we will use the notion of trace to prove some important properties of $BV$ functions: extension of $BV$ functions (Theorem 7.3.2), convergence of traces (Theorem 7.3.3) and the Gagliardo's extension Theorem (Theorem 7.3.4).

First of all we need to extend the classical Gauss-Green Theorem to sets with Lipschitz boundary

**Theorem 7.0.4.** *Let $U$ be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, and let $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$. Then*

$$\int_U \operatorname{div}(\varphi)\,\mathrm{d}x = \int_{\partial U} \langle \varphi, \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}$$

*where $\nu$ denotes the outer normal to $\partial U$.*

*Proof.* Using partitions of unity, and the fact that $\partial U$ is compact, we only need to prove the following fact: let $\alpha \in C^{0,1}(A)$, where

$$A := [a_i, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$$

for some $a_i < b_i \in \mathbb{R}$, and define

$$\Omega := \Big\{ x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \mid (x_1, \dots, x_{n-1}) \in A\,,\ 0 \le x_n \le \alpha(x_1, \dots, x_{n-1}) \Big\}$$

Then for each $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ and each $i = 1, \dots, n$ it holds

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = \int_{\partial\Omega} \langle \varphi, \langle \nu_\Omega, e_i \rangle e_i \rangle \,\mathrm{d}\mathcal{H}^{n-1} \tag{7.1}$$

where $\nu_\Omega$ is the outer normal to $\Omega$. Note that

$$\nu_\Omega(x) = \Big( -\frac{1}{\sqrt{1 + |D\alpha(x)|^2}}, \frac{D\alpha(x)}{\sqrt{1 + |D\alpha(x)|^2}} \Big)$$

if $x_n = \alpha(x_1, \dots, x_{n-1})$.
Let's prove formula (7.1): since we can suppose that $\alpha \in C^{0,1}(\mathbb{R}^n)$, we can consider a sequence of mollifiers $(\rho_\varepsilon)_\varepsilon$, and the mollified functions $\alpha_\varepsilon := \alpha * \rho_\varepsilon \in C^\infty(A)$. It hold

$$\alpha_\varepsilon \to \alpha \quad \text{uniformly on } A$$

and

$$\frac{\partial \alpha_\varepsilon}{\partial x_i}(x_1, \dots, x_{n-1}) \to \frac{\partial \alpha}{\partial x_i}(x_1, \dots, x_{n-1})$$

for a.e. $x \in A$ and each $i = 1, \dots, n-1$. Hence if we define

$$\Omega_\varepsilon := \Big\{ x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \mid (x_1, \dots, x_{n-1}) \in A\,,\ 0 \le x_n \le \alpha_\varepsilon(x_1, \dots, x_{n-1}) \Big\}$$

from the classical Gauss-Green Theorem it holds

$$\int_{\Omega_\varepsilon} \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = \int_{\partial\Omega_\varepsilon} \langle \varphi, \langle \nu_{\Omega_\varepsilon}, e_i \rangle e_i \rangle \,\mathrm{d}\mathcal{H}^{n-1}$$

where $\nu_{\Omega_\varepsilon}$ is the outer normal to $\partial\Omega_\varepsilon$. Note that

$$\nu_{\Omega_\varepsilon} = \Big( -\frac{1}{\sqrt{1 + |D\alpha_\varepsilon(x)|^2}}, \frac{D\alpha_\varepsilon(x)}{\sqrt{1 + |D\alpha_\varepsilon(x)|^2}} \Big)$$

if $x_n = \alpha_\varepsilon(x_1, \dots, x_{n-1})$. Hence, letting $\varepsilon \to 0$ we obtain the desired result. $\square$

## 7.1 The cartesian case

Let's start with a Lemma[1]:

**Lemma 7.1.1.** *Let* $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n > 0\}$*, and let* $\mu$ *be a positive Radon measure on* $\mathbb{R}^n_+$ *with* $\mu(\mathbb{R}^n_+) < \infty$*. For* $\rho > 0$ *and* $y \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$ *let*

$$\mathcal{C}^+_\rho(y) := \{x \in \mathbb{R}^n \mid x = (z,t),\ |y - z| < \rho,\ 0 < t < \rho\} = \mathcal{B}_\rho(y) \times (0, \rho)$$

*Then for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \mathbb{R}^{n-1}$

$$\lim_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \mu(\mathcal{C}^+_\rho(y)) = 0$$

*Proof.* For each $k$ define

$$A_k := \left\{ y \in \mathbb{R}^{n-1} \mid \limsup_{\rho \to 0^+} \frac{\mu(\mathcal{C}^+_\rho(y))}{\rho^{n-1}} > \frac{1}{k} \right\}$$

Then we show that $\mathcal{H}^{n-1}(A_k) = 0$ for all $k$. Fix $\epsilon > 0$; for each $y \in A_k$ there exists $\rho_y < \epsilon$ such that

$$\mu(\mathcal{C}^+_\rho(y)) > \frac{\rho_y^{n-1}}{2k}$$

Then $A_k \subset \bigcup_{y \in A_k} B_{\rho_y}(y)$. By the Vitali covering Theorem (see Theorem 2.6.1) we can find a countable subset $(y_i)_i \subset A_k$ such that

$$B_{\rho_{y_i}}(y_i) \cap B_{\rho_{y_j}}(y_j) = \emptyset\,, \quad \text{if } i \neq j$$

$$A_k \subset \bigcup_{i=0}^{\infty} B_{5\rho_{y_i}}(y_i)$$

Then

$$\mathcal{H}^{n-1}(A_k) \leq \omega_{n-1} \sum_{i=0}^{\infty} (5\rho_{y_i})^{n-1} < 2k\omega_{n-1} 5^{n-1} \sum_{i=0}^{\infty} \mu(\mathcal{C}^+_{\rho_{y_i}}(y_i)) \tag{7.2}$$

Setting

$$L_\epsilon := \{x \in \mathbb{R}^n \mid 0 < x_n < \epsilon\}$$

we have that $\mathcal{C}^+_{\rho_{y_i}}(y_i) \subset L_\epsilon$ for each $i$; moreover, since the sets $\mathcal{C}^+_{\rho_{y_i}}(y_i)$ are disjoint, because their basis are, from (7.2) we obtain

$$\mathcal{H}^{n-1}(A_k) \leq 2k\omega_{n-1} 5^{n-1} \mathcal{H}^{n-1}(L_\epsilon)$$

And since $\mu(\mathbb{R}^n_+) < \infty$, letting $\epsilon \to 0^+$ we have the desired result. □

[1] We recall that with the notation $\mathcal{B}_r(x)$ we denote the ball of center $x$ and radius $r$ contained in $\mathbb{R}^{n-1}$.

Next proposition is the foundamental brick to define traces of $BV$ functions on the Lipschitz boundary of an open set.

**Proposition 7.1.2.** *Let $A \subset \mathbb{R}^{n-1}$ be an open bounded set, $\omega : A \to \mathbb{R}$ a lipschitz function of constant $L$, and let $\overline{\delta} := \inf\{\omega(y) \mid y \in A\} > 0$. Let*

$$\begin{array}{ll} U & := \{x = (y, x_n) \in \mathbb{R}^n \mid y \in A,\ 0 < x_n < \omega(y)\} \\ S & := \{x = (y, x_n) \in \mathbb{R}^n \mid y \in A,\ x_n = \omega(y)\} \end{array}$$

*Let $u \in BV(U)$. Then there exists a function $u^+ \in L^1(S)$ such that*

$$(1) \quad \int_S |u^+| \, \mathrm{d}\mathcal{H}^{n-1} \leq \sqrt{1+L^2}|Du|(U) + c(U) \int_U |u| \, \mathrm{d}x$$

*where $c(U)$ is a positive constant depending only on $U$.*

$$(2) \quad \int_U u\mathrm{div}(\varphi) \, \mathrm{d}x = -\int_U \varphi \cdot \mathrm{d}[Du] + \int_S u^+ \langle \varphi, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}$$

*for each $\varphi \in C_c^1(A \times \mathbb{R}^{n-1}; \mathbb{R}^n)$, where $\nu$ denotes the outer normal to $S$.*

$$(3) \quad \lim_{\rho \to 0} \frac{1}{\rho^n} \int_{B_\rho(\overline{x}) \cap U} |u(x) - u^+(\overline{x})| \, \mathrm{d}x = 0$$

*for $\mathcal{H}^{n-1}$-a.e. $\overline{x} \in S$.*

*Proof.* Suppose first $u \in BV(U) \cap C^\infty(U)$; fix $\delta \in (0, \overline{\delta})$ and for $t \in (0, \delta)$ we define the functions

$$\begin{array}{rccl} \omega_t : & A & \to & \mathbb{R} \\ & y & \mapsto & \omega(y) - t \end{array}$$

$$\begin{array}{rccl} u_t : & S & \to & \mathbb{R} \\ & (y, \omega(y)) & \mapsto & u(y, \omega_t(y)) \end{array}$$

and the sets

$$U_t := \{x = (y, t) \in A \times \mathbb{R} \mid 0 < x_n < \omega_t(y)\}$$

$$S_t := \{x = (y, t) \in A \times \mathbb{R} \mid x_n = \omega_t(y)\}$$

We note that

$$\begin{aligned} \int_S u_t(x) \, \mathrm{d}\mathcal{H}^{n-1} &= \int_A u(y, \omega_t(y)) \sqrt{1 + |D\omega(y)|^2} \, \mathrm{d}y \\ &= \int_A u(y, \omega_t(y)) \sqrt{1 + |D\omega_t(y)|^2} \, \mathrm{d}y = \int_{S_t} u(x) \, \mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

We want to prove that $(u_t)_t$ is a Cauchy sequence in $L^1(S)$: so, let $0 < t_1 < t_2 < \delta$; then

$$\begin{aligned}
\int_S |u_{t_2} - u_{t_1}|\,\mathrm{d}\mathcal{H}^{n-1} &= \int_A |u(y,\omega_{t_2}(y)) - u(y,\omega_{t_1}(y))|\sqrt{1+|D\omega(y)|^2}\,\mathrm{d}y \\
&\leq \sqrt{1+L^2}\int_A \Big|\int_{\omega(y)-t_2}^{\omega(y)-t_1} \frac{\partial u}{\partial x_n}(y,x_n)\,\mathrm{d}x_n\Big|\,\mathrm{d}y \\
&\leq \sqrt{1+L^2}\int_A \mathrm{d}y\int_{\omega(y)-t_2}^{\omega(y)-t_1} |Du_n(y,x_n)|\,\mathrm{d}x_n \\
&\leq \sqrt{1+L^2}|Du|(U_{t_1}\setminus \overline{U}_{t_2}) \overset{t_1,t_2\to 0}{\longrightarrow} 0
\end{aligned}$$

Hence $(u_t)_t$ is a Cauchy sequence in $L^1(S)$; then there exists a function $u^+ \in L^1(S)$ such that $u_t \to u^+$ in $L^1(S)$.

Now we want to prove the local estimate for the trace. We note that from the inequality above we have, in particular, that

$$\int_S |u_{t_2} - u_{t_1}|\,\mathrm{d}\mathcal{H}^{n-1} \leq \sqrt{1+L^2}|Du|(U_{t_1}\setminus \overline{U}_{t_2})$$

So if we take $t_2 = t$, passing to the limit for $t_1 \to 0$ we obtain that

$$\int_S |u_t - u^+|\,\mathrm{d}\mathcal{H}^{n-1} \leq \sqrt{1+L^2}|Du|(U\setminus \overline{U}_t)$$

Hence

$$\begin{aligned}
\int_S |u^+|\,\mathrm{d}\mathcal{H}^{n-1} &\leq \int_S |u^+ - u_t|\,\mathrm{d}\mathcal{H}^{n-1} + \int_S |u_t|\,\mathcal{H}^{n-1} \\
&\leq \sqrt{1+L^2}|Du|(U\setminus \overline{U}_t) + \int_{S_t} |u|\,\mathrm{d}\mathcal{H}^{n-1} \\
&\leq \sqrt{1+L^2}|Du|(U\setminus \overline{U}_\delta) + \int_{S_t} |u|\,\mathrm{d}\mathcal{H}^{n-1}
\end{aligned}$$

Integrating from 0 to $\delta$ we obtain

$$\begin{aligned}
\delta\int_S |u^+|\,\mathrm{d}\mathcal{H}^{n-1} &\leq \delta\sqrt{1+L^2}|Du|(U\setminus \overline{U}_\delta) + \int_0^\delta \mathrm{d}t\int_{S_t} |u|\,\mathrm{d}\mathcal{H}^{n-1} \\
&\leq \delta\sqrt{1+L^2}|Du|(U\setminus \overline{U}_\delta) + \sqrt{1+L^2}\int_0^\delta \mathrm{d}t\int_A |u(y,\omega_t(y))|\,\mathrm{d}y \\
&= \delta\sqrt{1+L^2}|Du|(U\setminus \overline{U}_\delta) + \sqrt{1+L^2}\int_{U\setminus\overline{U}_t} |u|\,\mathrm{d}x
\end{aligned}$$

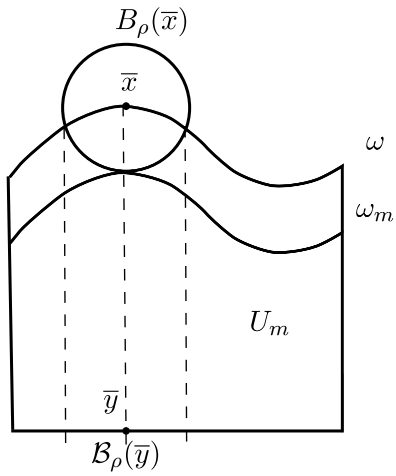



Figure 7.1: Graphic situation

So we have obtain the estimate

$$\int_S |u^+| \, \mathrm{d}\mathcal{H}^{n-1} \leq \sqrt{1+L^2}|Du|(U) + \frac{\sqrt{1+L^2}}{\delta} \int_U |u| \, \mathrm{d}x$$

Now we want to prove assertion (2): let $\varphi \in C_c^1(A \times \mathbb{R}; \mathbb{R}^n)$; then from the Gauss-Green theorem

$$\int_{U_t} u\mathrm{div}(\varphi) \, \mathrm{d}x = -\int_{U_t} \langle \varphi, Du \rangle \, \mathrm{d}x + \int_{S_t} u \langle \varphi, \nu_t \rangle \, \mathrm{d}\mathcal{H}^{n-1}$$

Now since $\nu_t = \nu$, and

$$\int_{S_t} u \langle \varphi, \nu_t \rangle \, \mathrm{d}\mathcal{H}^{n-1} = \int_S u_t \langle \varphi_t, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}$$

passing to the limit fro $t \to 0$ and using the continuity of $\varphi$ we obtain

$$\int_U u\mathrm{div}(\varphi) \, \mathrm{d}x = -\int_U \varphi \cdot \mathrm{d}[Du] + \int_S u^+ \langle \varphi, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}$$

Finally we prove the limit in (3): consider $\overline{x} := (\overline{y}, \omega(\overline{y})) \in S$, where $\overline{y} \in A$, and $0 < \rho < d(\overline{x}, \partial A \times \mathbb{R}^+ \cup A \times \{0\})$. since we want to estimate

$$\int_{U \cap B_\rho(\overline{x})} |u(x) - u^+(\overline{x})| \, \mathrm{d}x$$

we want, fixed $\rho$, to determine $m$ in such a way that $B_\rho(\overline{x}) \cap U_m = \emptyset$ in order to simplify the calculate.

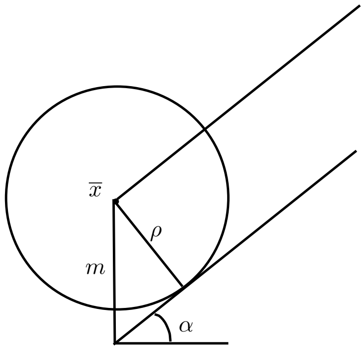



Figure 7.2: The construction of $m$

Since $\omega$ is a lipschitz function the graph of $\omega$ lies under the cone generated by the ray starting from $\overline{x}$ and with slope $L$; hence, as we can see in Figure 7.2, we have to take $m = \rho\sqrt{1+L^2}$; since $m$ must be less than $\delta$, we must take $\rho < \frac{\delta}{\sqrt{1+L^2}}$.
So if we take $m = \rho\sqrt{1+L^2}$, and we noting that $\mathcal{B}_\rho(\overline{y})$ is the projection of $B_\rho(\overline{x})$ on $A$, we have that

$$\begin{aligned}
\int_{U\cap B_\rho(\overline{x})} |u(x) - u^+(\overline{x})| \,\mathrm{d}x &\leq \int_{\mathcal{B}_\rho(\overline{y})} \mathrm{d}y \int_{\omega(y)-m}^{\omega(y)} |u(y,t) - u^+(\overline{y},\omega(\overline{y}))| \,\mathrm{d}t \\
&\leq \int_{\mathcal{B}_\rho(\overline{y})} \mathrm{d}y \int_{\omega(y)-m}^{\omega(y)} |u(y,t) - u^+(y,\omega(y))| \,\mathrm{d}t + \\
&\quad \int_{\mathcal{B}_\rho(\overline{y})} \mathrm{d}y \int_{\omega(y)-m}^{\omega(y)} |u^+(y,\omega(y)) - u^+(\overline{y},\omega\overline{y})| \,\mathrm{d}t
\end{aligned}$$

We study separately the two integral on the right: for the second integral we have

$$\int_{\mathcal{B}_\rho(\overline{y})} \mathrm{d}y \int_{\omega(y)-m}^{\omega(y)} |u^+(y,\omega(y)) - u^+(\overline{y},\omega(\overline{y}))| \,\mathrm{d}t = m \int_{\mathcal{B}_\rho(\overline{y})} |u^+(y,\omega(y)) - u^+(\overline{y},\omega(\overline{y}))| \,\mathrm{d}t$$

Since $m = \rho\sqrt{1+L^2}$, from the Lebesgue's point Theorem (see Theorem 2.7.10) we have that

$$\lim_{\rho\to 0} \sqrt{1+L^2} \frac{1}{\rho^{n-1}} \int_{\mathcal{B}_\rho(\overline{y})} |u^+(y,\omega(y)) - u^+(\overline{y},\omega(\overline{y}))| \,\mathrm{d}t = 0$$

For the first integral

$$\begin{aligned}
& \int_{\mathcal{B}_\rho(\overline{y})} \mathrm{d}y \int_{\omega(y)-m}^{\omega(y)} |u(y,t) - u^+(y,t)| \,\mathrm{d}t \\
= & \int_0^m \mathrm{d}s \int_{\mathcal{B}_\rho(\overline{y})} |u(y,\omega(y)-s) - u^+(y,\omega(y)-s)| \,\mathrm{d}y \\
\leq & \int_0^m \mathrm{d}s \int_{\mathcal{B}_\rho(\overline{y})} |u(y,\omega_s(y)) - u^+(y,\omega_s(y))| \sqrt{1+|D\omega(y)|^2} \,\mathrm{d}y \\
= & \int_0^m \mathrm{d}s \int_{S \cap (\mathcal{B}_\rho(\overline{y}) \times \mathbb{R}^+)} |u_s(x) - u^+(x)| \,\mathrm{d}x \\
\leq & \sqrt{1+L^2} \int_0^m \mathrm{d}s |Du|((U \setminus \overline{U}_s) \cap (\mathcal{B}_\rho(\overline{y}) \times \mathbb{R}^+)) \\
\leq & m\sqrt{1+L^2} |Du|((U \setminus \overline{U}_m) \cap (\mathcal{B}_\rho(\overline{y}) \times \mathbb{R}^+)) \\
\leq & m\sqrt{1+L^2} |Du|(\mathcal{B}_\rho(\overline{y}) \times (0,M))
\end{aligned}$$

for some $M > 0$; hence, apply the previous Lemma we obtain that

$$\begin{aligned}
0 \ \leq & \lim_{\rho \to 0} \frac{1}{\rho^n} \int_{\mathcal{B}_\rho(\overline{y})} \mathrm{d}y \int_{\omega(y)-m}^{\omega(y)} |u(y,t) - u^+(y,t)| \,\mathrm{d}t \\
\leq & \lim_{\rho \to 0} \frac{1}{\rho^{n-1}} |Du|(\mathcal{B}_\rho(\overline{y}) \times (0,M)) \to 0
\end{aligned}$$

for $\mathcal{H}^{n-1}$-a.e. $\overline{y} \in A$.

Now take $u \in BV(U)$; from the Anzellotti-Giaquinta theorem (see Theorem 5.2.1) there exists $(u_k)_k \in BV(U) \cap C^\infty(U)$ such that

- $u_k \to u$ in $L^1(U)$
- $|Du_k|(U) \to |Du|(U)$
- $\int_U \varphi \cdot \mathrm{d}[Du_k] \to \int_U \varphi \cdot \mathrm{d}[Du] \quad \forall \varphi \in C_c^1(A \times \mathbb{R}^+; \mathbb{R}^n)$
- $\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{U \cap B_\rho(\overline{y})} |u(x) - u_k(x)| \,\mathrm{d}x = 0 \quad \forall k, \forall \overline{y} \in S$

Now, since $\omega$ is Lipschitz, there exists a constant $c$ indipendent from $\rho$ and $\overline{y}$ such that, for $\rho$ suffficiently small,

$$c\mathcal{L}^n(B_\rho(\overline{y})) \leq \mathcal{L}^n(U \cap B_\rho(\overline{y})) \leq \mathcal{L}^n(B_\rho(\overline{y}))$$

Hence, from $\rho$ sufficiently small, we have that $\mathcal{L}^n(U \cap B_\rho(\overline{y}))$ has the same behavior of $\rho^n$; hence

$$\lim_{\rho\to 0} \frac{1}{\mathcal{L}^n(U\cap B_{\rho(\overline{y})})} \int_{U\cap B_\rho(\overline{y})} |u(x) - u_k^+(\overline{y})| \,\mathrm{d}x \leq$$
$$\lim_{\rho\to 0} \frac{1}{\rho^n}\Big( \int_{U\cap B_\rho(\overline{y})} |u(x) - u_k(x)| \,\mathrm{d}x + \int_{U\cap B_\rho(\overline{y})} |u_k(x) - u_k^+(\overline{y})| \,\mathrm{d}x \Big) = 0$$

Hence we obtain that **all** the traces of the functions $u_k$ coincides, and are equal to

$$u_k^+(\overline{y}, \omega(\overline{y})) = \lim_{\rho\to 0} \frac{1}{\mathcal{L}^n(U\cap B_{\rho(\overline{y})})} \int_{U\cap B_{\rho(\overline{y})}} u(x) \,\mathrm{d}x$$

for $\mathcal{H}^{n-1}$-a.e. $(\overline{y}, \omega(\overline{y})) \in S$. So we define

$$u^+((\overline{y}, \omega(\overline{y}))) := u_k^+((\overline{y}, \omega(\overline{y})))$$

for $\mathcal{H}^{n-1}$-a.e. $y \in A$. Hence we obtain (1) and (2) as limit for $k \to \infty$ of (1) and (2) written for $u_k$. Finally we obtain (3) as follows

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{U\cap B_\rho(\overline{y})} |u(x) - u^+(\overline{y}, \omega(\overline{y}))| \,\mathrm{d}x \leq$$
$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{U\cap B_\rho(\overline{y})} |u(x) - u_k(x)| \,\mathrm{d}x + \lim_{\rho\to 0} \frac{1}{\rho^n} \int_{U\cap B_\rho(\overline{y})} |u_k(x) - u_k^+(\overline{y}, \omega(\overline{y}))| \,\mathrm{d}x = 0$$

□

## 7.2 The general case

Now we present the general case of the theorem above, but first we need a definition

**Definition 7.2.1.** *Let $U \subset \mathbb{R}^n$ be a bounded open set. We say that $U$ has* ***Lipschitz boundary of constant*** *$L$, if we can find open sets $V_1, \ldots, V_k$ and functions $\omega_1, \ldots, \omega_k$ such that each $\omega_i : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, $L$ is the maximum of the Lipschitz constants of the functions $\omega_i$, and, upon rotation and traslation, for each $i$ it holds*

$$\partial U \cap A_i = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y = \omega_i(x)\}$$

$$U \cap A_i = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid 0 < y < \omega_i(x)\}$$

The generalization to bounded open sets with Lipschitz boundary of the previous theorem is give in the following

**Theorem 7.2.2.** *Let $U \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary of constant $L$. Let $u \in BV(U)$; then there exists a function $u^+ \in L^1(\partial U)$ such that*

1. *there exists a positive constant $c(U)$ depending only on $U$ such that*
$$\int_{\partial U} |u^+| \, \mathrm{d}\mathcal{H}^{n-1} \leq \sqrt{1+L^2}|Du|(U) + c(U) \int_U |u| \, \mathrm{d}x$$
2. *for each $\varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ it holds*
$$\int_U u\mathrm{div}(\varphi) \, \mathrm{d}x = -\int_U \varphi \cdot \, \mathrm{d}[Du] + \int_{\partial U} u^+ \langle \varphi, \nu \rangle \, \mathrm{d}\mathcal{H}^{n-1}$$
*where $\nu$ denotes the outer normal to $\partial U$.*
3. $\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{U \cap B_\rho(z)} |u(x) - u^+(z)| \, \mathrm{d}x = 0 \quad \mathcal{H}^{n-1} - a.e.\ z \in \partial U$
4. $\|u^+\|_{L^\infty(\partial U)} \leq \|u\|_{L^\infty(U)}$

*Proof.* Since $U$ is a bounded open set with lipschitz boundary of constant $L$, we can find $p$ open sets $\Omega_i := A_i \times (0, M_i)$ where $A_i$ is a open set in $\mathbb{R}^{n-1}$, $M_i > 0$, and lipschitz functions $\omega_i : A_i \to (0, M_i)$ of constant $L_i$ such that

$$\overline{\delta}_i := \inf\{\omega_i(y) \mid y \in A_i\} > 0 \qquad L_i \leq L$$
$$U_i := U \cap \Omega_i = \{x = (y, x_n) \in \mathbb{R}^n \mid 0 < x_n < \omega_i(y)\}$$
$$S_i := \partial U \cap \Omega_i = \{x = (y, x_n) \in \mathbb{R}^n \mid x_n = \omega_i(y)\}$$

Let $u_i := u_{|U_i}$; then $u_i \in BV(U_i)$. So, for each $i = 1, \ldots, p$ we are in the same hypothesis of the previous Theorem; so there exists functions $u_i^+ \in L^1(S_i)$ satisfying the thesis of the previous Theorem. In particular

$$u_i^+(z) = \lim_{\rho \to 0} \frac{1}{\rho^n} \int_{U_i \cap B_\rho(z)} u(x) \, \mathrm{d}x = \lim_{\rho \to 0} \frac{1}{\rho^n} \int_{U \cap B_\rho(z)} u(x) \, \mathrm{d}x$$

So if we define
$$u^+(z) = u_i^+(z) \quad \text{if } z \in U_i$$
we have that $u^+$ is well defined. Moreover we have immediately point (3) of the Theorem, since it is a "local" property, and point (4); in fact

$$\begin{aligned} |u^+(z)| &\leq \lim_{\rho \to 0} \fint_{U \cap B_\rho(z)} |u^+(z) - u(x)| \, \mathrm{d}z + \lim_{\rho \to 0} \fint_{U \cap B_\rho(z)} |u(x)| \, \mathrm{d}x \\ &\leq 0 + \|u\|_{L^\infty(U)} \fint_{U \cap B_\rho(z)} 1 \, \mathrm{d}x \leq \|u\|_{L^\infty(U)} \end{aligned}$$

Now we want to prove point (2): let $\Omega_0 \Subset U$ such that $\overline{U} \subset \cup_{i=0}^{p}\Omega_i$, and let $(\alpha_i)_{i=0}^{p}$ be a partition of unity subordinate of the covering $(\Omega_i)_{i=0}^{p}$. Then, if $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$

$$\int_U u\mathrm{div}(\varphi)\,\mathrm{d}x = \int_U u\mathrm{div}(\alpha_0\varphi)\,\mathrm{d}x + \sum_{i=1}^{p}\int_U u\mathrm{div}(\alpha_i\varphi)\,\mathrm{d}x$$

Since $\alpha_i\varphi \in C_c^1(\Omega_i;\mathbb{R}^n)$, we have that

$$\int_U u\mathrm{div}(\alpha_i\varphi)\,\mathrm{d}x = -\int_U \alpha_i\varphi\cdot\mathrm{d}[Du] + \int_{S_i} u_i^+\alpha_i\langle\varphi,\nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}$$

for all $i = 1,\dots,p$, and

$$\int_U u\mathrm{div}(\alpha_0\varphi)\,\mathrm{d}x = -\int_U \alpha_0\varphi\cdot\mathrm{d}[Du]$$

since $\mathrm{supp}(\alpha_0\varphi) \Subset U$. Hence

$$\int_U u\mathrm{div}(\varphi)\,\mathrm{d}x = -\int_U \varphi\cdot\mathrm{d}[Du] + \int_{\partial U}\langle\varphi,\nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}$$

Finally we prove point (1): let $\Omega_0$ and $(\alpha_i)_{i=0}^{p}$ as above, and let $\overline{\delta} := \min\{\overline{\delta_1},\dots,\overline{\delta}_p\}$. Since $\alpha_i u \in BV(U_i)$, from the local estimate of the trace in each $U_i$, we obtain that, for each $\delta \in (0,\overline{\delta})$

$$\begin{aligned}\int_{S_i}|(\alpha_i u)^+|\,\mathrm{d}\mathcal{H}^{n-1} &\le \sqrt{1+L_i^2}|D(\alpha_i u)|(U_i\setminus(\overline{U}_i)_\delta) + \frac{\sqrt{1+L_i^2}}{\delta}\int_{U_i\setminus(\overline{U}_i)_\delta}|\alpha_i u|\,\mathrm{d}x\\ &\le \sqrt{1+L^2}\int_{U_i\setminus(\overline{U}_i)_\delta}|u|\,\mathrm{d}|D\alpha_i| + \sqrt{1+L^2}\int_{U_i\setminus(\overline{U}_i)_\delta}\alpha_i\,\mathrm{d}|Du| +\\ &\quad \frac{\sqrt{1+L^2}}{\delta}\int_{U_i\setminus(\overline{U}_i)_\delta}\alpha_i|u|\,\mathrm{d}x \end{aligned} \tag{7.3}$$

where in the last step we have used the fact that

$$|D(\alpha_i u)|(U) \le \int_U |\alpha_i|\,\mathrm{d}|Du| + \int_U |u||D\alpha_i|\,\mathrm{d}x$$

Now if $u \in BV(U)$ and $\alpha \in C_c^1(\mathbb{R}^n)$, we have that

$$|(\alpha u)^+(\overline{x}) - \alpha u^+(\overline{x})| \le \lim_{\rho\to 0} \fint_{U\cap B_\rho(\overline{x})}|\alpha(x)-\alpha(\overline{x})||u(x)|\,\mathrm{d}x$$

From the countinuity of $\alpha$, if we fix $\varepsilon > 0$, we have that there exists $\rho_\varepsilon > 0$ such that if $x \in B_{\rho_\varepsilon}(\overline{x})$, then $|\alpha(x) - \alpha(\overline{x})| < \varepsilon$; then

$$|(\alpha u)^+(\overline{x}) - \alpha u^+(\overline{x})| \leq \varepsilon \lim_{\rho\to 0} \fint_{U\cap B_\rho(\overline{x})} |u(x)|\,\mathrm{d}x = C(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$$

for $\mathcal{H}^{n-1}$-a.e. $\overline{x} \in \partial U$. Hence

$$(\alpha u)^+ = \alpha u^+ \qquad \text{in } L^1(\partial U)$$

Hence, from (7.3) we obtain that

$$\begin{aligned}\int_{\partial U} \alpha_i |u^+|\,\mathrm{d}\mathcal{H}^{n-1} \quad &\leq \quad \sqrt{1+L^2}\int_{U_i\setminus(\overline{U}_i)_\delta} |u|\,\mathrm{d}|D\alpha_i| + \sqrt{1+L^2}\int_{U_i-(\overline{U}_i)_\delta} \alpha_i\,\mathrm{d}|Du| + \\ &\quad\quad \frac{\sqrt{1+L^2}}{\delta}\int_{U_i\setminus(\overline{U}_i)_\delta} \alpha_i|u|\,\mathrm{d}x\end{aligned}$$

Now, since $U_i \setminus (\overline{U}_i)_\delta \subset U \setminus \overline{U}_\delta$, recalling that

$$\sum_{i=1}^p \alpha_i \equiv 1 \quad \text{on } \partial U\,, \qquad \sum_{i=1}^p \alpha_i \leq 1 \quad \text{on } U$$

we obtain that, for each $\delta \in (0, \overline{\delta})$,

$$\begin{aligned}\int_{\partial U} |u^+|\,\mathrm{d}\mathcal{H}^{n-1} \quad &\leq \quad \sqrt{1+L^2}\int_{U\setminus\overline{U}_\delta} \sum_{i=1}^p \alpha_i\,\mathrm{d}|Du| + \sqrt{1+L^2}\int_{U\setminus\overline{U}_\delta}\sum_{i=1}^p |D\alpha_i||u|\,\mathrm{d}x + \\ &\quad\quad \frac{\sqrt{1+L^2}}{\delta}\int_{U\setminus\overline{U}_\delta}\sum_{i=1}^p \alpha_i|u|\,\mathrm{d}x \\ &\leq \quad \sqrt{1+L^2}|Du|(U\setminus\overline{U}_\delta) + \\ &\quad\quad \underbrace{\sqrt{1+L^2}\Big(\Big(\max_{U\setminus\overline{U}_\delta}\sum_{i=1}^p |D\alpha_i|\Big) + \frac{1}{\delta}\Big)}_{c(U)}\int_{U\setminus\overline{U}_\delta} |u|\,\mathrm{d}x\end{aligned}$$

and the proof is complete. $\square$

## 7.3 Some applications

Now we present some important applications of the Theorem above. First of all we prove that the trace operator is linear and bounded.

**Theorem 7.3.1.** *Let $U \subset \mathbb{R}^n$ be an open bounded set. Then the trace operator*

$$\begin{array}{rlll} Tr: & BV(U) & \to & L^1(\partial U) \\ & u & \mapsto & u^+ \end{array}$$

*is linar and bounded.*

*Proof.* Trivial. □

Now we want to understand what happend if we paste two $BV$ functions.

**Notation**: let $U, A$ be open bounded subset of $\mathbb{R}^n$ such that $\partial A \cap U$ is Lipschitz. Let $u_1 \in BV(U \setminus \overline{A})$ and $u_2 \in BV(U \cap A)$. We denote by $u_1^+$ and $u_2^-$ the traces of $u_1$ and $u_2$ on $\partial A \cap U$ respectively.

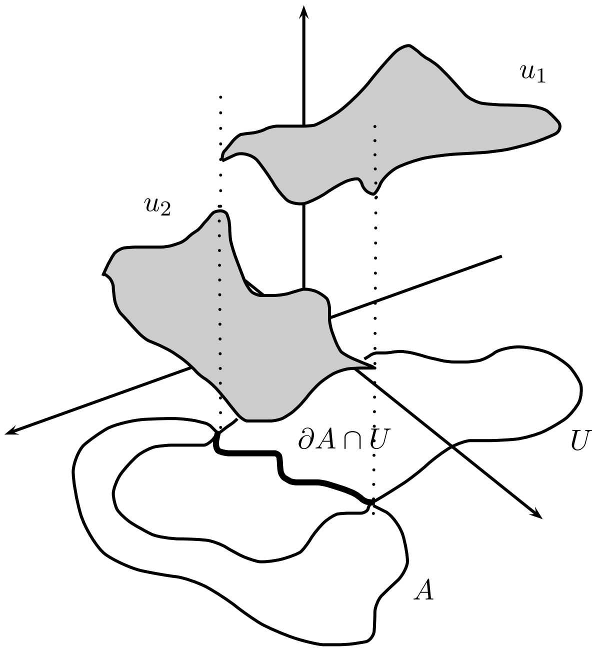
**Theorem 7.3.2.** *Let $U, A$ be open bounded subset of $\mathbb{R}^n$ such that $\partial A \cap U$ is Lipschitz. Let $u_1 \in BV(U \setminus \overline{A})$ and $u_2 \in BV(U \cap A)$. Define*

$$u := \begin{cases} u_1 & ,in\ U \setminus \overline{A} \\ u_2 & ,in\ U \cap A \end{cases}$$

*Then $u \in BV(U)$ and*

$$|Du|(U) = |Du_1|(U \setminus \overline{A}) + |Du_2|(U \cap A) + \int_{\partial A \cap U} |u_1^+ - u_2^-| \, \mathrm{d}\mathcal{H}^{n-1}$$

**Note:** this theorem says that we can measure the “jump”of a function $u \in BV$ in a set of Lebesgue measure 0 with the measure $|Du|$. An important difference between $BV$ functions and Sobolev functions is that in this last case, we cannot expect a similar result, unless $u_1^+ = u_2^+$ $\mathcal{H}^{n-1}$–a.e., since the derivates of a Sobolev function are absolutely continous with respect to the Lebesgue measure.




*Proof.* Let $\varphi \in C_c^1(U\ \mathbb{R}^n)$; we can eventually consider an open set $B$ with lipscitz boundary such that supp$\varphi \Subset B \Subset U$. Hence, denoting with $\nu$ the outer normal to $U \setminus \overline{A}$, we have that

$$\int_{U-\overline{A}} u_1 \mathrm{div}(\varphi)\,\mathrm{d}x = -\int_{U\setminus\overline{A}} \varphi \cdot \mathrm{d}[Du_1] + \int_{\partial A\cap U} u_1^+ \langle \varphi, \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}$$

and

$$\int_{U\cap A} u_1 \mathrm{div}(\varphi)\,\mathrm{d}x = -\int_{U\cap A} \varphi \cdot \mathrm{d}[Du_2] - \int_{\partial A\cap U} u_2^+ \langle \varphi, \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}$$

So

$$\begin{aligned}\int_U u\mathrm{div}(\varphi)\,\mathrm{d}x &= -\int_{U\setminus\overline{A}} \varphi \cdot \mathrm{d}[Du_1] - \int_{U\cap A} \varphi \cdot \mathrm{d}[Du_2] + \\ &\quad \int_{\partial A\cap U} (u_1^+ - u_2^+)\langle \varphi, \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

If we take $\varphi$ such that $|\varphi| \le 1$ we obtain that$\int_U u\mathrm{div}(\varphi)\,\mathrm{d}x < \infty$, and hence $u \in BV(U)$. In particular we have that, for every $\varphi \in C_c^1(U;\mathbb{R}^n)$

$$\int_{\partial A\cap U} (u_1^+ - u_2^+)\langle \varphi, \nu\rangle\,\mathrm{d}\mathcal{H}^{n-1} = -\int_{\partial A\cap U} \varphi \cdot [Du]$$

Defining the vector measures

$$\lambda := (u_1^+ - u_2^+)\nu\,\mathrm{d}\mathcal{H}^{n-1} \llcorner (\partial A \cap U)$$

$$\mu := Du \llcorner (\partial A \cap U)$$

we obtain that $\lambda = -\mu$; hence, passing to the total variation we obtain that

$$\int_{\partial A \cap U} |u_1^+ - u_2^+| \, \mathrm{d}\mathcal{H}^{n-1} = - \int_{\partial A \cap U} \mathrm{d}|Du|$$

□

The trace operator has also a good behaviour with respect to the convergence of $BV$ functions.

**Theorem 7.3.3.** *Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary; let $u \in BV(U)$ and $(u_j)_j \subset BV(U)$ such that*

$$u_j \to u \quad \text{in } L^1(U)$$

*and*

$$|Du_j|(U) \to |Du|(U)$$

*Then*

$$u_j^+ \to u^+ \quad \text{in } L^1(\partial U)$$

*Proof.* Since $u - u_j \in BV(U)$ for each $j$, we can apply the local estimate of the trace, obtaining

$$\int_{\partial U} |(u-u_j)^+| \, \mathrm{d}\mathcal{H}^{n-1} \leq \sqrt{1+L^2}|D(u-u_j)|(U \backslash \overline{U}_\delta) + c(U,\delta) \int_{U \backslash \overline{U}_\delta} |u-u_j| \, \mathrm{d}x$$

Since $(u - u_j)^+ = u^+ - u_j^+$ and for each open set $A \subset \mathbb{R}^n$

$$|D(u-u_j)|(A) \leq |Du|(A) + |Du_j|(A)$$

Hence

$$\begin{aligned} \int_{\partial U} |u^+ - u_j^+| \, \mathrm{d}\mathcal{H}^{n-1} \quad &\leq \quad \sqrt{1+L^2}\Big( |Du|(U \setminus \overline{U}_\delta)|Du_j|(U \setminus \overline{U}_\delta) \Big) + \\ &\quad + c(U,\delta) \int_{U \backslash \overline{U}_\delta} |u-u_j| \, \mathrm{d}x \end{aligned}$$

Since from Theorem 5.2.4

$$\limsup_{j \to \infty} |Du_j|(U \setminus \overline{U}_\delta) \leq |Du_j|(U \setminus U_\delta) \leq |Du|(U \setminus U_\delta)$$

we have that

$$\limsup_{j \to \infty} \int_{\partial U} |u^+ - u_j^+| \, \mathrm{d}\mathcal{H}^{n-1} \quad \leq \quad 2\sqrt{1+L^2}|Du|(U \setminus U_\delta) = 0$$

□

Now we wanto to state a converse of Theorem 7.2.2

**Theorem 7.3.4.** ***(Gagliardo Extension Theorem)*** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary, $\varphi \in L^1(\partial\Omega)$ and $\varepsilon \in (0,1)$. Then there exists a function $u \in W^{1,1}(\Omega) \subset BV(\Omega)$ such that*

- $u^+ = \varphi$, $\mathcal{H}^{n-1}$-*a.e. on* $\partial\Omega$
- $\|u\|_{L^1(\Omega)} \leq \varepsilon \|\varphi\|_{L^1(\partial\Omega)}$
- $\|Du\|_{L^1(\Omega)} \leq C(\varepsilon, \partial\Omega)\|\varphi\|_{L^1(\partial\Omega)}$

*Moreover $u$ is continous and locally Lipschitz in $\Omega$, $\|u\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\partial\Omega)}$. Moreover if $\partial\Omega$ is of class $C^1$, then we can choose $C(\varepsilon, \partial\Omega) = 1 + \varepsilon$.*

Now we present some important properties concerning the trace of a $BV$ functions that will be useful later.

**Remark 7.3.5.** *Let $U$ be an open set in $\mathbb{R}^n$, and $f \in BV(U)$; let $A \Subset U$ be and open set with Lipschitz boundary. Then $f_{|A} \in BV(A)$ and $f_{|_{U\setminus\overline{A}}} \in BV(U \setminus \overline{A})$. Denoting with $f_A^+$ and $f_A^-$ respectively the traces of $f_{|_{U\setminus\overline{A}}}$ and $f_{|A}$, we have that*

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)\cap A} |f(z) - f_A^-(x)| \,\mathrm{d}z = 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial A$$

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B_\rho(x)\setminus A} |f(z) - f_A^+(x)| \,\mathrm{d}z = 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial A$$

*From Theorem 7.3.2 it follows that*

$$\int_{\partial A} |f_A^+ - f_A^-| \,\mathrm{d}\mathcal{H}^{n-1} = |Df|(\partial A) \tag{7.4}$$

*Moreover, from the proof of Theorem 7.3.2, we have that*

$$Df = (f_A^+ - f_A^-)\nu \mathrm{d}\mathcal{H}^{n-1} \qquad \text{on } \partial A$$

*where $\nu$ is the outer normal to $\partial A$.*

*In what follows we will deal with balls; so now we consider the special case of $U = U_R(y)$, $A = U_\rho(y)$, with $0 < \rho < R$ and $y \in \mathbb{R}^n$. For simplicity we suppose $y = 0$. We write $f_\rho^+$ and $f_\rho^-$ instead of $f_A^+$ and $f_A^-$ respectively. Since $|Df|$ is a Radon measure on $U_R(0)$, we have, from (7.4) that for almost every $\rho$*

$$f_\rho^+(x) = f_\rho^-(x) = f(x) \quad \mathcal{H}^{n-1} - \text{a.e. on } \partial U_\rho(0)$$

*Moreover, looking at how we have constructed the trace of a function $f$ in the proof of Proposition 7.1.2, we have that*

$$f^{-}(\rho x) = \lim_{\substack{t\to\rho^{-} \\ t\notin N}} f(tx) \quad \text{in } L^{1}(\partial U_1(0))$$

*where $N$ is a set of measure $0$. Similary for $f^{+}$.*

**Remark 7.3.6.** *Now, if we take $f \in BV(A)$ and define*

$$F := \begin{cases} f & ,\text{in } A \\ 0 & ,\text{in } U \setminus A \end{cases}$$

*from (7.4) it follows that*

$$|Df|(U) = |Df|(A) + \int_{\partial A\cap U} |f_A^{-}|\, \mathrm{d}\mathcal{H}^{n-1}$$

*In particular, if we take $A$ and $U$ as above, and $f = \chi_E$, where $E \subset \mathbb{R}^n$ is a set of finite perimeter in $U$, we have that for the $\rho$'s such that $\chi_{E,\rho}^{-} = \chi_E$ $\mathcal{H}^{n-1}$-a.e. on $\partial U_\rho(x)$, for $x \in \mathbb{R}^n$, (for simplicity we will omitt the point $x$ when we will write balls)*

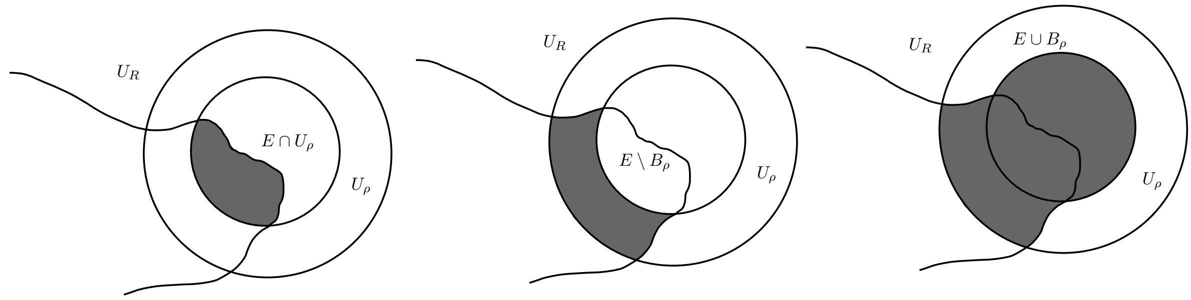
$$P(E\cap U_\rho, U_R) = P(E, U_R\setminus U_\rho) + \mathcal{H}^{n-1}(\partial U_\rho \cap E)$$

*Similary, putting $A := U_R \setminus U_\rho$ and $U := U_R$*

$$P(E\setminus B_\rho, U_R) = P(E, U_R\setminus B_\rho) + \mathcal{H}^{n-1}(\partial U_\rho \cap E)$$

*and*

$$\begin{aligned} &P(E\cup B_\rho, U_R) = P(U_R\setminus(E\cup U_\rho), U_R) = P((U_R\setminus E)\cap(U_R\setminus B_\rho), U_R) \\ = \;& P(U_R\setminus E, U_R\setminus U_\rho) + \mathcal{H}^{n-1}(\partial U_\rho\setminus E) = P(E, U_R\setminus U_\rho) + \mathcal{H}^{n-1}(\partial U_\rho\setminus E)\end{aligned}$$



# Chapter 8

# Some inequalities for minimizing perimeter sets in $\mathbb{R}^n$

The aim of this chapter is to obtain some estimates concerning $BV$ functions, that will lead us to some important inequalities for sets of finite perimeter. In particular in Section 8.2 we will prove that the function

$$r \to \frac{1}{r^{n-1}}|\partial E|(B_r)$$

is non decreasing; moreover we will prove an upper and a lower estimate estimate for the perimeter and a lower estimate for the Lebesgue measure (Proposition 8.2.1) of minimal sets. These estimates will be very useful in chapter 9, where we will study the regularity of minimal surface, and will be foundamental for solving the Bernstein Problem.

## 8.1 Technical results

**Definition 8.1.1.** *Let $E$ be a Caccioppoli set, and let $U$ be an open set. We define*

$$\nu(E,U) := \inf\{|\partial F|(U) \mid F \text{ Caccioppoli set }, F\triangle E \Subset U\}$$

$$\psi(E,U) := |\partial E|(U) - \nu(E,U)$$

*Let $f \in BV(U)$, with $U$ open set in $\mathbb{R}^n$. Define*

$$\nu(f,U) := \inf\left\{|Dg|(U) \mid g \in BV(U),\ \operatorname{supp}(g-f) \subset U\right\}$$

$$\psi(f,U) := |Df|(U) - \nu(f,U)$$

*If $U = U_\rho$ we write $\nu(f,\rho)$ and $\psi(f,\rho)$ in place of $\nu(f,U)$ and $\psi(f,U)$ respectively.*

**Note:** $\psi(f,U)$ is a measure of how close $f$ is to being minimal in $U$. Clearly, if $E$ is a minimal set in $U$, we have $\psi(E,U)=0$.

First of all we want to estimate the distance of the trace of a $BV$ function on the boundary of two balls in terms of the gradient of the function.

**Lemma 8.1.2.** *Let $f \in BV(U_R)$, $0<\rho<r<R$. Then*

$$\int_{\partial U_1} |f^-(rx) - f^-(\rho x)| \,\mathrm{d}\mathcal{H}^{n-1} \le \int_{U_r \setminus U_\rho} \mathrm{d}\Big|\frac{x}{|x|^n} \cdot [Df]\Big|$$

$$\int_{\partial U_1} |f^+(rx) - f^+(\rho x)| \,\mathrm{d}\mathcal{H}^{n-1} \le \int_{B_r \setminus B_\rho} \mathrm{d}\Big|\frac{x}{|x|^n} \cdot [Df]\Big|$$

*Proof.* First of all we consider

$$\int_{\partial U_1} h(x)\Big(f^-(rx) - f^+(\rho x)\Big)\,\mathrm{d}\mathcal{H}^{n-1}$$

where $h$ is a $C^1$ function. So, if we define $\alpha(x) := h\Big(\frac{x}{|x|}\Big)$ we have that

$$\begin{aligned}
& \int_{\partial U_1} h(x)\Big(f^-(rx) - f^+(\rho x)\Big)\,\mathrm{d}\mathcal{H}^{n-1} \\
= \;& \frac{1}{r^{n-1}} \int_{\partial U_r} \alpha f^- \,\mathrm{d}\mathcal{H}^{n-1} - \frac{1}{\rho^{n-1}} \int_{\partial U_\rho} \alpha f^+ \,\mathrm{d}\mathcal{H}^{n-1} \\
= \;& \int_{\partial U_r} \alpha f^- \langle g, \frac{x}{|x|}\rangle \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial U_\rho} \alpha f^+ \langle g, \frac{x}{|x|}\rangle \,\mathrm{d}\mathcal{H}^{n-1} \\
= \;& \int_{U_r \setminus B_\rho} \alpha g \cdot d[Df]
\end{aligned}$$

where in the last step we have take into account that $\mathrm{div}(\alpha g) = 0$ in $\mathbb{R}^n \setminus \{0\}$: in fact

$$\begin{aligned}
\mathrm{div}(\alpha g) &= \alpha \mathrm{div}(g) + \langle \nabla \alpha, g\rangle \\
&= \alpha \sum_{i=1}^n \Big(\frac{1}{|x|^n} - n\frac{x_i^2}{|x|^{n+2}}\Big) + \frac{1}{|x|^{n+1}} \langle \nabla h(\frac{x}{|x|}), x\rangle \\
&\quad - \frac{1}{|x|^{n+3}} |x|^2 \langle \nabla h(\frac{x}{|x|}), x\rangle = 0
\end{aligned}$$

So by Theorem 7.2.2 we obtain the last step of the equalities above. Since if we define the linear functional on $C_c(A)$

$$L_\mu(h) := \int_A h \,\mathrm{d}(g \cdot [Df]) = \int_A hg \cdot \mathrm{d}[Df]$$

from the Riesz Representation Theorem (Theorem 2.8.5) and from the density of $C_c^1(A)$ in $C_c(A)$, we have that

$$|g\cdot[Df]|(A)=\|L_\mu\|=\sup\{\int_A f\mathrm{div}(\alpha g)\,\mathrm{d}x \mid \alpha\in C_c^1(A)\,,\ |\alpha|\leq\}$$

Then if we restrict $h$ such that $|h|\leq 1$, and hence $|\alpha|\leq 1$, we have that

$$|g\cdot[Df]|(U_r\setminus B_\rho)\geq\int_{U_r\setminus B_\rho} f\mathrm{div}(\alpha g)\,\mathrm{d}x$$

and hence

$$\int_{\partial U_1} h(x)[\,f^-(rx)-f^+(\rho x)\,]\,\mathrm{d}\mathcal{H}^{n-1}\leq|g\cdot[Df]|(U_r\setminus B_\rho)$$

From Remark 7.3.5 we have that for almost every $\rho<r$, $|Df|(\partial U_\rho)=0$ and $f^+=f^-=f$. So

$$\int_{\partial U_1} h\Big(f^-(rx)-f^-(\rho x)\Big)\,\mathrm{d}\mathcal{H}^{n-1}\leq\int_{U_r\setminus U_\rho}\mathrm{d}\Big|g\cdot[Df]\Big| \tag{8.1}$$

for almost every $\rho<r$.

Now fix a $\rho<r$; from Remark 7.3.5 we can find a sequence $(\rho_i)_i$ such that $\rho_i\to\rho$, (8.1) holds, and $f^-(\rho_i\cdot)\to f^-(\rho\cdot)$ in $L^1(\partial U_1)$. Taking the limit in (8.1) we obtain that (8.1) holds for every $\rho<r$. Finally, taking the supremum over all $h\in C^1$ with $|h|\leq 1$ we obtain the desired result.

The proof of the second inequaility is similar to the proof of the first one. $\square$

Now we want to obtain a covergence results for $\nu$ and $\psi$ when we calculate them in balls that converges to a bigger ball.

**Lemma 8.1.3.** *Let $f\in BV(U_R)$, $\rho<R$. Let $(\rho_i)_i$ such that $\rho_i\leq\rho$ and $\rho_i\to\rho$. Then*

$$\lim_{i\to\infty}\nu(f,\rho_i)=\nu(f,\rho)$$

*and*

$$\lim_{i\to\infty}\psi(f,\rho_i)=\psi(f,\rho)$$

*Proof.* Fix $\epsilon>0$; then there exists a function $g\in BV(U_\rho)$ such that $\mathrm{supp}(f-g)\subset U_\rho$ and

$$|Dg|(U_\rho)\leq\nu(f,\rho)+\epsilon$$

For $j$ large enough we have that $\mathrm{supp}(f-g)\subset U_{\rho_i}$. Hence

$$|Dg|(U_\rho)\geq|Dg|(U_{\rho_i})\geq\nu(f,\rho_i)$$

Since $\epsilon$ is arbitrary we obtain that

$$\limsup_{i\to\infty} \nu(f,\rho_i) \leq \nu(f,\rho)$$

To prove the other inequailty, for each $i$ we can find a function $g_i \in BV(U_\rho)$ such that $\mathrm{supp}(g_i - f) \subset U_{\rho_i}$ and

$$\nu(f,\rho_i) + \frac{1}{i} \geq |Dg_i|(U_{\rho_i})$$

Hence

$$|Dg_i|(U_\rho) = |Dg_i|(U_{\rho_i}) - |Df|(U_\rho \setminus B_{\rho_i}) \geq \nu(f,\rho) - |Df|(U_\rho \setminus B_{\rho_i})$$

and therefore, since $|Df|(U_\rho \setminus B_{\rho_i}) \to 0$,

$$\liminf_{i\to\infty} \nu(f,\rho_i) \geq \nu(f,\rho)$$

The second statement follows immediately from the first one. $\square$

Next result is very important, because it estimates the difference of $\nu(f,\rho)$ and $\nu(g,\rho)$ in terms of the integral difference of the traces of $f$ and $g$ on the boundary of $U_\rho$. This results tells us that if $f$ and $g$ have $\mathcal{H}^{n-1}$-a.e. the same trace on $\partial U_\rho$, then $\nu(f,\rho) = \nu(g,\rho)$. So we can think $\nu(f,\rho)$ as

$$\inf\{|Dg|(U_\rho) \mid g^- = f^- \text{ in } L^1(\partial U_\rho)\}$$

**Lemma 8.1.4.** *Let $f, g \in BV(U_R)$ and $\rho < R$. Then*

$$|\nu(f,\rho) - \nu(g,\rho)| \leq \int_{\partial U_\rho} |f^- - g^-| \, \mathrm{d}\mathcal{H}^{n-1}$$

*Proof.* Since the inequality is simmetric, we can just prove that

$$\nu(f,\rho) - \nu(g,\rho) \leq \int_{\partial U_\rho} |f^- - g^-| \, \mathrm{d}\mathcal{H}^{n-1}$$

Fix $\epsilon > 0$; then there exists a function $\varphi \in BV(U_R)$ such that $\mathrm{supp}(f - \varphi) \subset U_\rho$ and

$$|D\varphi|(U_\rho) \leq \nu(f,\rho) + \epsilon$$

Let $(\rho_i)_i$ be a sequence such that $\rho_i \leq \rho$, $\rho_i \to \rho$ and

$$|Df|(\partial U_\rho) = |Dg|(\partial U_\rho) = 0$$

and $\mathrm{supp}(f - \varphi) \subset U_{\rho_i}$. Define, forevery $i$

$$g_i := \begin{cases} \varphi & , \text{in } U_{\rho_j} \\ g & , \text{in } U_R \setminus B_{\rho_j} \end{cases}$$

Then by Proposition 7.3.2 we have that $g_i \in BV(U_R)$, and

$$\begin{aligned}
\nu(g,\rho) &\leq |Dg_i|(U_\rho) \\
&= |D\varphi|(U_{\rho_i}) + |Dg|(U_\rho \setminus B(\rho_i)) + \int_{\partial U_{\rho_i}} |f^- - g^-| \, \mathrm{d}\mathcal{H}^{n-1} \\
&\leq |D\varphi|(U_\rho) + |Dg|(U_\rho \setminus B(\rho_i)) + \int_{\partial U_{\rho_i}} |f^- - g^-| \, \mathrm{d}\mathcal{H}^{n-1} \\
&\leq \nu(f,\rho) + \epsilon + |Dg|(U_\rho \setminus U_{\rho_i}) + \int_{\partial U_{\rho_i}} |f^- - g^-| \, \mathrm{d}\mathcal{H}^{n-1}
\end{aligned}$$

Since $\epsilon$ is arbitrary we obtain

$$\nu(g,\rho) - \nu(f,\rho) - |Dg|(U_\rho \setminus U_{\rho_i}) \leq \int_{\partial U_{\rho_i}} |f^- - g^-| \, \mathrm{d}\mathcal{H}^{n-1}$$

Now, letting $i \to \infty$ we obtain the desired result. □

**Remark 8.1.5.** *if $\psi(f,R) = 0$, from the previous result it follows that, for every $g \in BV(U_R)$,*

$$|Df|(U_\rho) \leq |Dg|(U_\rho) + \int_{\partial U_\rho} |f^- - g^-| \, \mathrm{d}\mathcal{H}^{n-1}$$

Next two results are thecnical results we will use to obtain an useful formulation of the estimate of Lemma 8.1.2.

**Lemma 8.1.6.** *Let $f \in BV(U_R)$ and $0 < \rho < r < R$. Then*

$$\Big[ \int_{U_r \setminus U_\rho} \mathrm{d}\Big| \frac{x}{|x|^n} \cdot Df \Big| \Big]^2 \leq 2 \Big[ \int_{U_r \setminus U_\rho} \frac{1}{|x|^{n-1}} \, \mathrm{d}|Df| \Big] \cdot$$

$$\cdot \Big[ \frac{1}{r^{n-1}} |Df|(U_r) - \frac{1}{\rho^{n-1}} |Df|(U_\rho) + (n-1) \int_\rho^r t^{-n} \psi(f,t) \, \mathrm{d}t \Big]$$

*Proof.* Suppose first that $f \in C^1(U_R)$. Define, for $0 < t < R$,

$$f_t(x) := \begin{cases} f(x) & , t < |x| < R \\ f\Big(t\dfrac{x}{|x|}\Big) & , |x| < t \end{cases}$$

Then

$$Df_t(x) := \begin{cases} Df(x) & , t < |x| < R \\ \dfrac{t}{|x|} \Big[ Df\Big(t\dfrac{x}{|x|}\Big) - \dfrac{x}{|x|^2} \langle Df\Big(t\dfrac{x}{|x|}\Big), x \rangle \Big] & , |x| < t \end{cases}$$

Then, for $|x| < t$,

$$\Big|Df\Big(t\frac{x}{|x|}\Big)\Big| = \frac{t}{|x|}\Big| Df\Big(t\frac{x}{|x|}\Big) - \frac{x}{|x|^2}\langle Df\Big(t\frac{x}{|x|}\Big), x\rangle \Big|$$

To calculate it we computed

$$\begin{aligned}\Big| Df\Big(t\frac{x}{|x|}\Big) - \frac{x}{|x|^2}\langle Df\Big(t\frac{x}{|x|}\Big), x\rangle \Big|^2 &= \Big|Df\Big(t\frac{x}{|x|}\Big)\Big|^2 + \frac{1}{|x|^2}\langle Df\Big(t\frac{x}{|x|}\Big), x\rangle^2 - \\ &\quad \frac{2}{|x|^2}\langle Df\Big(t\frac{x}{|x|}\Big), x\rangle^2 \\ &= \Big|Df\Big(t\frac{x}{|x|}\Big)\Big|^2 - \frac{1}{|x|^2}\langle Df\Big(t\frac{x}{|x|}\Big), x\rangle^2 \\ &= \Big|Df\Big(t\frac{x}{|x|}\Big)\Big|^2 \left[1 - \frac{\langle x, Df\Big(t\frac{x}{|x|}\Big)\rangle^2}{|x|^2\Big|Df\Big(t\frac{x}{|x|}\Big)\Big|^2}\right]\end{aligned}$$

Hence, if $|x| < t$

$$\Big|Df\Big(t\frac{x}{|x|}\Big)\Big| = \frac{t}{|x|}\Big|Df\Big(t\frac{x}{|x|}\Big)\Big| \left[1 - \frac{\langle x, Df\Big(t\frac{x}{|x|}\Big)\rangle^2}{|x|^2\Big|Df\Big(t\frac{x}{|x|}\Big)\Big|^2}\right]^{\frac{1}{2}}$$

Now

$$\nu(f,t) = |Df|(U_t) - \psi(f,t) \leq |Df_t|(U_t) = \int_{U_t} |Df_t| \,\mathrm{d}x \tag{8.2}$$

where in the last step we have take into account that $f \in C^1$. From the Change of Variable Formula, and recalling the definition of $Df_t$, we have that

$$\int_{U_t} |Df_t| \,\mathrm{d}x = t\int_0^1 s^{n-1}\,\mathrm{d}s\Big(\int_{\partial U_t} |Df_t(sz)| \,\mathrm{d}\mathcal{H}^{n-1}(z)\Big)$$

Hence

$$\int_{U_t} |Df_t(x)| \,\mathrm{d}x = \frac{t}{n-1}\int_{\partial U_t} |Df(z)| \left[1 - \frac{\langle z, Df(z)\rangle^2}{\Big|Df(z)\Big|^2}\right]^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}(z)$$

Since if $|a| < 1$ it holds $(1-a)^{\frac{1}{2}} \leq 1 - \frac{1}{2}a$, from (8.2) we obtain that

$$\nu(f,t) \leq \frac{t}{n-1}\int_{\partial U_t} |Df| \,\mathrm{d}\mathcal{H}^{n-1} - \frac{t}{2(n-1)}\int_{\partial U_t} \frac{\langle z, Df\rangle^2}{|z|^2|Df|} \,\mathrm{d}\mathcal{H}^{n-1}$$

hence

$$\begin{aligned}
\frac{1}{2}t^{1-n}\int_{\partial U_t}\frac{\langle z,Df\rangle^2}{|z|^2|Df|}\,\mathrm{d}\mathcal{H}^{n-1} &\leq t^{1-n}\int_{\partial U_t}|Df|\,\mathrm{d}\mathcal{H}^{n-1}-(n-1)t^{-n}\int_{U_t}|Df|\,\mathrm{d}x+\\
&\quad+(n-1)t^{-n}\psi(f,t)\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\Big(t^{1-n}\int_{U_t}|Df|\,\mathrm{d}x\Big)+(n-1)t^{-n}\psi(f,t)
\end{aligned}\tag{8.3}$$

Then, integrating with respect to $t$ from 0 to $\rho$ we obtain

$$\begin{aligned}
\frac{1}{2}\int_{U_r\setminus U_\rho}\frac{\langle x,Df\rangle^2}{|x|^{n+1}|Df|}\,\mathrm{d}x &\leq r^{1-n}\int_{U_r}|Df|\,\mathrm{d}x-\rho^{1-n}\int_{U_\rho}|Df|\,\mathrm{d}x\\
&+(n-1)\int_\rho^r t^{-n}\psi(f,t)\,\mathrm{d}t
\end{aligned}\tag{8.4}$$

From the Schwartz inequality we have that

$$\begin{aligned}
\Big[\int_{U_r\setminus U_\rho}\Big|\langle\frac{x}{|x|^n},Df\rangle\Big|\,\mathrm{d}x\Big]^2 &\leq \Big(\int_{U_r\setminus U_\rho}|x|^{1-n}|Df|\,\mathrm{d}x\Big)\cdot\\
&\cdot\Big(\int_{U_r\setminus U_\rho}\frac{|\langle x,Df\rangle|}{|x|^n|Df|}\cdot\Big|\langle\frac{x}{|x|},Df\rangle\Big|\,\mathrm{d}x\Big)
\end{aligned}$$

So, from (8.4), we have obtained the desired result for $f\in C^1$.
Now, let $f\in BV(U_R)$; we can approximate $f$ by $C^1$ functions $f_i$ such that for almost every $t$

$$\int_{U_t}|Df_i|\,\mathrm{d}x\to|Df|(U_t)\,,\qquad\int_{\partial U_t}|f-f_i|\,\mathrm{d}\mathcal{H}^{n-1}\to 0$$

Now from Lemma 8.1.4, $\psi(f_i,t)\to\psi(f,t)$; moreover, since

$$\Big|\frac{x}{|x|^n}\cdot[Df_i]\Big|=\Big|\frac{x}{|x|^n}\Big|\cdot|Df_i|$$

we obtain that the result holds for $f\in BV(U_R)$ and for almost every $r,\rho$. Finally, if we fix $0<r<\rho<R$, we can find increasing sequences $(r_i)_i$ and $(\rho_i)_i$ such that the result holds for each $\rho_i<r_i$. Then from Lemma 8.1.3 we obtain that the result holds for every $r,\rho$. □

**Lemma 8.1.7.** *Let $f\in BV(U_R)$ and $0<\rho<r<R$. Then*

$$\begin{aligned}
\int_{U_r\setminus U_\rho}|x|^{1-n}\,\mathrm{d}|Df| &\leq \Big[1+(n-1)\log\Big(\frac{r}{\rho}\Big)\Big]r^{1-n}|Df|(U_r)\\
&+(n-1)^2\int_\rho^r s^{-n}\log\Big(\frac{s}{\rho}\Big)\psi(f,s)\,\mathrm{d}s
\end{aligned}$$

*Proof.* First suppose $f \in C^1(U_R)$. In this case, from the Change of Variable Formula we have

$$\int_{U_r\setminus U_\rho} |x|^{1-n}|Df|\,\mathrm{d}x = \int_\rho^r t^{1-n}\Big(\int_{\partial U_t} |Df|\,\mathrm{d}\mathcal{H}^{n-1}\Big)\,\mathrm{d}t = \int_\rho^r t^{1-n}\nu'(t)\,\mathrm{d}t$$

where we have set $\nu(t) := \int_{U_t} |Df|\,\mathrm{d}x$. Integrating by parts

$$\begin{aligned}\int_\rho^r t^{1-n}\nu'(t)\,\mathrm{d}t &= [t^{1-n}\nu(t)]_\rho^r + (n-1)\int_\rho^r t^{-n}\Big(\int_{U_t} |Df|\,\mathrm{d}x\Big)\,\mathrm{d}t \\ &\le r^{1-n}\int_{U_r} |Df|\,\mathrm{d}x + (n-1)\int_\rho^r t^{-n}\Big(\int_{U_t} |Df|\,\mathrm{d}x\Big)\,\mathrm{d}t\end{aligned}$$

Using the fact that the last term in the inequality of the previous Lemma is positive, we have that

$$t^{-n}\int_{U_t} |Df|\,\mathrm{d}x \le t^{-1}\Big[r^{1-n}\int_{U_r} |Df|\,\mathrm{d}x + (n-1)\int_t^r s^{-n}\psi(f,s)\,\mathrm{d}s\Big]$$

Hence

$$\begin{aligned}\int_\rho^r t^{-n}\Big(\int_{U_t}|Df|\,\mathrm{d}x\Big)\,\mathrm{d}t &\le \int_\rho^r t^{-1}\Big[r^{1-n}\int_{U_r}|Df|\,\mathrm{d}x + (n-1)\int_t^r s^{-n}\psi(f,s)\,\mathrm{d}s\Big] \\ &= r^{1-n}\log\Big(\frac{r}{\rho}\Big)\int_{U_r}|Df|\,\mathrm{d}x + (n-1)\int_\rho^r \frac{\mathrm{d}t}{t}\int_t^r s^{-n}\psi(f,s)\,\mathrm{d}s \\ &= r^{1-n}\log\Big(\frac{r}{\rho}\Big)\int_{U_r}|Df|\,\mathrm{d}x + \\ &\quad (n-1)\Big[-\log\rho\int_\rho^r s^{-n}\psi(f,s)\,\mathrm{d}s + \int_\rho^r (\log s)s^{-n}\psi(f,s)\,\mathrm{d}s\Big] \\ &= r^{1-n}\log\frac{r}{\rho}\int_{U_r}|Df|\,\mathrm{d}x + (n-1)\Big[\int_\rho^r \log\Big(\frac{s}{\rho}\Big)s^{-n}\psi(f,s)\,\mathrm{d}s\Big]\end{aligned}$$

Hence

$$\begin{aligned}\int_{U_r\setminus U_\rho}|x|^{1-n}|Df|\,\mathrm{d}x \le r^{1-n}\int_{U_r}|Df|\,\mathrm{d}x + (n-1)r^{1-n}\log\Big(\frac{r}{\rho}\Big)\int_{U_r}|Df|\,\mathrm{d}x + \\ (n-1)^2\int_\rho^r s^{-n}\log\Big(\frac{s}{\rho}\Big)\psi(f,s)\,\mathrm{d}s\end{aligned}$$

That is the desired estimate for $f \in C^1(U_R)$. To prove the result for $f \in BV(U_R)$ and for every $\rho, r$ just reasoning as in the previous Lemma. $\square$

Putting together all the lemmas we obtain the following

**Proposition 8.1.8.** *Let $f \in BV(U_R)$, $0 < \rho < r < R$. Then*

$$\Big|\frac{1}{r^{n-1}}\int_{U_r} \mathrm{d}[Df] - \frac{1}{\rho^{n-1}}\int_{U_\rho} \mathrm{d}[Df]\Big|^2 \leq \Big[\frac{1}{r^{n-1}}|Df|(U_r) - \frac{1}{\rho^{n-1}}|Df|(U_\rho)$$
$$+ (n-1)\int_\rho^r s^{-n}\psi(f,s)\,\mathrm{d}s\Big] \cdot \Big[\frac{2}{r^{n-1}}\Big(1 + (n-1)\log\frac{r}{\rho}\Big)\int_{U_r}\mathrm{d}|Df|$$
$$+ 2(n-1)^2\int_\rho^r s^{-n}\log\frac{s}{\rho}\psi(f,s)\,\mathrm{d}s\Big]$$

*Proof.* From Remark 7.2.2 we have that

$$\int_{U_t} \mathrm{d}[Df] = \int_{\partial U_t} f^-(x)\frac{x}{|x|}\,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{t^{n-1}}\int_{\partial U_1} f^-(tx)x\,\mathrm{d}\mathcal{H}^{n-1}$$

Hence

$$\Big|\frac{1}{r^{n-1}}\int_{U_r} \mathrm{d}[Df] - \frac{1}{\rho^{n-1}}\int_{U_\rho} \mathrm{d}[Df]\Big| \leq \int_{\partial U_1}\Big|f^-(rx) - f^-(\rho x)\Big|\,\mathrm{d}\mathcal{H}^{n-1}$$

The result follows by putting together all the previous estimates. □

## 8.2 Estimates for minimal sets

In this section we want to obtain some estimate for the perimeter and the Lebesgue measure of minimal sets, usign the results of the previous section. So we consider the thesis of Proposition 8.1.8 when $f = \chi_E$ and $E$ is a set of minimizing boundary in $U_R$, that is $\psi(E,R) = 0$. It hold:

**Fact** 1: from the previous proposition we obtain that

$$\Big[\int_{\partial U_1} |\chi_E^-(\rho x) - \chi_E^-(rx)|\mathcal{H}^{n-1}\Big]^2 \leq \Big[\int_{U_r\setminus U_\rho} \mathrm{d}\Big|\frac{x}{|x|^n}\cdot[\partial E]\Big|\Big]^2$$
$$\leq 2\int_{U_r\setminus U_\rho} |x|^{1-n}\,\mathrm{d}|\partial E|\Big[\frac{1}{r^{n-1}}|\partial E|(U_r) - \frac{1}{\rho^{n-1}}|\partial E|(U_\rho)\Big] \tag{8.5}$$

and hence, for every $\rho < r < R$,

$$\frac{1}{\rho^{n-1}}|\partial E|(U_\rho) \leq \frac{1}{r^{n-1}}|\partial E|(U_r) \tag{8.6}$$

that is the function

$$\rho \mapsto \frac{1}{\rho^{n-1}}|\partial E|(U_\rho)$$

is a non decreasing function.

**Fact** 2: now let $0 < s < r \leq R$ and consider the sets[1] $E \setminus B_s$ and $E \cup B_s$. Since $E$ is minimal in $U_R$, $E$ is also minimal in $U_r$; hence

$$P(E \setminus B_s, U_r) \geq |\partial E|(U_r)$$

and

$$P(E \cap B_s, U_r) \geq |\partial E|(U_r)$$

Recalling Remark 7.3.6 we obtain

$$P(E \setminus B_s, U_r) = P(E, U_r \setminus B_s) + \mathcal{H}^{n-1}(\partial U_s \cap E)$$

and

$$P(E \cup B_s, U_r) = P(E, U_r \setminus B_s) + \mathcal{H}^{n-1}(\partial U_s \setminus E)$$

for almost all $s < r$. Hence

$$\begin{aligned} P(E, U_r) &\leq P(E, U_r \setminus B_s) + \min(\mathcal{H}^{n-1}(\partial U_s \cap E), \mathcal{H}^{n-1}(\partial U_s \setminus E)) \\ &\leq P(E, U_r \setminus B_s) + \frac{1}{2} s^{n-1} n\omega_n \qquad (8.7) \end{aligned}$$

for almost all $s$. So, if we take a sequence $(s_i)_i$, $s_i \to r$, for which (8.7) holds we obtain that

$$|\partial E|(U_r) \leq \frac{1}{2} r^{n-1} n\omega_n \qquad (8.8)$$

**Fact** 3: now, if we take $x \in \partial^* E$, we have that

$$\frac{|\partial E|(U_\rho)}{\rho^{n-1}} \xrightarrow{\rho \to 0} \omega_{n-1}$$

Hence, letting $\rho \to 0$ in (8.6)

$$|\partial E|(U_r) \geq r^{n-1} \omega_{n-1} \qquad (8.9)$$

Since $\overline{\partial^* E} = \partial E$, this estimate holds for each $x \in \partial E$.

A similar inequality holds for the $\mathcal{L}^n$ measure of $E \cap U_r(x)$.

**Proposition 8.2.1.** *Suppose $\psi(E, U) = 0$, and let $x_0 \in E$. Then, for every $r < \mathrm{d}(x_0, \partial U)$ we have*

$$\mathcal{L}^n(E \cap U_r(x_0)) \geq \frac{r^n}{2nC_1}$$

*where $C_1$ is the constant of the isoperimetric inequality (see Theorem 5.4.2).*

[1] For simplicity we omitt the center of the balls.

*Proof.* Let $\rho < \mathrm{d}(x_0, \partial U)$; since $\mathrm{supp}(\chi_E - \chi_{E\setminus U_\rho}) \subset U$

$$|\partial E|(U) \leq |\partial(E \setminus U_\rho)|(U)$$

hence

$$|\partial E|(U_\rho) \leq \int_{\partial U_\rho} \chi_E \, \mathrm{d}\mathcal{H}^{n-1} \tag{8.10}$$

On the other hand, from almost every $\rho$, it holds

$$|\partial(E \cap U_\rho)|(U) = |\partial E|(U_\rho) + \int_{\partial U_\rho} \chi_E \, \mathrm{d}\mathcal{H}^{n-1} \tag{8.11}$$

Hence from (8.10) and (8.11) it follows

$$|\partial(E \cap U_\rho)|(U) \leq 2\int_{\partial U_\rho} \chi_E \, \mathrm{d}\mathcal{H}^{n-1} = 2\frac{\mathrm{d}}{\mathrm{d}\rho}\mathcal{L}^n(E \cap U_\rho)$$

where the last step follows from the Coarea Formula. Recalling the isoperimetric inequality (see Theorem 5.4.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\mathcal{L}^n(E \cap U_\rho) \geq \frac{1}{2C_1}\Big(\mathcal{L}^n(E \cap U_\rho)\Big)^{\frac{n-1}{n}}$$

Integrating from 0 to $r$ we obtain

$$\begin{aligned}
\mathcal{L}^n(E \cap U_r) &\geq \frac{1}{2C_1}\int_0^r \Big(\mathcal{L}^n(E \cap U_\rho)\Big)^{\frac{n-1}{n}} \, \mathrm{d}\rho \\
&\geq \frac{1}{2C_1}\int_0^r \Big(\frac{1}{\rho^n}\Big)^{-\frac{1-n}{n}} = \frac{1}{2C_1}\frac{r^n}{n}
\end{aligned}$$

□

**Remark 8.2.2.** *Since $E$ minimize the perimeter in $U$, also $U \setminus E$ minimize the perimeter in $U$. Hence if $x_0 \in \partial E$ and $U_r(x_0) \subset U$ we obtain*

$$\frac{1}{2n\omega_n C_1}\mathcal{L}^n(U_r) \leq \mathcal{L}^n(E \cap U_r(x_0)) \leq \Big(1 - \frac{1}{2n\omega_n C_1}\Big)\mathcal{L}^n(U_r)$$

*These inequalities tell us that if we look at the minimal set $E$ from an its boundary point, the set $E$, measurally speaking, cannot be too many, nor too much with respect to a ball.*

# Chapter 9

# Regularity of minimal surfaces in $\mathbb{R}^n$

In this chapter we will study the regularity of minimal surfaces: in particular we will prove (see Theorem 9.3.5) that minimal surfaces in $\mathbb{R}^n$ are regular for $n \leq 7$, while in higher dimensions there exist minimal surfaces with singularities (see Section 9.4). We start by stating in Section 9.1 that the only possible singularities for a minimal surface $E$ must occour in $\partial E \setminus \partial^* E$ (Theorem 9.1.2). Then in the following two sections we will prove that there are no singularity for minimal surfaces in $\mathbb{R}^n$ for $n \leq 7$. The idea to do this is the following one: given a minimal set $E$ we blow-up it in a point $x \in \partial E$, obtaining a minimal cone $C$ (see Theorem 9.2.2). Then $C$ will be an hyperplane if and only if $\partial E$ is regular in $x$. So the problem of the regularity of minimal surfaces in $\mathbb{R}^n$ is turned into the problem of existence of singular minimal cone in $\mathbb{R}^n$. We will show that we can concentrate on minimal cones that have only a singularity (see Theorem 9.2.5). In Section 9.3 we will prove that such a cones cannot exist in $\mathbb{R}^n$ for $n \leq 7$, proving the regularity of minimal surfaces for $n \leq 7$: we will obtain this result calculating the first and the second variation of the area functional (subsections 9.3.1 and 9.3.2) and then showing that the mean curvature of a minimal cone in $\mathbb{R}^n$ with the only possible singularity at the origin, is 0 for $n \leq 7$ (Theorem 9.3.4) and hence that such a minimal cone must be an half space.
In Section 9.4 we will give an example of a minimal surface in $\mathbb{R}^8$ having a singularity at the origin (the so called *Simons cone*), thus proving that the regularity result obtained is the best possible. Finally, to understand the behaviour of minimal surfaces in higher dimension we will state that the singular set of a minimal surface has bounded Hausdorff dimension (see Theorem 9.3.6).

## 9.1 Partial regularity of minimal surfaces

In this section we state the theorem of partial regularity of minimal surfaces, showing that the reduced boundary $\partial^* E$ of minimal surfaces is analytic and the only possible singularities must occur in $\partial E \setminus \partial^* E$. For the proof of these results see [Giu84, chapters 6, 7, 8]

The principal tool in regularity theorey is the following De Giorgi Lemma

**Lemma 9.1.1.** *For every $n \geq 2$ and every $\alpha$, $0 < \alpha < 1$, there exists a constant $\sigma(n, \alpha)$ such that if $E$ is a Caccippoli set in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, $\rho > 0$ and*

$$\psi(E, B_\rho(x)) = 0$$

$$|\partial E|(B_\rho(x)) - \Big| \int_{B_\rho(x)} \mathrm{d}[\partial E] \Big| < \sigma(n, \alpha)\rho^{n-1}$$

*then*

$$|\partial E|(B_{\alpha\rho}(x)) - \Big| \int_{B_{\alpha\rho}(x)} \mathrm{d}[\partial E] \Big| \leq \alpha^n \Big[ |\partial E|(B_\rho(x)) - \Big| \int_{B_\rho(x)} \mathrm{d}[\partial E] \Big| \Big]$$

The meaning of this lemma is the following one: suppose $x = 0$; the term

$$\begin{aligned} \Lambda(E, \rho) \;&:=\; |\partial E|(B_\rho) - \Big| \int_{B_\rho} \mathrm{d}[\partial E] \Big| \\ &= \rho^{n-1} \Big[ \mathcal{H}^{n-1}(B_\rho \cap \partial^* E) - \Big| \int_{B_\rho \cap \partial^* E} \nu_E(y) \, \mathrm{d}\mathcal{H}^{n-1}(y) \Big| \Big] \end{aligned}$$

is called the *excess*, and it is the measure of how much the direction of $\nu_E$ change in $B_\rho \cap \partial^* E$. So if we can estimate the excess $\Lambda(E, \rho)$ with $\sigma(n, \alpha)\rho^{n-1}$, then we can estimate the excess $\Lambda(E, \alpha\rho)$ in terms of $\Lambda(E, \rho)$.

The following result shows that $\partial E$ is analytic in a neighbourhood of every point $x$ that satisfied the hypothesis of the previous lemma. In particular it can be shown that all the points of the reduced boundary satisfied the hypothesis of De Giorgi Lemma.

**Theorem 9.1.2.** *Suppose $E$ is a Caccippoli set in $\mathbb{R}^n$, $x \in \partial E$, $\rho > 0$ and $0 < \alpha < 1$ are such that*

$$\psi(E, B_\rho(x)) = 0$$

$$|\partial E|(B_\rho(x)) - \Big| \int_{B_\rho(x)} \mathrm{d}[\partial E] \Big| < \sigma(n, \alpha)\rho^{n-1}$$

*Then $\partial E \cap B_r(x)$ is an analtic hypersurface for $r = \rho(\alpha - \alpha^{\frac{n}{n-1}})$.*

So we have state that the singular set is cointained in $\partial E \setminus \partial^* E$. This set can be nonempty, as we will see in Theorem 9.4.7, but we will find an upper bound for its the Hausdorff dimension (see Theorem 9.3.6).

## 9.2 Minimal Cones

The aim of this section is to prove that the existence of singularity for minimal surfaces is equaivalent to the existence of minimal cone with singularity, and in particular of minimal cones whith singularity at the origin. To obtain this results we have to blow up a minimal set: this procedure will produce a minimal cone (Theorem 9.2.2). Moreover we will prove, again blowing up such a minimal cone (Proposition 9.2.6) and proving a relation between the minimality of the cone and its exploded (Proposition 9.2.8), that we can "exclude the dimensions that have more that a singular point" (Theorem 9.2.5).

Since we have to deal with exploded sets, we start by studing the behaviour of a sequence of a minimal sets converging to a set.

**Lemma 9.2.1.** *Let $U \subset \mathbb{R}^n$ be an open set, and let $(E_j)_j$ be a sequence of Caccippoli sets of least area in $U$. Suppose that there exists a set $E$ such that $E_j \to E$. Then $E$ has least perimeter in $U$.*
*Moreover, if $L \Subset U$ is an open set with $|\partial E|(\partial L) = 0$, then*

$$\lim_{j\to\infty} |\partial E_j|(L) = |\partial E|(L)$$

*Proof.* We have to prove that, if $A \Subset U$, then $\psi(E, A) = 0$. Since if $\psi(E, B) = 0$ and $B \supset A$ then $\psi(E, A) = 0$, we can suppose $\partial A$ smooth[1]. Hence

$$|\partial E_j|(A) \le \mathcal{H}^{n-1}(\partial A)$$

From the semicontinuity (see Theorem 5.1.4)

$$|\partial E|(A) \le \mathcal{H}^{n-1}(\partial A)$$

We want to apply Lemma 8.1.4 to the functions $\chi_{E_j}$ and $\chi_E$; but we are not sure that

$$\lim_{i\to\infty} \int_{\partial A} |\chi^-_{E_j} - \chi^-_E| \, \mathrm{d}\mathcal{H}^{n-1} = 0$$

So we have tot do in this way: for $t > 0$ and define

$$A_t := \{x \in U \mid \mathrm{d}(x, A) < t\}$$

Let $T > 0$ such that $A_T \subset U$. Since $E_j \to E$ we obtain that

$$\lim_{j\to\infty} \int_{A_T} |\chi_{E_j} - \chi_E| \, \mathrm{d}x = 0$$

[1] Let $A \Subset U$; if we take $\varepsilon < \frac{d(\partial A, U)}{2}$ and we consider a mollifier $\rho$, we have that $\chi_A * \rho_\varepsilon$ is of class $C^\infty$, and $\mathrm{supp}(\chi_A * \rho) \Subset U$. So, by Sard's Lemma[2] there exists a $t \in (0, 1)$ such that $\partial A_t$ is smooth, where $A_t := \{x \in \mathbb{R}^n \mid (\chi_A * \rho_\varepsilon)(x) \le t\} \Subset U$.

Then there exists a subsequence $(E_{j_k})_{k\to\infty}$ such that $\chi_{E_{j_k}} \to \chi_E$ pointwise a.e. in $A_T$. Hence, for almost every $0 < t < T$ we have

$$\lim_{k\to\infty} \int_{\partial A_t} |\chi_{E_{j_k}} - \chi_E| \, \mathrm{d}\mathcal{H}^{n-1} = 0 \tag{9.1}$$

Since $E_{j_k}$ and $E$ are sets of finite perimeter in $A_t$, from Remark 7.3.5 we have that for almost every $0 < t < T$

$$(\chi_{E_{j_k}})^-_{A_t} = \chi_{E_{j_k}} \qquad (\chi_E)^-_{A_t} = \chi_E \tag{9.2}$$

Hence for almost all $0 < t < T$ we have that (9.1) and (9.2) hold. For these $t$, from Lemma 8.1.4, we obtain that

$$\lim_{k\to\infty} \nu(E_{j_k}, A_t) = \nu(E, A_t)$$

Then

$$\psi(E, A_t) = |\partial E|(A_t) - \nu(E, A_t) \leq \liminf_{k\to\infty} \Big( |\partial E_{j_k}|(A_t) - \nu(E_{j_k}, A_t) \Big) = 0$$

and hence

$$\psi(E, A_t) = 0$$

Since $A \subset A_t$ we obtain that

$$\psi(E, A) = 0$$

Now, let $L \Subset U$ such that $|\partial E|(\partial L) = 0$; we can find a smooth open set $A$ such that $L \Subset A \Subset U$; let $(E_{j_k})_k$ be any subsequence of $(E_j)_j$. Reasoning as above we can find a $t > 0$ and a subsequence, also denoting with $(E_{j_k})_k$ such that

$$\lim_{k\to\infty} \nu(E_{j_k}, A_t) = \nu(E, A_t)$$

Since $\psi(E_{j_k}, A_t) = \psi(E, A_t) = 0$ we have that

$$\lim_{k\to\infty} |\partial E_{j_k}|(A_t) = |\partial E|(A_t)$$

Hence by Theorem 5.2.4 we have the desired result. □

We note that, from the Compactness Theorem (see Theorem 5.3.2), the condition $E_j \to E$ is not restrictive.

**Theorem 9.2.2.** *Let $E$ be a minimal set in $B_1$, $0 \in \partial E$. Fot $t > 0$ define*

$$E_t := \{x \in \mathbb{R}^n \mid tx \in E\}$$

*Then, for every $(t_j)_j$, $t_j \to 0$ there exists a subsequence denoted by $(s_j)_j$ such that $E_{s_j} \to C$ for some set $C \subset \mathbb{R}^n$. Moreover $C$ is a minimal cone.*

*Proof.* First we show that for each $R > 0$ there exists a subsequence $(\sigma_j)_j$ such that $E_{\sigma_j}$ converges in $B_R$. Since

$$|\partial E_t|(B_R) = t^{1-n}|\partial E|(B_{Rt}) \leq \frac{1}{2}n\omega_n R^{n-1}$$

where in the last step we hav eused the estimate in (8.8). We have that, for $t$ such that $Rt < 1$, $E_t$ is minimal in $B_R$, and frome the Compactness Theorem (see Theorem 5.3.2) there exists a subsequence $(E_{\sigma_j})_j$ and a set $C_R \subset B_R$ such that $E_{\sigma_j} \to C_R$ in $B_R$. Using a diagonal process we find a set $C \subset \mathbb{R}^n$ and a subsequence $(E_{s_j})_j$ such that $E_{s_j} \to C$ locally. From the preceing lemma we obtain that $C$ is minimal.
Now we prove that $C$ is a cone. To do this, from the proof of the previous Lemma, we obtain that for almost all $R > 0$

$$|\partial E_{\sigma_j}|(B_R) \to |\partial C|(B_R) \tag{9.3}$$

Define

$$f(t) := \frac{1}{t^{n-1}}|\partial E|(B_t) = |\partial E_t|(B_1)$$

From (9.3) we have that for almost all $R > 0$

$$\lim_{j\to\infty} f(s_j R) = \lim_{j\to\infty} \frac{1}{R^{n-1}}|\partial E_{s_j}|(B_R) = \frac{1}{R^{n-1}}|\partial C|(B_R)$$

Recalling (8.6) we also have that $f$ is an increasing function, since $E$ is minimal. Let $\rho < R$ for which the limit above holds. Since for every $j$ we can find an integer $m_j > 0$ such that

$$s_j\rho > s_{j+m_j}R$$

we have that

$$f(s_{j+m_j}) \leq f(s_j\rho) \leq f(s_j R)$$

and hence

$$\frac{1}{\rho^{n-1}}|\partial C|(B_\rho) = \lim_{j\to\infty} f(s_j\rho) = \lim_{j\to\infty} f(s_j R) = \frac{1}{R^{n-1}}|\partial C|(B_R)$$

So we have proved that

$$\frac{1}{\rho^{n-1}}|\partial C|(B_\rho)$$

is indipendent from $\rho$, for almost every $\rho$. Then from Proposition 8.1.8 apply to $\chi_C$ we obtain that

$$\int_{\partial B_1} |\chi_C(\rho x) - \chi_C(rx)|\,\mathcal{H}^{n-1} = 0$$

for almost every $r, \rho$. Hence $C$ differs only by a set of measure 0 from a cone with vertex at the origin. □

It is clear that if $E$ is regular in 0, then $C$ is a half space. From the regularity of the boundary of minimal set it can be prove also the converse. We only state this result, because its proof is based on some thecnical results needed to prove Theorem 9.1.2.

**Theorem 9.2.3.** *Let $(E_j)_j$ be a sequence of minimal sets in $B_1$ such that $E_j \to E$ for some set $E \in \mathbb{R}^n$. Let $x \in \partial^* E$ and $(x_j)_j$ such that $x_j \in \partial E_j$, $x_j \to x$. Then for $j$ sufficiently large $x_j$ is a regular point of $\partial E_j$ and*

$$\lim_{j\to\infty} \nu_{E_j}(x_j) = \nu_E(x)$$

**Remark 9.2.4.** *From this theorem and the regularity theory for minimal sets, we have that if there is no minimal singular cones in $\mathbb{R}^n$, then for every set $E \subset \mathbb{R}^n$ with $\psi(E,\rho) = 0$, $\partial E \cap B_\rho$ is an analytic hypersurface.*

Now, our aim is to show that no singular minimal cones exists in $\mathbb{R}^n$ for $n \leq 7$, thus proving the regularity of minimal surface in $\mathbb{R}^n$, $n \leq 7$. To do this we will restrict our attenction to singular minimal cones which only have singularity at the origin. This is possible thanks to the following

**Theorem 9.2.5.** *Let $C$ be a minimal cone in $\mathbb{R}^n$, singular at the origin. Then there exists $k \leq n$ and a minimal cone $A \subset \mathbb{R}^k$ such that $A$ is a minimal cone which is singular only at the origin.*

This theorem follows by the following three results.

First of all we want to understand what we obtain if we explode a minimal cone in a boundary point different from its vertex.

**Proposition 9.2.6.** *Let $C$ be a minimal cone with vertex at the origin, and $x_0 \in \partial C \setminus \{0\}$. For $t > 0$ define*

$$C_t := \{x \in \mathbb{R}^n \mid x_0 + t(x - x_0) \in C\}$$

*Then there exists a sequence $(t_j)_j$, $t_j \to 0$ such that $C_{t_j} \to Q$, $Q$ minimal cone. Moreover $Q$ is a cylinder with axis through 0 and $x_0$.*

*Proof.* We can suppose $x_0 = (0, \dots, 0, a)$ $a \neq 0$. Since

$$\chi_{C_t}(x) = \chi_C(x_0 + t(x - x_0))$$

and $C$ is a cone, we have that

$$|\partial C_t|(B(x_0,\rho)) = \frac{1}{t^{n-1}} |\partial C|(B(x_0,\rho t)) = \rho^{n-1} |\partial C|(B_{x_0}, 1)$$

Arguing as in Theorem 9.2.2 we obtain that there exists a sequence $(t_j)_j$, $t_j \to 0$ such that $C_{t_j} \to Q$, $Q$ minimal cone.

Now we prove that there exists a set $A \subset \mathbb{R}^{n-1}$ such that $Q = A \times \mathbb{R}$. Consider the measure

$$x \cdot [D\chi_C] = \langle x, \nu_C \rangle |\partial C|$$

If $x$ is in the interior of $C$, then $|\partial C| = 0$; if $x \in \partial C$ since $C$ is a cone, we have $\langle x, \nu_C \rangle = 0$. Hence $x \cdot [D\chi_C] = 0$. So, for every $x \in C$

$$a D_n \chi_C = -(x - x_0) \cdot [D\chi_C]$$

and hence, using the Riesz Representation Theorem (Theorem 2.8.5)

$$|a D_n \chi_C| = |\, \langle x - x_0, \nu_C \rangle \cdot |\partial C|\,| \leq |x - x_0||\partial C|$$

So

$$\int_{B(x_0,\rho)} \mathrm{d}|D\chi_{C_t}| = \frac{1}{t^{n-1}} \int_{B(x_0,t\rho)} |D_n \chi_C| \leq \frac{t^{2-n}\rho}{|x_0|} \int_{B(x_0,t\rho)} |\partial C| \leq \frac{1}{2} \frac{n\omega_n \rho^n}{|x_0|} t$$

where in the last step we have take into account that $C$ is minimal, and hence used the estimate of Remark 8.8. Hence from Theorem 2.9.5 we have that

$$D_n \chi_Q = \lim_{j \to \infty} D_n \chi_{C_{t_j}} = 0$$

Now we want to estimate $\chi_Q(y, r) - \chi_Q(y, s)$, $0 < s < r$, in terms of $|D_n \chi_Q|$. Let $f$ be a smooth function defined in $U_R$; if we defined, for each $t > 0$, the function $f_t(y) := f(y, t)$, it holds

$$\int_{\mathcal{B}_R} |f_s - f_t| \,\mathrm{d}\mathcal{H}^{n-1} \leq \int_{\mathcal{B}_R \times (s,t)} |D_n f| \,\mathrm{d}x$$

Now, taking an approximating sequence $(f_j)_j \in BV(U_R) \cap C^\infty(U_R)$ of $\chi_Q$, we obtain that, for almost all $s < t$ (in particular for those $s, t$ such that $(f_j)_s \to (\chi_Q)_s$) that

$$\int_{\mathcal{B}_R} |(\chi_Q)_s - (\chi_Q)_t| \,\mathrm{d}\mathcal{H}^{n-1} \leq \int_{\mathcal{B}_R \times (s,t)} \mathrm{d}|D_n \chi_Q| = 0$$

we obtain that there exists a set $A \subset \mathbb{R}^{n-1}$ such that for almost all $r, s$

$$\chi_Q(y, s) = \chi_Q(y, r) = \chi_A(y)$$

for almost all $y \in \mathbb{R}^{n-1}$. So we have obtained that $Q = A \times \mathbb{R}$. □

Since $Q$ is a cone, also $A$ is a cone: in fact, for $t, s > 0$ and $y \in \mathbb{R}^{n-1}$, recalling that $x_0$ belongs to the $x_n$ axis

$$\chi_A(ty) = \chi_Q(ty, (1 - t)a + ts) = \chi_Q(y, s) = \chi_A(y)$$

Now we prove that $Q$ is minimal if and only if $A$ is minimal. To do it we need the following

**Lemma 9.2.7.** *Let $f \in BV_{loc}(\mathbb{R}^n)$, that is $f \in BV(V)$ for each compact subset $V$. Let $U \subset \mathbb{R}^{n-1}$ be an open bounded set. Then, for each $T > 0$,*

$$\int_{U\times(-T,T)} \mathrm{d}|Df| \geq \int_{-T}^{T} \Big( \int_U \mathrm{d}|Df_t| \Big)\, \mathrm{d}t$$

*where $f_t(y) := f(y,t)$. Equality holding when $f$ is indipendent from $x_n$.*

*Proof.* Suppose first $f \in C^1(U \times (-T,T))$. Since

$$|Df(y,t)| = \Big(\sum_{i=1}^{n} D_i f(y,t)\Big)^{\frac{1}{2}} \geq \Big(\sum_{i=1}^{n-1} D_i f(y,t)\Big)^{\frac{1}{2}} = |Df_t(y)|$$

from the Fubini's Theorem we obtain the desired inequality.
Now let $f \in BV(U \times (-T,T))$, and let $(f_j)_j$ be a sequence of $C^1$ functions such that $f_j \to f$ in $L^1(U \times (-T,T))$ and

$$\lim_{j\to\infty} |Df_j|(U \times (-T,T)) = |Df|(U \times (-T,T))$$

Possibily passing to a subsequence, we can suppose that for almost all $t \in (-T,T)$ we have

$$f_{j,t} \to f_t \qquad L^1(U)$$

Then, by the semicontinuity we have

$$\liminf_{j\to\infty} |Df_{j,t}|(U) \geq |Df_t|(U)$$

That is, for almost all $|t| < T$ $f_t \in BV(U)$, and the desired inequality holds for $f \in BV(U \times (-T,T))$.
Now suppose that $f$ is indipendent from $x_n$. Approximating $f$ with $C^1$ functions we obtain that

$$\int_{U\times(-T,T)} f D_n g \,\mathrm{d}x = 0$$

for all $g \in C_c^1(U \times (-T,T))$. Then, if we take $g \in C_c^1(U \times (-T,T))$ with $|g| \leq 1$, we obtain

$$\begin{aligned}
\int_{U\times(-T,T)} f \mathrm{div}(g)\,\mathrm{d}x &= \int_{U\times(-T,T)} \sum_{i=1}^{n-1} f D_i g \,\mathrm{d}x \\
&= \int_{-T}^{T} \mathrm{d}t \int_U f_t \mathrm{div}(g)_t \,\mathrm{d}y \leq \int_{-T}^{T} \mathrm{d}t \int_U |Df_t|
\end{aligned}$$

Hence we obtain that

$$\int_{(U\times(-T,T))} |Df| = \int_{-T}^{T} \mathrm{d}t \int_U |Df_t|$$

□

**Proposition 9.2.8.** *Let $Q = A \times \mathbb{R}$ be a cylinder. Then $Q$ is minimal in $\mathbb{R}^n$ if and only if $A$ is minimal in $\mathbb{R}^{n-1}$.*

*Proof.* Suppose $A$ is minimal in $\mathbb{R}^{n-1}$. Let $M$ be a Caccioppoli set coinciding with $Q$ outside a compact set $K$. Let $T > 0$ such that

$$K \subset \Delta := \mathcal{B}_T \times (-T, T)$$

From the previous Lemma we have that

$$|\partial M|(\Delta) \geq \int_{-T}^{T} \mathrm{d}t \int_{\mathcal{B}_T} \mathrm{d}|\partial M_t|$$

where $\chi_{M_t}(y) := \chi_M(y, t)$. Then $M_t$ coinciding with $A$ outside a compact set $H \subset \mathcal{B}_T$. Since $A$ is minimal

$$|\partial A|(\mathcal{B}_T) \leq |\partial M_t|(\mathcal{B}_T)$$

Hence

$$|\partial M|(\Delta) \geq \int_{-T}^{T} \mathrm{d}t \int_{\mathcal{B}_T} \mathrm{d}|\partial A| = |\partial Q|(\Delta)$$

where in the last step we have take into account that $\chi_Q$ is indipendent from $x_n$ since $Q = A \times \mathbb{R}$.

Now suppose $Q$ is minimal in $\mathbb{R}^n$. If $A$ is not minimal in $\mathbb{R}^{n-1}$ there exists $\varepsilon, R > 0$ and a set $E$ coinciding with $A$ outside a compact set $H \subset \mathcal{B}_R$ such that

$$|\partial E|(\mathcal{B}_R) \leq |\partial A| - \epsilon$$

Let $T > 0$ and define the set

$$M := \begin{cases} E \times (-T, T) & , |x_n| < T \\ Q & , \text{otherwise} \end{cases}$$

Then $M = Q$ outside $H \times [-T, T]$. Since $Q$ is minimal

$$|\partial Q|(\mathcal{B}_R \times [-T, T]) \leq |\partial M|(\mathcal{B}_R \times [-T, T]) \tag{9.4}$$

On the other hand from the previous Lemma

$$|\partial Q|(\mathcal{B}_R \times [-T, T]) = 2T|\partial A|(\mathcal{B}_R)$$

Moreover, since $\chi_M$ is indipendent from $x_n$ in $\mathcal{B}_R \times (-T, T)$ we have

$$\begin{aligned} |\partial M|(\mathcal{B}_R \times [-T, T]) &= |\partial M|(\mathcal{B}_R \times (-T, T)) + |\partial M|(\mathcal{B}_R \times \{-T, T\}) \\ &\leq 2T|\partial E|(\mathcal{B}_R) + 2\omega_{n-1}R^{n-1} \qquad (9.5) \\ &\leq 2T|\partial A|(\mathcal{B}_R) - 2T\epsilon + 2\omega_{n-1}R^{n-1} \qquad (9.6) \\ &= |\partial Q|(\mathcal{B}_R \times (-T, T)) - 2T\epsilon + 2\omega_{n-1}R^{n-1} \end{aligned}$$

where in step (9.5) we have used the fact that $\mathcal{B}_R \times \{-T, T\}$ is regular, while in step (9.6) we have used the fact that $A$ is not minimal. Now, taking $T$ sufficiently large, we obtain a contradiction with (9.4). □

Now we have all the elements to prove Proposition 9.2.5:

*Proof.* (of Proposition 9.2.5) let $C$ be a minimal cone in $\mathbb{R}^n$ singular in 0 and in $x_0 \neq 0$. Hence $C$ is singular in all the points in the half line through 0 and $x_0$. We can suppose that this half line is the positive $x_n$ axis. Now, if we blow-up $C$ near $x_0$ we obtain a minimal cylinder $Q$ with the axis $x_n$ through $\partial Q$ and all the points in the $x_n$ axis are singular, because limits of singular points. Since we can write $Q = A \times \mathbb{R}$, with $A$ minimal cones in $\mathbb{R}^{n-1}$ singular at the origin. Repeting the argument above as many times as necessary, we obtain Proposition 9.2.6. $\square$

## 9.3 First and second variation of the area

In this section we want to prove that no minimal singular cones can exist in $\mathbb{R}^n$ for $n \leq 7$, and hence, using Proposition 9.2.6 of the previous section, we prove the regularity of minimal sets in $\mathbb{R}^n$ for $n \leq 7$. In the following section, we will prove that this result is the best possible, showing a minimal cone in $\mathbb{R}^8$ singular at the origin.

We consider a cone in $\mathbb{R}^{n+1}$ such that $C$ has locally finite perimeter, and $\partial C$ is smooth everywhere except possibly at the origin. We want to show that $\partial C$ is regular, or $n \geq 7$. This result is due to Simons, but we will follow the proof due to Massari and Miranda (see [Mir06]).

Our framework is the following one: let $A \subset \mathbb{R}^n$ be an open set, and $u \in C^2(A)$; the hypersurface $\mathcal{S}$ we consider will be the graph of the function $u$. In this case we have that the normal $\nu$ to $\mathcal{S}$ is

$$\nu(y) = \Big( - \frac{Du(y)}{\sqrt{1+|Du|^2}}, \frac{1}{\sqrt{1+|Du|^2}} \Big) \quad y \in A$$

We will work in the cylinder $\Omega := A \times \mathbb{R}$, and so we will extend $\nu$ to all the cylinder by

$$\nu(x) := \nu(y)$$

where $x := (y, x_{n+1}) \in \Omega$.
Now we introduce the differential operator $\delta$, introduced by Miranda: for $x \in \mathcal{S}$ define $\delta(x) := (\delta_1(x), \dots, \delta_{n+1}(x))$ where

$$\delta_i(x) := D_i(x) - \langle \nu(x), D(x) \rangle \nu(x)$$

Explain in words, the differential operator $\delta$ is nothing but the projection of the differential operator $D$ on the tangent hyperplane to $\mathcal{S}$.
Finally we define the Laplace operator

$$\mathcal{D} := \sum_{h=1}^{n+1} \delta_h \delta_h$$

and the two functions

$$\mathrm{H} := \sum_{h=1}^{n+1} \delta_h \nu_h$$

and

$$c^2 := \sum_{i,j=1}^{n+1} (\delta_i \nu_j)^2$$

We have that $\mathcal{H}(x)$ is the mean curvature of the hypersurface $\mathcal{S}$ in $x$ and that $c^2(x)$ is the sum of the squares of the principal curvatures of $\mathcal{S}$ in $x$.

**Note**: in this two sections we will work for simplicity in $\mathbb{R}^{n+1}$ instead that in $\mathbb{R}^n$.

Now we prove some useful identities we will use:

- it holds
$$\sum_{i=1}^{n+1} \nu_i \delta_i = 0$$
in fact
$$\sum_{i=1}^{n+1} \nu_i \delta_i = \langle \nu, D\rangle - \sum_{i=1}^{n+1} \nu_i^2 \langle \nu, D\rangle = 0$$

- for each $i = 1, \dots, n+1$ it holds
$$\sum_{h=1}^{n+1} \nu_h \delta_i \nu_h = 0$$
in fact
$$\sum_{h=1}^{n+1} \nu_h \delta_i \nu_h = \frac{1}{2} \delta_i |\nu|^2 = 0$$

- it holds
$$\delta_i \nu_j = \delta_j \nu_i$$
in fact, if $i, j \leq n$
$$\begin{aligned}
\delta_i \nu_j &= D_i \nu_j - \nu_i \sum_{h=1}^{n+1} \nu_k D_k \nu_j \\
&= D_i \Big( - \frac{D_j g}{\sqrt{1+|Dg|^2}} \Big) - \nu_i \sum_{k=1}^{n} (D_k(\nu_k \nu_j) - \nu_j D_k \nu_k) \\
&= -\frac{D_i g D_j g}{\sqrt{1+|Dg|^2}} + \frac{D_j g}{(1+|Dg|^2)^{\frac{3}{2}}} \sum_{k=1}^{n} D_k g D_i D_k g \\
&\quad -\frac{D_i g}{(1+|Dg|^2)^{\frac{5}{2}}} \sum_{k=1}^{n} \Big[ (D_k D_k g D_j g + D_k g D_k D_j g) \\
&\quad -2 D_k g D_j g \sum_{h=1}^{n} D_h g D_k D_h g \Big]
\end{aligned}$$

Now, noting that $D_k D_i g = D_i D_k g$ since $g$ is $C^2$, we obtain that

$$\delta_i \nu_j - \delta_j \nu_i = 0$$

Now, if $i = n+1$ and $j \leq n$ we have that

$$\begin{aligned}
\delta_{n+1}\nu_j &= -\nu_{n+1}\sum_{k=1}^{n}\nu_k D_k \nu_j \\
&= \frac{1}{(1+|Dg|^2)^{\frac{5}{2}}}\sum_{k=1}^{n} D_k g\Big[\, D_k D_j g(1+|Dg|^2) + D_j g \sum_{h=1}^{n} D_h g D_k D_h g\,\Big]
\end{aligned}$$

and

$$\begin{aligned}
\delta_j \nu_{n+1} &= D_j\Big(\frac{1}{\sqrt{1+|Dg|^2}}\Big) - \nu_j \sum_{k=1}^{n}\nu_k D_k\Big(\frac{1}{\sqrt{1+|Dg|^2}}\Big) \\
&= -\frac{1}{(1+|Dg|^2)^{\frac{3}{2}}}\sum_{k=1}^{n} D_k g D_j D_k g \\
&\quad + \frac{1}{(1+|Dg|^2)^{\frac{5}{2}}}\sum_{k=1}^{n} D_j g D_k g \sum_{h=1}^{n} D_h g D_k D_h g
\end{aligned}$$

Hence

$$\delta_{n+1}\nu_j - \delta_j \nu_{n+1} = 0$$

- for each $i, j = 1, \ldots, n+1$ the commutation formula holds

$$\delta_i \delta_j = \delta_j \delta_i + \sum_{h=1}^{n+1}(\nu_i \delta_j \nu_h - \nu_j \delta_i \nu_h)\delta_h$$

in fact

$$\delta_i \delta_j = D_i D_j - \sum_{h=1}^{n+1}\nu_i \nu_h D_h D_j - \sum_{h=1}^{n+1}\big[\,(\delta_i \nu_j \nu_h + \nu_j \delta_i \nu_h) D_h - \nu_j \nu_h \delta_i D_h\,\big]$$

Hence, since $\delta_i \nu_j = \delta_j \nu_i$, we obtain that

$$\delta_i \delta_j - \delta_j \delta_i = \sum_{h=1}^{n+1}(\nu_i \delta_j \nu_h - \nu_j \delta_i \nu_h)\delta_h$$

- for each $j = 1, \ldots, n+1$ it holds

$$\mathcal{D}\nu_j = -c^2 \nu_j + \delta_j \mathrm{H}$$

in fact

$$\begin{aligned}
\mathcal{D}\nu_j &= \sum_{i=1}^{n+1} \delta_i\delta_i\nu_j = \sum_{i=1}^{n+1} \delta_i\delta_j\nu_i \\
&= \sum_{i=1}^{n+1} \Big[ \delta_j\delta_i\nu_i + \sum_{h=1}^{n+1} (\nu_i\delta_j\nu_h - \nu_j\delta_i\nu_h)\delta_h\nu_i \Big] \\
&= \sum_{i=1}^{n+1} \delta_j\delta_i\nu_i - \nu_j c^2 = \delta_j \mathrm{H} - \nu_j c^2
\end{aligned}$$

where we have used the fact that $\delta_i\nu_h = \delta_h\nu_i$ and $\sum_{i=1}^{n+1} \nu_i\delta_h\nu_i = 0$.

Firstly we want to calculate the first and the second variation of the area of $\mathcal{S}$. To do this we consider a function $g \in C_c^2(\Omega)$, and the deformation of $\mathcal{S}$ given by

$$G_t(x) := x + tg(x)\nu(x)$$

where $t \in (-\varepsilon, \varepsilon)$ is such that $G_t(x) \in \Omega$ for each $x \in \mathcal{S}$. Since $\mathcal{S}$ is a $C^2$ hypersurface, we have that $|\partial\mathcal{S}| = \mathcal{H}^n \llcorner \mathcal{S}$. So we want to calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^n(G_t\mathcal{S})_{|t=0}\,, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}^n(G_t\mathcal{S})_{|t=0}$$

### 9.3.1 First variation of the area

We need a parameterization of $G_t\mathcal{S}$: so we define the function $\phi : A \to \mathbb{R}^{n+1}$ as

$$\phi(y) := (y, u(y)) + tg(y, u(y))\nu(y)$$

So, by the Area Formula, we have that

$$\mathcal{H}^n(G_t\mathcal{S}) = \int_A \sqrt{\det(\lambda_{ij})}\,\mathrm{d}y$$

where

$$\lambda_{ij} := \sum_{h=1}^{n+1} \Big( \frac{\partial\phi_h}{\partial y_i}\frac{\partial\phi_h}{\partial y_j} \Big)$$

So

$$\frac{\partial\phi_j}{\partial y_i} = \varepsilon_{ij} + t\Big(D_i g + D_{n+1} g \frac{\partial u}{\partial y_i}\Big)\nu_j + tg\frac{\partial \nu_j}{\partial y_i}\,, \qquad j = 1, \ldots, n$$

$$\frac{\partial\phi_{n+1}}{\partial y_i} = \frac{\partial u}{\partial y_i} + t\Big(D_i g + D_{n+1} g \frac{\partial u}{\partial y_i}\Big)\nu_{n+1} + tg\frac{\partial \nu_{n+1}}{\partial y_i}$$

Hence, taking into account that $|\nu| \equiv 1$ and hence $\langle D_i\nu, \nu\rangle = \frac{1}{2}D_j(|\nu|^2) = 0$, and the definition of $\nu$, we have

$$
\begin{aligned}
\lambda_{ij} &= \sum_{h=1}^{n} \Big[\Big(\varepsilon_{ih} + t\Big(D_i g + D_{n+1} g\frac{\partial u}{\partial y_i}\Big)\nu_h + tg\frac{\partial \nu_h}{\partial y_i}\Big)\cdot \\
&\quad \cdot\Big(\varepsilon_{hj} + t(D_j g + D_{n+1} g\frac{\partial u}{\partial y_j})\nu_h + tg\frac{\partial \nu_h}{\partial y_j}\Big)\Big] \\
&\quad +\Big[\frac{\partial u}{\partial y_i} + t\Big(D_i g + D_{n+1} g\frac{\partial u}{\partial y_i}\Big)\nu_{n+1} + tg\frac{\partial \nu_{n+1}}{\partial y_i}\Big] \\
&\quad \Big[\frac{\partial u}{\partial y_j} + t\Big(D_j g + D_{n+1} g\frac{\partial u}{\partial y_j}\Big)\nu_{n+1} + tg\frac{\partial \nu_{n+1}}{\partial y_j}\Big] \\
&= \varepsilon_{ij} + \frac{\partial u}{\partial y_i}\frac{\partial u}{\partial y_j} \\
&\quad +tg\Big[\frac{\partial \nu_i}{\partial y_j} + \frac{\partial \nu_j}{\partial y_i} + \frac{\partial u}{\partial y_i}\frac{\partial \nu_{n+1}}{\partial y_j} + \frac{\partial u}{\partial y_j}\frac{\partial \nu_{n+1}}{\partial y_i}\Big] \\
&\quad +t^2\Big[\Big(D_i g + D_{n+1} g\frac{\partial u}{\partial y_i}\Big)\Big(D_j g + D_{n+1} g\frac{\partial u}{\partial y_j}\Big) + g^2\sum_{h=1}^{n+1}\frac{\partial \nu_h}{\partial y_i}\frac{\partial \nu_h}{\partial y_j}\Big] \\
&\quad +t\Big[\Big(D_j g\nu_i + \frac{\partial u}{\partial y_i}D_j g\nu_{n+1}\Big) + \Big(D_{n+1} g\frac{\partial u}{\partial y_j}\nu_i + D_{n+1} g\frac{\partial u}{\partial y_i}\frac{\partial u}{\partial y_j}\nu_{n+1}\Big) \\
&\quad +\Big(D_i g\nu_j + \frac{\partial u}{\partial y_j}D_i g\nu_{n+1}\Big) + \Big(D_{n+1} g\frac{\partial u}{\partial y_i}\nu_j + D_{n+1} g\frac{\partial u}{\partial y_i}\frac{\partial u}{\partial y_j}\nu_{n+1}\Big)\Big] \\
&\quad +t^2 g\Big[\langle D_j\nu, \nu\rangle D_i g + \langle D_j\nu, \nu\rangle D_{n+1} g\frac{\partial u}{\partial y_i} + \langle D_i\nu, \nu\rangle D_j g + \langle D_i\nu, \nu\rangle D_{n+1} g\frac{\partial u}{\partial y_j}\Big] \\
&= \varepsilon_{ij} + \frac{\partial u}{\partial y_i}\frac{\partial u}{\partial y_j} \\
&\quad +tg\Big[\frac{\partial \nu_i}{\partial y_j} + \frac{\partial \nu_j}{\partial y_i} + \frac{\partial u}{\partial y_i}\frac{\partial \nu_{n+1}}{\partial y_j} + \frac{\partial u}{\partial y_j}\frac{\partial \nu_{n+1}}{\partial y_i}\Big] \\
&\quad +t^2\Big[\Big(D_i g + D_{n+1} g\frac{\partial u}{\partial y_i}\Big)\Big(D_j g + D_{n+1} g\frac{\partial u}{\partial y_j}\Big) + g^2\sum_{h=1}^{n+1}\frac{\partial \nu_h}{\partial y_i}\frac{\partial \nu_h}{\partial y_j}\Big] \\
&= \frac{1}{\nu_{n+1}^2}\Big[\nu_{n+1}^2\varepsilon_{ij} + \nu_i\nu_j \\
&\quad +tg\Big(\nu_{n+1}^2\Big(\frac{\partial \nu_i}{\partial y_j} + \frac{\partial \nu_j}{\partial y_i}\Big) - \nu_{n+1}\Big(\nu_i\frac{\partial \nu_{n+1}}{\partial y_j} + \nu_j\frac{\partial \nu_{n+1}}{\partial y_i}\Big)\Big) \\
&\quad +t^2\Big((\nu_{n+1}D_i g - \nu_i D_{n+1} g)(\nu_{n+1}D_j g - \nu_j D_{n+1} g) + g^2\nu_{n+1}^2\sum_{h=1}^{n+1}\frac{\partial \nu_h}{\partial y_i}\frac{\partial \nu_h}{\partial y_j}\Big)\Big]
\end{aligned}
$$

Now

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^n(G_t\mathcal{S}) = \frac{1}{2}\int_A \frac{1}{\sqrt{\det(\lambda_{ij})}}\frac{\mathrm{d}}{\mathrm{d}t}\det(\lambda_{ij})\,\mathrm{d}y
$$

We recall the formula for the derivate of a determinant

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(\lambda_{ij}) = \det(\lambda_{ij})\sum_{i,j=1}^{n}\lambda^*_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}$$

where $(\lambda^*_{ij})_{ij}$ is the inverse of the symmetric matrix $(\lambda_{ij})_{ij}$. Now using the fact that

$$(\lambda^*_{ij})_{|t=0} = \varepsilon_{ij} - \nu_i\nu_j$$

and that

$$\begin{aligned}
&\sum_{i,j=1}^{n}\lambda^*_{ij}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}\Big)_{|t=0}\\
=\ &\sum_{i,j=1}^{n}(\varepsilon_{ij}-\nu_i\nu_j)\Big(g\Big(\frac{\partial\nu_i}{\partial y_j}+\frac{\partial\nu_j}{\partial y_i}\Big)-g\nu_{n+1}^{-1}\Big(\nu_i\frac{\partial\nu_{n+1}}{\partial y_j}+\nu_j\frac{\partial\nu_{n+1}}{\partial y_i}\Big)\Big)\\
=\ &2g\sum_{i=1}^{n}\Big(\frac{\partial\nu_i}{\partial y_i}-\nu_{n+1}^{-1}\nu_i\frac{\partial\nu_{n+1}}{\partial y_i}\Big)-g\nu_{n+1}\Big(\sum_{j=1}^{n}\nu_j\Big[\langle\nu,D_j\nu\rangle-\frac{\partial\nu_{n+1}}{\partial y_j}\Big]\\
&+\sum_{i=1}^{n}\nu_i\Big[\langle\nu,D_i\nu\rangle-\frac{\partial\nu_{n+1}}{\partial y_i}\Big]\Big)+g\nu_{n+1}^{-1}\Big[(1-\nu_{n+1}^2)\sum_{j=1}^{n}\nu_j\frac{\partial\nu_{n+1}}{\partial y_j}\\
&+(1-\nu_{n+1}^2)\sum_{i=1}^{n}\nu_j\frac{\partial\nu_{n+1}}{\partial y_i}\Big]\\
=\ &2g\sum_{i=1}^{n}\frac{\partial\nu_i}{\partial y_i}-2g\nu_{n+1}^{-1}\sum_{i=1}^{n}\nu_i\frac{\partial\nu_{n+1}}{\partial y_i}+2g\nu_{n+1}\sum_{i=1}^{n}\nu_i\frac{\partial\nu_{n+1}}{\partial y_i}\\
&+2g\nu_{n+1}^{-1}\Big[\sum_{i=1}^{n}\nu_i\frac{\partial\nu_{n+1}}{\partial y_i}-\nu_{n+1}^2\sum_{i=1}^{n}\nu_i\frac{\partial\nu_{n+1}}{\partial y_i}\Big]\\
=\ &2g\sum_{i=1}^{n}\frac{\partial\nu_i}{\partial y_i}=2g\sum_{i=1}^{n}(\delta_i\nu_i+\nu_i\langle\nu,D\nu_i\rangle)=2g\Big(\sum_{i=1}^{n}\delta_i\nu_i-\nu_{n+1}\langle\nu,D\nu_{n+1}\rangle\Big)\\
=\ &2g\sum_{i=1}^{n+1}\delta_i\nu_i
\end{aligned}$$

and that

$$(\det(\lambda_{ij}))_{|t=0} = \det\Big(\varepsilon_{ij}+\frac{\partial u}{\partial y_i}\frac{\partial u}{\partial y_j}\Big)=\frac{1}{\nu_{n+1}^2}$$

we obtain that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^n(G_t\mathcal{S})_{|t=0} &= \frac{1}{2}\int_A \sqrt{\det(\lambda_{ij})}\sum_{i,j=1}^{n} \lambda^*_{ij}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}\Big)_{|t=0} \\
&= \frac{1}{2}\int_A \sum_{h=1}^{n+1} g(\delta_h\nu_h)\nu_{n+1}^{-1}\,\mathrm{d}y
\end{aligned}
$$

Then, using the Change of Variable Formula, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^n(G_t\mathcal{S})_{|t=0} = \int_{\mathcal{S}} \mathrm{H}g\,\mathrm{d}\mathcal{H}^n
$$

### 9.3.2 Second variation of the area

Now we want to calcultate the second variation of the area, that is

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^n(G_t\mathcal{S})\Big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_A \sqrt{\det(\lambda_{ij})}\sum_{i,j=1}^{n}\lambda^*_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}\,\mathrm{d}y\Big) \\
&= \frac{1}{2}\int_A \sqrt{\det(\lambda_{ij})}\Big[\frac{1}{2}\Big(\sum_{i,j=1}^{n}\lambda^*_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}\Big)^2 \\
&\quad + \sum_{i,j=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\lambda^*_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij} + \sum_{i,j=1}^{n}\lambda^*_{ij}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda_{ij}\Big]\,\mathrm{d}y
\end{aligned}
$$

Now we want to calculate each of the three terms in the integral. Let's start with the simple one, since we have already calculate in the previous section

$$
\Big(\sum_{i,j=1}^{n}\lambda^*_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ij}\Big)^2_{|t=0} = 4g^2\Big(\sum_{h=1}^{n+1}\delta_h\nu_h\Big)^2
$$

For the second term we have

$$\begin{aligned}
\sum_{i,j=1}^n \lambda^*_{ij} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda_{ij}{}_{|t=0} &= \sum_{i,j=1}^n 2(\varepsilon_{ij}-\nu_i\nu_j)\Big[\frac{1}{\nu_{n+1}^{-2}}\Big(\nu_{n+1}D_i g-\nu_i D_{n+1}g\Big)\cdot \\
&\quad \cdot\Big(\nu_{n+1}D_j g-\nu_j D_{n+1}g\Big)+g^2\nu_{n+1}^2\sum_{h=1}^{n+1}\frac{\partial\nu_h}{\partial y_i}\frac{\partial\nu_h}{\partial y_j}\Big] \\
&= 2\nu_{n+1}^{-2}\Big[\sum_{i=1}^n \nu_{n+1}^2(\delta_i g)^2-2\nu_{n+1}\sum_{i=1}^n\nu_i(\delta_i g)(\delta_{n+1}g)+(1-\nu_{n+1}^2)(\delta_{n+1}g)^2\Big] \\
&\quad -2\nu_{n+1}^{-2}\sum_{i,j=1}^n \nu_i\nu_j(\nu_{n+1}\delta_i g-\nu_i\delta_{n+1}g)(\nu_{n+1}\delta_j g-\nu_j\delta_{n+1}g) \\
&\quad +2g^2\nu_{n+1}^{-2}\nu_{n+1}^2\sum_{i,j=1}^n\sum_{h=1}^{n+1}(\varepsilon_{ij}-\nu_i\nu_j)\frac{\partial\nu_h}{\partial y_i}\frac{\partial\nu_h}{\partial y_j}
\end{aligned}$$

Now, since

$$\nu_{n+1}\sum_{i=1}^n \nu_i(\delta_i g)(\delta_{n+1}g) = -\nu_{n+1}^2(\delta_{n+1}g)^2$$

and

$$\begin{aligned}
&\sum_{i,j=1}^n \nu_i\nu_j(\nu_{n+1}\delta_i g-\nu_i\delta_{n+1}g)(\nu_{n+1}\delta_j g-\nu_j\delta_{n+1}g) \\
&= \nu_{n+1}^2\Big(\sum_{i=1}^n\nu_i\delta_i g\Big)^2-2\nu_{n+1}\delta_{n+1}g\sum_{i=1}^n\nu_i\delta_i g\Big(\sum_{j=1}^n\nu_j^2\Big)+(\delta_{n+1}g)^2\Big(\sum_{i=1}^n\nu_i^2\Big)\Big(\sum_{j=1}^n\nu_{ij}^2\Big) \\
&= \nu_{n+1}^2(-\nu_{n+1}\delta_{n+1}g)^2-2\nu_{n+1}\delta_{n+1}g(-\nu_{n+1}\delta_{n+1}g)(1-\nu_{n+1}^2)+(\delta_{n+1}g)^2(1-\nu_{n+1}^2)^2 \\
&= (\delta_{n+1}g)^2
\end{aligned}$$

and

$$\nu_{n+1}^2\sum_{i,j=1}^n\sum_{h=1}^{n+1}(\varepsilon_{ij}-\nu_i\nu_j)\frac{\partial\nu_h}{\partial y_i}\frac{\partial\nu_h}{\partial y_j} = \nu_{n+1}^2\sum_{i=1}^n\sum_{h=1}^{n+1}\Big(\frac{\partial\nu_h}{\partial y_i}\Big)^2-\sum_{h=1}^{n+1}(\delta_{n+1}\nu_h)^2$$

we obtain that

$$\begin{aligned}
\sum_{i,j=1}^{n} \lambda^*_{ij} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda_{ij}{}_{|t=0} &= 2|\delta g|^2 + 2g^2\Big( \sum_{i=1}^{n}\sum_{h=1}^{n+1}\Big(\frac{\partial \nu_h}{\partial y_i}\Big)^2 - \nu_{n+1}^2 \sum_{h=1}^{n+1}(\delta_{n+1}\nu_h)^2\Big) \\
&= 2|\delta g|^2 + 2g^2\Big( \sum_{h=1}^{n+1}\sum_{i=1}^{n+1}(\delta_i\nu_h - \nu_i\nu_{n+1}^{-1}\delta_{n+1}\nu_h)^2 - \nu_{n+1}^{-2} \sum_{h=1}^{n+1}(\delta_{n+1}\nu_h)^2\Big) \\
&= 2|\delta g|^2 + 2g^2\Big( \sum_{h=1}^{n+1}\sum_{i=1}^{n+1}(\delta_i\nu_h)^2 + \sum_{h=1}^{n+1}\nu_{n+1}^{-2}(\delta_{n+1}\nu_h)^2 \\
&\quad -\sum_{h=1}^{n+1} 2\delta_{n+1}\nu_h\nu_{n+1}^{-1}\sum_{i=1}^{n+1}\nu_i\delta_i\nu_h - \nu_{n+1}^{-2}\sum_{h=1}^{n+1}(\delta_{n+1}\nu_h)^2\Big) \\
&= 2|\delta g|^2 + 2g^2 \sum_{h,i=1}^{n+1}(\delta_i\nu_h)^2
\end{aligned}$$

where in the last step we have used the fact that $\sum_{i=1}^{n+1}\nu_i\delta_i = 0$.
For the third term: from the equality

$$\sum_{h=1}^{n+1}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\lambda^*_{ih}\Big)\lambda_{hj} = -\sum_{h=1}^{n+1}\lambda^*_{ih}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{hj} =: -b_{ij}$$

Multipling for $\lambda^*_{jk}$ and summing over $j$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda^*_{ik} = -\sum_{j=1}^{n} b_{ij}\lambda^*_{jk}$$

Hence

$$\sum_{i,k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\lambda^*_{ik}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ki} = -\sum_{i,j,k=1}^{n} b_{ij}\lambda^*_{jk}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_{ki} = -\sum_{i,j=1}^{n} b_{ij}b_{ji}$$

Now we want to calculate explictly the coefficients in the summation:

$$
\begin{aligned}
b_{ij}{}_{|t=0} &= \sum_{h=1}^{n} \lambda^*_{ih} \frac{\mathrm{d}}{\mathrm{d}t} \lambda_{hj}{}_{|t=0} \\
&= \sum_{h=1}^{n} (\varepsilon_{ih} - \nu_i \nu_h) \nu_{n+1}^{-2} \Big[ \nu_{n+1}^2 g \Big( \frac{\partial \nu_j}{\partial y_h} + \frac{\partial \nu_h}{\partial y_j} \Big) - g \nu_{n+1} \Big( \nu_h \frac{\partial \nu_{n+1}}{\partial y_j} + \nu_j \frac{\partial \nu_{n+1}}{\partial y_h} \Big) \Big] \\
&= g \Big( \frac{\partial \nu_j}{\partial y_i} + \frac{\partial \nu_i}{\partial y_j} \Big) - g \nu_{n+1}^{-1} \nu_j \frac{\partial \nu_{n+1}}{\partial y_i} - g \nu_{n+1}^{-1} \nu_i \frac{\partial \nu_{n+1}}{\partial y_j} + \\
&\quad - g \nu_i \sum_{h=1}^{n} \Big( \nu_h \frac{\partial \nu_j}{\partial y_h} + \nu_h \frac{\partial \nu_h}{\partial y_j} \Big) \\
&\quad + g \nu_{n+1}^{-1} \Big[ \nu_j \nu_i \Big( - \nu_{n+1}^{-1} \delta_{n+1} \nu_{n+1} + (1 - \nu_{n+1}^2) \frac{\partial \nu_{n+1}}{\partial y_j} \Big) \Big] \\
&= g \Big( \frac{\partial \nu_j}{\partial y_i} + \frac{\partial \nu_i}{\partial y_j} \Big) - g \nu_{n+1}^{-1} \nu_j \frac{\partial \nu_{n+1}}{\partial y_i} \\
&\quad - g \nu_i \Big[ \langle \nu, D \nu_j \rangle + \langle \nu, D_j \nu \rangle \Big] \\
&\quad - g \nu_j \nu_i \nu_{n+1}^{-2} \delta_{n+1} \nu_{n+1} \\
&= g \Big( \frac{\partial \nu_j}{\partial y_i} + \frac{\partial \nu_i}{\partial y_j} \Big) - g \nu_{n+1}^{-1} \nu_j \delta_i \nu_{n+1} + g \nu_i \nu_{n+1}^{-1} \delta_{n+1} \nu_j \\
&= g \Big( \delta_i \nu_j + (\delta_j \nu_i + \nu_j \langle \nu, D \nu_i \rangle) - \nu_{n+1}^{-1} \nu_j \delta_i \nu_{n+1} \Big) \\
&= g \Big( 2 \delta_i \nu_j + \nu_j \langle \nu, D \nu_i \rangle - \nu_j \nu_{n+1}^{-1} \delta_{n+1} \nu_i \Big) \\
&= 2g \Big( \delta_i \nu_j + \nu_j \sum_{h=1}^{n} \nu_h \frac{\partial \nu_i}{\partial y_h} \Big)
\end{aligned}
$$

where in the last steps we have used the fact that $\delta_j \nu_i = \delta_i \nu_j$.
Hence

$$
\begin{aligned}
\sum_{i,j=1}^{n} b_{ij} b_{ji}{}_{|t=0} &= 4g^2 \sum_{i,j=1}^{n} \Big( \delta_i \nu_j + \nu_j \sum_{h=1}^{n} \nu_h \frac{\partial \nu_i}{\partial y_h} \Big) \Big( \delta_i \nu_j + \nu_i \sum_{k=1}^{n} \nu_k \frac{\partial \nu_j}{\partial y_k} \Big) \\
&= 4g^2 \Big( \sum_{i,j=1}^{n} (\delta_i \nu_j)^2 - \sum_{j=1}^{n} \nu_{n+1} \delta_{n+1} \nu_j \sum_{k=1}^{n} \nu_k \frac{\partial \nu_j}{\partial y_k} \\
&\quad - \sum_{i=1}^{n} \nu_{n+1} \delta_{n+1} \nu_i \sum_{h=1}^{n} \nu_h \frac{\partial \nu_i}{\partial y_h} + \sum_{i,j,h,k=1}^{n} \nu_i \nu_j \nu_h \nu_k \frac{\partial \nu_i}{\partial y_h} \frac{\partial \nu_j}{\partial y_k} \Big)
\end{aligned}
$$

$$\begin{aligned}
&= 4g^2\Big( \sum_{i,j=1}^{n} (\delta_i \nu_j)^2 - \sum_{j=1}^{n} \delta_{n+1}\nu_j \nu_{n+1}\langle \nu, D_j \nu\rangle - \sum_{i=1}^{n} \delta_{n+1}\nu_i \nu_{n+1}\langle \nu, D_i \nu\rangle \\
&\quad + \sum_{i,j=1}^{n} \Big( \delta_i \nu_i - \frac{\partial \nu_i}{\partial y_i}\Big)\Big( \delta_j \nu_j - \frac{\partial \nu_j}{\partial y_j}\Big)\Big) \\
&= 4g^2\Big( \sum_{i,j=1}^{n} (\delta_i \nu_j)^2 + \sum_{j=1}^{n} (\delta_{n+1}\nu_j)^2 + \sum_{i=1}^{n} (\delta_{n+1}\nu_i)^2 + (\delta_{n+1}\nu_{n+1})^2 \Big) \\
&= 4g^2 \sum_{i,j=1}^{n+1} (\delta_i \nu_j)^2
\end{aligned}$$

So we can write

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}^n(G_t\mathcal{S}) &= \int_{\mathcal{S}} \Big[ g^2 \Big( \sum_{h=1}^{n+1} \delta_h \nu_h \Big)^2 + \frac{1}{2}\Big( 2|\delta g|^2 + 2g^2 \sum_{i,h=1}^{n+1} (\delta_i \nu_h)^2 \Big) \\
&\quad - \frac{1}{2} 4g^2 \sum_{i,j=1}^{n+1} (\delta_i \nu_j)^2 \Big] \mathrm{d}\mathcal{H}^n
\end{aligned}$$

Hence we obtain that the second variation of the area is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}^n(G_t\mathcal{S}) = \int_{\mathcal{S}} \Big[ g^2 \Big( \mathrm{H}^2 - c^2 \Big) + |\delta g|^2 \Big] \mathrm{d}\mathcal{H}^n$$

### 9.3.3 Simons Theorem

In this section we want to prove the foundamental theorem due to Simons, that allow us to prove the regularity of minimal surfaces in $\mathbb{R}^n$, $n \le 7$. The main tools to prove this result are the first and the second variation of the area calculated in the previous sections, togheter with the following

**Theorem 9.3.1.** *Let $C$ be a cone in $\mathbb{R}^{n+1}$ such that $\partial E$ is regular in $\mathbb{R}^{n+1} \setminus \{0\}$. Suppose $\mathrm{H} \equiv 0$ on $\partial C$. Then*

$$\frac{1}{2}\mathcal{D}c^2 \ge \frac{2c^2}{|x|^2} - c^4 + |\delta c|^2$$

*in every point of $\partial C$ such that $c^2 > 0$.*

*Proof.* We have that

$$\begin{aligned}
\frac{1}{2}\mathcal{D}c^2 &= \frac{1}{2}\sum_{h,i,j=1}^{n+1} \delta_h\delta_h(\delta_i\nu_j)^2 = \sum_{h,i,j=1}^{n+1} \delta_h(\delta_i\nu_j\delta_h\delta_i\nu_j) \\
&= \sum_{h,i,j=1}^{n+1} (\delta_h\delta_i\nu_j)^2 + \sum_{h,i,j=1}^{n+1} \delta_i\nu_j\delta_h\delta_h\delta_i\nu_j
\end{aligned}$$

Using the commutation formula we have that

$$\begin{aligned}
&\sum_{h,i,j=1}^{n+1} \delta_i\nu_j\delta_h\delta_h\delta_i\nu_j = \sum_{h,i,j=1}^{n+1} \delta_i\nu_j\delta_h\Big( \delta_i\delta_h\nu_j + \sum_{k=1}^{n+1}(\nu_h\delta_i\nu_k - \nu_i\delta_h\nu_k)\delta_k\nu_j \Big) \\
= &\sum_{h,i,j=1}^{n+1} \delta_i\nu_j\delta_h\delta_i\delta_h\nu_j + \sum_{h,i,j,k=1}^{n+1} \delta_i\nu_j\delta_h\nu_h\delta_i\nu_j + \sum_{h,i,j,k=1}^{n+1} \delta_i\nu_j\nu_h\delta_h\delta_i\nu_k \\
&- \sum_{h,i,j,k=1}^{n+1} \delta_i\nu_j\delta_h\nu_i\delta_h\nu_k\delta_k\nu_j - \sum_{h,j,k=1}^{n+1} \delta_h\delta_h\nu_k\delta_k\nu_j\Big(\sum_{i=1}^{n+1}\nu_i\delta_i\Big)\nu_j \\
&- \sum_{h,j,k=1}^{n+1} \delta_h\nu_k\delta_h\delta_k\nu_j\Big(\sum_{i=1}^{n+1}\nu_i\delta_i\Big)\nu_j \\
= &\sum_{h,i,j=1}^{n+1} \delta_i\nu_j\delta_h\delta_i\delta_h\nu_j - \sum_{h,i,j,k=1}^{n+1} \delta_i\nu_j\delta_h\nu_i\delta_h\nu_k\delta_k\nu_j
\end{aligned}$$

Now, using again the commutation formula, we want to rewrite the first term of the sum above:

$$
\begin{aligned}
&\sum_{h,i,j=1}^{n+1} \delta_i\nu_j\delta_h\delta_i\delta_h\nu_j = \sum_{i,j=1}^{n+1} \delta_i\nu_j\delta_i\mathcal{D}\nu_j + \sum_{h,i,j,k=1}^{n+1} \delta_i\nu_j(\nu_h\delta_i\nu_k - \nu_i\delta_h\nu_k)\delta_k\delta_h\nu_j\\
= \ & -\sum_{i,j=1}^{n+1} \delta_i\nu_j\delta_i(c^2\nu_j) + \sum_{i,j,k=1}^{n+1} \delta_i\nu_j\delta_i\nu_k\Big(\sum_{h=1}^{n+1}\nu_h\delta_h\Big)\delta_k\nu_j +\\
&+\sum_{h,i,j,k,s=1}^{n+1} \delta_i\nu_j\nu_h\delta_i\nu_k(\nu_k\delta_h\nu_s - \nu_h\delta_k\nu_s)\delta_s\nu_j + \sum_{h,j,k=1}^{n+1}\Big(\sum_{i=1}^{n+1}\nu_i\delta_i\Big)\nu_j\delta_h\nu_k\delta_h\delta_k\nu_j\\
&+\sum_{h,i,,k,s=1}^{n+1} \delta_i\nu_j\nu_i\delta_h\nu_k(\nu_k\delta_h\nu_s - \nu_h\delta_k\nu_s)\delta_s\nu_j\\
= \ & -\sum_{i=1}^{n+1}\delta_i c^2\Big(\sum_{j=1}^{n+1}\nu_j\delta_i\nu_j\Big) - c^4 + \sum_{i,j,k,s=1}^{n+1}\delta_i\nu_j\delta_i\nu_k\nu_k\delta_s\nu_j\Big(\sum_{h=1}^{n+1}\nu_h\delta_h\Big)\nu_s\\
&-\sum_{i,j,k,s=1}^{n+1}\delta_i\nu_j\delta_i\nu_k\delta_k\nu_s\delta_s\nu_j\Big(\sum_{h=1}^{n+1}\nu_h^2\Big) + \sum_{h,j,k,s=1}^{n+1}\Big(\sum_{i=1}^{n+1}\nu_i\delta_i\Big)\nu_j\delta_h\nu_k\nu_k\delta_h\nu_s\delta_s\nu_j\\
&-\sum_{h,j,k,s=1}^{n+1}\Big(\sum_{i=1}^{n+1}\nu_i\delta_i\Big)\nu_j\delta_h\nu_k\nu_h\delta_k\nu_s\delta_s\nu_j\\
= \ & -c^4 - \sum_{i,j,k,s=1}^{n+1}\delta_i\nu_j\delta_i\nu_k\delta_k\nu_s\delta_s\nu_j
\end{aligned}
$$

Hence we have obtain that

$$
\begin{aligned}
\sum_{h,j,k=1}^{n+1}\delta_i\nu_j\delta_h\delta_h\nu_i\nu_j &= -c^4 - 2\sum_{k,i,j,s=1}^{n+1}\delta_i\nu_j\delta_i\nu_k\delta_k\nu_s\delta_s\nu_j\\
&= -c^4 - 2\sum_{k,i,j,s=1}^{n+1}\nu_j\nu_k\delta_i\delta_k\nu_s\delta_i\delta_j\nu_s
\end{aligned}
$$

where in the last step we have used the fact that

$$
\sum_{k=1}^{n+1}\delta_i\nu_k\delta_k\nu_s = -\sum_{k=1}^{n+1}\nu_k\delta_i\delta_k\nu_s\,,\qquad \sum_{j=1}^{n+1}\delta_i\nu_j\delta_j\nu_s = -\sum_{j=1}^{n+1}\nu_j\delta_i\delta_j\nu_s
$$

that hold since $\sum_{k=1}^{n+1}\nu_k\delta_k = 0$ and $\sum_{j=1}^{n+1}\nu_j\delta_j = 0$.
Hence we have that

$$
\frac{1}{2}\mathcal{D}c^2 + c^4 = \sum_{h,i,j=1}^{n+1}(\delta_i\delta_h\nu_j)^2 - 2\sum_{k,i,j,s=1}^{n+1}\nu_j\nu_k\delta_i\delta_k\nu_s\delta_i\delta_j\nu_s
$$

We note that

$$\begin{aligned} c^2|\delta c|^2 &= c^2 \sum_{i=1}^{n+1} \Big( \delta_i \sqrt{\sum_{h,j=1}^{n+1} (\delta_h \nu_j)^2} \Big)^2 = \frac{1}{4} \sum_{i=1}^{n+1} \Big( \sum_{h,j=1}^{n+1} \delta_i (\delta_h \nu_j)^2 \Big)^2 \\ &= \sum_{i=1}^{n+1} \Big( \sum_{h,j=1}^{n+1} \delta_h \nu_j \delta_i \delta_h \nu_j \Big)^2 \end{aligned}$$

Hence, if $c^2 > 0$ we can write

$$\begin{aligned} \frac{1}{2} \mathcal{D} c^2 + c^4 - |\delta c|^2 &= \sum_{h,i,j=1}^{n+1} (\delta_h \delta_i \nu_j)^2 - 2 \sum_{k,i,j,s=1}^{n+1} \nu_j \nu_k \delta_i \delta_k \nu_s \delta_i \delta_j \nu_s \\ &\quad - c^{-2} \sum_{i=1}^{n+1} \Big( \sum_{h,j=1}^{n+1} \delta_h \nu_j \delta_i \delta_h \nu_j \Big)^2 \end{aligned} \tag{9.7}$$

Now we want to give an upper bound of this quantity in a point $x \neq 0$. We can suppose that $\nu_x = e_{n+1}$; under this assumption we have that for each function $\alpha \in C^1 \mathbb{R}^{n+1}$

$$\delta_{n+1} \alpha(x) = 0, \quad \delta_i \alpha(x) = D_i \alpha(x) \ \ \forall i = 1, \ldots, n$$

Moreover, for each $i = 1, \ldots, n+1$, using the commutation formula, we have that

$$\begin{aligned} (\delta_i \delta_{n+1} \nu_{n+1})(x) &= (\delta_{n+1} \delta_i \nu_{n+1})(x) + \Big( \sum_{h=1}^{n+1} (\nu_i \delta_{n+1} \nu_h - \nu_{n+1} \delta_i \nu_h) \delta_h \nu_{n+1} \Big)(x) \\ &= - \sum_{h=1}^{n+1} \delta_i \nu_h \delta_h \nu_{n+1}(x) \\ &= -\delta_i \Big( \sum_{h=1}^{n+1} \nu_h \delta_h \Big) \nu_{n+1}(x) + \Big( \sum_{h=1}^{n+1} \nu_h \delta_i \delta_h \Big) \nu_{n+1}(x) = 0 \end{aligned}$$

So, in the point $x$, we have that the the first two terms on the right of (9.7) can be write as

$$\begin{aligned} \sum_{h,i,j=1}^{n+1} (\delta_h \delta_i \nu_j)^2 - 2 \sum_{k,i,j,s=1}^{n+1} \nu_j \nu_k \delta_i \delta_k \nu_s \delta_i \delta_j \nu_s &= \sum_{h=1}^{n} \sum_{i,j=1}^{n+1} (\delta_h \delta_i \nu_j)^2 - 2 \sum_{i,s=1}^{n} (\delta_i \delta_{n+1} \nu_s)^2 \\ &= \sum_{i,j,h=1}^{n} (\delta_h \delta_i \nu_j)^2 \end{aligned}$$

where in the last step we have used the fact that $\delta_i\nu_{n+1} = \delta_{n+1}\nu_i$.

Now we want to estimate the last term on the right of (9.7). For this, we choose $e_n := x|x|^{-1}$; since $\langle x, \nu(x)\rangle = 0$ for each $x \neq 0$, in the point $x$ we have that

$$0 = \delta_i\langle x, \nu(x)\rangle = \sum_{h=1}^{n+1}\big(\,(\delta_i x_h)\nu_h + x_h\delta_i\nu_h\,\big) = \delta_i x_{n+1} + |x|\delta_i x_n$$

So, since for $i \leq n \quad \delta_i x_{n+1}(x) = D_i x_{n+1}(x) = 0$, we obtain that

$$\delta_i\nu_n(x) = 0 \qquad \forall i \leq n$$

Hence, recalling that $\delta_{n+1}\alpha(x) = 0$ and $\delta_n\nu_h = \delta_h\nu_h = 0$ for $h = 1, \ldots, n$, we have that

$$\begin{aligned}
c^2|\delta c|^2 &= \sum_{i=1}^{n+1}\Big(\sum_{h,j=1}^{n+1}\delta_h\nu_j\delta_i\delta_h\nu_j\Big)^2 = \sum_{i=1}^{n}\Big(\sum_{h,j=1}^{n}\delta_h\nu_j\delta_i\delta_h\nu_j\Big)^2 \\
&\leq \sum_{i=1}^{n}\Big(\sum_{h,j=1}^{n-1}(\delta_h\nu_j)^2\Big)\Big(\sum_{h,j=1}^{n-1}(\delta_i\delta_h\nu_j)^2\Big) = c^2\sum_{i=1}^{n}\sum_{h,j=1}^{n-1}(\delta_i\delta_h\nu_j)^2
\end{aligned}$$

So, if $c^2 > 0$ we obtain that

$$|\delta c|^2 \leq \sum_{i=1}^{n}\sum_{h,j=1}^{n-1}(\delta_i\delta_h\nu_j)^2$$

So we can conclude that in the point $x$ we have

$$\frac{1}{2}\mathcal{D}c^2 + c^4 - |\delta c|^2 \geq 2\sum_{i=1}^{n}\sum_{j=1}^{n-1}(\delta_i\delta_n\nu_j)^2 + \sum_{i=1}^{n}(\delta_i\delta_n\nu_n)^2$$

But in the point $x$

$$\begin{aligned}
\delta_i\delta_n\nu_j &= \delta_n\delta_i\nu_j + \sum_{h=1}^{n+1}(\nu_i\delta_n\nu_h - \nu_n\delta_i\nu_h)\delta_h\nu_j \\
&= \delta_n\delta_i\nu_j = D_n\delta_i\nu_j = \frac{\partial}{\partial\Big(\frac{x}{|x|}\Big)}(\delta_i\nu_j) = \sum_{h=1}^{n+1}\frac{x_h}{|x|}D_h(\delta_i\nu_j) = -|x|^{-1}\delta_i\nu_j
\end{aligned}$$

where we have used that fact that $\delta_i\nu_j$ is homogeneous of degree $-1$, and hence the Euler's Theorem on homogeneous functions[3].

[3]A function $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree $k$ if for each $\alpha > 0$ $f(\alpha x) = \alpha^k f(x)$. Euler's Theorem on homogeneous functions says that a differentiable function $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$ is a cone, is homogeneous of degree $k$ if and only if

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}x_i = kf(x) \qquad \forall\, x \in A$$

Moreover

$$\sum_{i,j=1}^{n-1} (\delta_i \delta_n \nu_j)^2 = |x|^{-2} \sum_{i,j=1}^{n-1} (\delta_i \nu_j)^2 = |x|^{-2} c^2$$

Hence we obtain that

$$\frac{1}{2}\mathcal{D}c^2 + c^4 - |\delta c|^2 \geq 2c^2 |x|^{-2}$$

□

For the last result of this section we need the following

**Lemma 9.3.2 (Integration by parts).** *Let $\partial E$ be a smooth hypersurface and let $\varphi \in C^1_c(\mathbb{R}^{n+1})$. Then*

$$\int_{\partial E} \delta_i \varphi \, \mathrm{d}\mathcal{H}^n = -\int_{\partial E} \mathrm{H} \varphi \nu_i \, \mathrm{d}\mathcal{H}^n$$

*Proof.* First suppose that there exists $g \in C^\infty(U)$, $U \subset \mathbb{R}^n$ open set, such that

$$\partial E \cap \operatorname{supp} \varphi \subset \{ (\overline{x}, x_{n+1}) \mid n_{n+1} = g(\overline{x}) \}$$

Since $\nu_{n+1} = (1 + |Dg|^2)^{-\frac{1}{2}}$, $\nu_i = -\nu_{n+1} D_i g$ for $i = 1, \ldots, n$, we have that

$$\begin{aligned}
\int_{\partial E} \delta_i \varphi \, \mathrm{d}\mathcal{H}^n &= \sum_{h=1}^{n+1} \int_U (\varepsilon_{ih} - \nu_i \nu_h) \nu_{n+1}^{-1} D_h \varphi \, \mathrm{d}\overline{x} \\
&= -\sum_{h=1}^{n+1} \int_U \varphi D_h [ (\varepsilon_{ih} - \nu_i \nu_h) \nu_{n+1}^{-1} ] \, \mathrm{d}\overline{x} \\
&= -\int_U \varphi \Big[ \nu_{n+1}^{-2} \Big( - D_i \nu_{n+1} + \sum_{h=1}^{n+1} \nu_{n+1}^2 \nu_h D_h \Big( \frac{\nu_i}{\nu_{n+1}} \Big) \Big) \\
&\quad + \sum_{h=1}^{n+1} \nu_{n+1}^{-1} \nu_i D_h \nu_h \Big] \, \mathrm{d}\overline{x} \\
&= -\int_U \varphi \Big[ \nu_{n+1}^{-2} \Big( - \nu_{n+1}^3 \sum_{h=1}^{n+1} D_h g D_i D_h g + \nu_{n+1}^3 \sum_{h=1}^{n+1} D_h g D_h D_i g \Big) \\
&\quad + \nu_i \nu_{n+1}^{-1} \Big( \mathrm{H} + \langle \nu, \frac{1}{2} D |\nu|^2 \rangle \Big) \Big] \, \mathrm{d}\overline{x} \\
&= -\int_{\partial E} \mathrm{H} \varphi \nu_i \, \mathrm{d}\mathcal{H}^n
\end{aligned}$$

The genral case follows easly from partition of unity. □

**Remark 9.3.3.** *In particular, if $\partial E$ is stationary, that is* $\mathrm{H} \equiv 0$*, we have the standard formula by parts*

$$\int_{\partial E} u\delta_i v \,\mathrm{d}\mathcal{H}^n = -\int_{\partial E} v\delta_i u \,\mathrm{d}\mathcal{H}^n$$

*provided $uv \in C^1_c(\mathbb{R}^{n+1})$. Now we want to integrating by parts the Laplace operator $\mathcal{D}$: if $uv \in C^1_c(\mathbb{R}^{n+1})$ we have that*

$$\int_{\partial E} u\mathcal{D}v \,\mathrm{d}\mathcal{H}^n = -\int_{\partial E} \sum_{h=1}^{n+1} \delta_h u \delta_h v \,\mathrm{d}\mathcal{H}^n = \int_{\partial E} v\mathcal{D} \,\mathrm{d}\mathcal{H}^n$$

Now we can prove the fundamental theorem of this section

**Theorem 9.3.4 (Simons Theorem).** *Let $C$ be a cone in $\mathbb{R}^{n+1}$ such that $\mathcal{S} := \partial C$ is regular, except possibly at the origin. Suppose that for every $g \in C^1_c$ such that* $\operatorname{supp} g \cap \{0\} = \emptyset$ *it hold*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^n(G_t\mathcal{S}) = 0$$

*and*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{H}^n(G_t\mathcal{S}) \geq 0$$

*Then either $\mathcal{S}$ is an hyperplane or $n + 1 \geq 8$.*

*Proof.* Since the first variation of the area is 0 for each $g \in C^1_c$, we obtain that

$$\mathrm{H} \equiv 0$$

Now, using the fact that the second variation is non negative we obtain that

$$\int_{\mathcal{S}} g^2 c^2 \,\mathrm{d}\mathcal{H}^n \leq \int_{\mathcal{S}} |\delta g|^2 \,\mathrm{d}\mathcal{H}^n \tag{9.8}$$

Now let $g \in C^1_c$ such that $\operatorname{supp} g \cap \{0\} = \emptyset$, and write the inequality above for the function $gc$ in place of $g$, obtaining

$$\int_{\mathcal{S}} g^2 c^4 \,\mathrm{d}\mathcal{H}^n \leq \int_{\mathcal{S}} |g\delta c + c\delta g|^2 \,\mathrm{d}\mathcal{H}^n \tag{9.9}$$

Moreover it holds

$$\begin{aligned}
\int_{\mathcal{S}} |g\delta c + c\delta g|^2 \,\mathrm{d}\mathcal{H}^n &= \int_{\mathcal{S}} (\, g^2|\delta c|^2 + c^2|\delta g|^2 + 2gc\langle \delta g,\ \delta c\rangle \,) \,\mathrm{d}\mathcal{H}^n \\
&= \int_{\mathcal{S}} \Big( g^2|\delta c|^2 + c^2|\delta g|^2 + \frac{1}{2}\langle \delta g^2, \delta c^2\rangle \Big) \,\mathrm{d}\mathcal{H}^n \\
&= \int_{\mathcal{S}} \Big( c^2|\delta g|^2 + g^2\Big( |\delta c|^2 - \frac{1}{2}\mathcal{D}c\Big) \Big) \,\mathrm{d}\mathcal{H}^n
\end{aligned}$$

where in the last step we have used integration by parts. From the previous Theorem we obtain that

$$\int_{\mathcal{S}} g^2 c^4\, \mathcal{H}^n \le \int_{\mathcal{S}} c^2|\delta g|^2 + g^2(\, c^4 - 2c^2|x|^{-2}\,)\,\mathrm{d}\mathcal{H}^n$$

that is

$$\int_{\mathcal{S}} (\, c^2|\delta g|^2 - 2c^2|x|^{-2} g^2\,)\,\mathcal{H}^n \ge 0 \tag{9.10}$$

By approximation inequality (9.10) holds for every $g \in C^1(\mathbb{R}^{n+1} \setminus \{0\})$ such that

$$\int_{\mathcal{S}} g^2 c^2 |x|^{-2}\,\mathrm{d}\mathcal{H}^n < +\infty \tag{9.11}$$

Since $C$ is a cone we have that $\nu$ is homogeneous of degree $-1$, and hence $c^2$ is homogeneous of degree $-2$. Hence condition (9.11) becomes

$$\int_{\mathcal{S}} g^2(x) c^2\Big(\frac{x}{|x|}\Big)|x|^{-4}\,\mathrm{d}\mathcal{H}^n < +\infty$$

Since $C$ is regular in $\mathbb{R}^{n+1} \setminus \{0\}$ we have that $c^2$ is continous in the compact set $K := \mathcal{S} \cap \partial B_1$, and hence $\|c^2\|_{C^0(K)} < \infty$. Hence we can rewrite condition (9.11) as

$$\int_{\mathcal{S}} \frac{g^2}{|x|^4}\,\mathrm{d}\mathcal{H}^n < +\infty \tag{9.12}$$

Now we consider the function

$$g(x) := \begin{cases} |x|^{\alpha} & , |x| < 1 \\ |x|^{\alpha+\beta} & , |x| \ge 1 \end{cases}$$

We want to determine $\alpha$ and $\beta$ in order to satisfied condition (9.12), that is such that

$$\begin{cases} \displaystyle\int_{\mathcal{S} \cap B_1} |x|^{2\alpha - 4}\,\mathrm{d}\mathcal{H}^n < +\infty \\ \displaystyle\int_{\mathcal{S} - B_1} |x|^{2(\alpha+\beta)-4}\,\mathrm{d}\mathcal{H}^n < +\infty \end{cases}$$

From the Coarea formula we can write the integrals as

$$\begin{cases} \displaystyle\mathcal{H}^{n-1}(\mathcal{S} \cap B_1) \int_0^1 r^{2\alpha-4}\,\mathrm{d}r < +\infty \\ \displaystyle\mathcal{H}^{n-1}(\mathcal{S} - B_1) \int_1^{+\infty} r^{2(\alpha+\beta)-4}\,\mathrm{d}r < +\infty \end{cases}$$

So we have to impose that

$$\alpha > \frac{4-n}{2}\,, \qquad \alpha + \beta < \frac{4-n}{2}$$

If we choose $\alpha$ and $\beta$ satisfactory the system above, from our choise of $g$, inequality (9.10) becomes

$$\int_{\mathcal{S}\cap B_1} c^2|\delta(|x|^\alpha)|^2 - 2c^2|x|^{2\alpha-2}\,\mathrm{d}\mathcal{H}^n + \int_{\mathcal{S}-B_1} c^2|\delta(|x|^{\alpha+\beta})|^2 - 2c^2|x|^{2(\alpha+\beta)-2}\,\mathrm{d}\mathcal{H}^n \geq 0 \tag{9.13}$$

But

$$\begin{aligned} \delta(|x|^p) &= D(|x|^p) - \langle D(|x|^p), \nu\rangle\nu = p|x|^{p-2}x - \langle p|x|^{p-2}x, \nu\rangle\nu \\ &= p|x|^{p-2}(x - \langle x, \nu\rangle\nu) \end{aligned}$$

and hence

$$|\delta(|x|^p)|^2 = p^2|x|^{2(p-2)}(|x|^2 - \langle x\nu\rangle^2) = p^2|x|^{2p-2}$$

where in the last step we haave used the fact that $C$ is a cone. Hence (9.13) becomes

$$(\alpha^2-2)\int_{\mathcal{S}\cap B_1} c^2|x|^{2\alpha-2}\,\mathrm{d}\mathcal{H}^n + [(\alpha+\beta)^2-2]\int_{\mathcal{S}-B_1} c^2|x|^{2(\alpha+\beta)-2}\,\mathrm{d}\mathcal{H}^n \geq 0$$

So, if we choose $\alpha$ and $\beta$ such that

$$\begin{cases} \alpha^2 - 2 & \geq 0 \\ (\alpha+\beta)^2 - 2 & \geq 0 \end{cases}$$

we obtain that $c \equiv 0$, and so the whole $\mathcal{S}$ is an hyperplane. All the conditions about $\alpha$ and $\beta$ can be satisfied if

$$\left(\frac{4-n}{2}\right)^2 < 2$$

that is if $n = 2, 3, 4, 5, 6$. □

So we have proved the following regularity theorem

**Theorem 9.3.5.** *Suppose $n \leq 7$, and let $E$ be a minimal set in $B_\rho$. Then $\partial E \cap B_\rho$ is an analytic hypersurface.*

As we will see in the next section the above theorem is the best possible, since in higer dimensions there exist minimal surfaces with singularity, that is the set $\partial E \setminus \partial^* E$ is nonempty. But we can give an upper bound for the dimension of the singular set, as we will state in the next theorem

**Theorem 9.3.6.** *Suppose $E$ is a minimal set in $U \subset \mathbb{R}^n$, and let $\Sigma := (\partial E \setminus \partial^* E) \cap U$. Then*

$$\mathcal{H}^s(\Sigma) = 0 \quad \textit{for all } s > n-8$$

## 9.4 Minimality of the Simons cone

In this section we show that the Simons cone

$$\mathcal{C}_{\mathcal{S}} := \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid |x|^2 \leq |y|^2\}$$

is minimal in $\mathbb{R}^8$. This result is fundamental, because it says that Theorem 9.3.5 is the best possible.
To prove this result we will follow a simple thecnique due to De Philippis and Paolini (see [DPP09]).

We begin with the definition of sub-minimal sets, and two proposition that we will use later.

**Definition 9.4.1.** *Let $U \subset \mathbb{R}^n$ be an open set. We say that a measurable set $E$ is a **sub-minimal** in $U$ if for each open bounded set $A \subset U$*

$$P(E, A) \leq P(F, A)$$

*for each measurable set $F \subset E$ such that $E \setminus F \Subset A$.*

**Proposition 9.4.2.** *Let $U$ be an open set in $\mathbb{R}^n$ and $E$ a measurable set. If both $E$ and $E^c := U \setminus E$ are sub-minimal in $U$, then $E$ is minimal in $U$.*

*Proof.* Let $A$ an open bounded subset of $U$, and $F$ be a measurable set such that $E \Delta F \Subset A$. We want to show that

$$P(E, A) \leq P(F, A)$$

So, let

$$F' := E \cap F \subset E$$

$$F'' := (E \cup F)^c = E^c \setminus F \subset E^c$$

Then

$$E \setminus F' \subset E \Delta F \Subset A\,, \qquad E^c \setminus F'' \subset E \Delta F \Subset A$$

From the sub-minimality of $E$ in $U$ we have

$$P(E, A) \leq P(F', A) = P(E \cap F, A)$$

and from the sub-minimality of $E^c$ in $U$ we have

$$P(E^c, A) \leq P(F'', A) = P((F'')^c, A) = P(E \cup F, A)$$

Hence

$$P(E \cap F, A) + P(E \cup F, A) \geq 2P(E, A) \tag{9.14}$$

From Lemma 6.4.1 we obtain

$$P(E \cap F, A) + P(E \cup F, A) \leq P(E, A) + P(F, A) \tag{9.15}$$

Using (9.14) and (9.15) we obtain

$$P(E, A) \leq P(F, A)$$

□

**Proposition 9.4.3.** *Let $(E_k)_k, E \subset U$, where $U \subset \mathbb{R}^n$ is an open set. Suppose $E_k$ is sub-minimal in $U$ for each $k$, that $E_k \subset E$ and $E_k \to E$. Then $E$ is sub-minimal in $U$.*

*Proof.* Let $A \subset U$ be on open bounded set, $F \subset E$ be a measurable set such that $E \setminus F \Subset A$. Consider the sets $F'_k := F \cap E_k$; then

$$F'_k \subset E_k$$

$$E_k \setminus F_k \subset E \setminus F \Subset A$$

Since $E_k$ is sub-minimal in $U$ we have

$$P(E_k, A) \leq P(F'_k, A) = P(E \cap F'_k, A) \tag{9.16}$$

Since $E_k \subset E_k \cup F \subset E$, from the convergenge $E_k \to E$ we obtain that $E_k \cup F \to E$. Hence, from the semicontinuity

$$P(E, A) \leq \liminf_{k\to\infty} P(E_k \cup F, A) \tag{9.17}$$

From Lemma 6.4.1

$$P(E_k \cup F, A) \leq P(E_K, A) + P(F, A) - P(E_k \cap E, A)$$

Passing to the lim inf and using (9.16) on the right, and (9.17) on the left we obtain the result. □

Now we present a simple method to prove that a set is sub-minimal, the method of *sub-calibration*.

**Definition 9.4.4.** *Let $E \subset U$ be a measurable set such that $\partial E \cap U$ is $C^2$. A vector field $\xi \in C^1(U; \mathbb{R}^n)$ such that*

1. $\xi_{|\partial E \cap U} \equiv \nu_E$
2. $\mathrm{div}(\xi)(x) \leq 0 \quad \forall x \in E \cap U$
3. $|\xi| = 1$

*is called a **sub-calibration** of $E$ in $U$.*

**Theorem 9.4.5.** *Let $\xi$ be a sub-calibration of $E$ in $U$; suppose $\partial E \cap U$ is $C^2$. Then $E$ is sub-minimal in $U$.*

*Proof.* Let $A \subset U$ be an open bounded set, $F \subset E$ be a measurable set such that $E \setminus F \Subset A$. We want to show that

$$P(E, A) \leq P(F, A)$$

Since

$$P(F, A) := \sup \Big\{ \int_{F \cap A} \operatorname{div}(\varphi) \mid \varphi \in C_c^1(A; \mathbb{R}^n),\ |\varphi| \leq 1 \Big\}$$

we have to take functions $\nu_j \in C_c^1(A; \mathbb{R}^n)$ such that

$$(\nu_j)_{|_{E \setminus F}} \equiv 1\,, \quad 0 \leq \nu_j \leq 1 \quad \text{in } A$$

$$A_j := \{x \in A \mid \nu_j(x) = 1\} \uparrow A$$

Define $\xi_j := \nu_j \xi$. Then

$$\int_{E \cap A} \operatorname{div}(\xi_j)\,\mathrm{d}x - \int_{F \cap A} \operatorname{div}(\xi_j)\,\mathrm{d}x = \int_{E \setminus F} \operatorname{div}(\xi_j)\,\mathrm{d}x = \int_{E \setminus F} \operatorname{div}(\xi)\,\mathrm{d}x \leq 0$$

Hence

$$\int_{E \cap A} \operatorname{div}(\xi_j)\,\mathrm{d}x \leq \int_{F \cap A} \operatorname{div}(\xi_j)\,\mathrm{d}x \tag{9.18}$$

Since $\partial E \cap A$ is $C^2$

$$\int_{E \cap A} \operatorname{div}(\xi_j)\,\mathrm{d}x = \int_{\partial E \cap A} \langle \xi_j, \nu_E \rangle\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\partial E \cap A} \nu_j\,\mathrm{d}\mathcal{H}^{n-1} \geq \mathcal{H}^{n-1}(\partial E \cap A_j)$$

Since $A_j \uparrow A$ we have

$$\liminf_j \int_{E \cap A} \operatorname{div}(\xi_j)\,\mathrm{d}x \geq \mathcal{H}^{n-1}(\partial E \cap A) = P(E, A)$$

Hence, using (9.18)

$$P(E, A) \leq \int_{E \cap A} \operatorname{div}(\xi_j)\,\mathrm{d}x \leq \int_{F \cap A} \operatorname{div}(\xi_j)\,\mathrm{d}x \leq P(F, A)$$

□

Now we can prove that the Simons cone in $\mathbb{R}^n$, $n = 2m$, defined by

$$\mathcal{C}_{\mathcal{S}} := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mid |x| \leq |y|\}$$

is minimal in $\mathbb{R}^n$. To do it we reason as follows: we see at $\mathcal{C}_{\mathcal{S}}$ as the zero-sublevel of the function

$$f(x, y) := \frac{|x|^4 - |y|^4}{4}$$

Clearly $\mathcal{C}_{\mathcal{S}} = \{f \leq 0\}$. For $k \setminus \{0\}$ let

$$E_k := \left\{f(x, y) \leq -\frac{1}{k}\right\}, \qquad F_k := \left\{f(x, y) \leq \frac{1}{k}\right\}$$

We have that $E_k \subset \mathcal{C}_{\mathcal{S}}$, $\mathcal{C}_{\mathcal{S}} \subset F_k$ and $E_k \to \mathcal{C}_{\mathcal{S}}$, $F_k \to \mathcal{C}_{\mathcal{S}}$. We want to prove that $E_k$ and $F_k^c$ are sub-minimal in $\mathbb{R}^n$, and hence apply Proposition 9.4.3 to obtain that $\mathcal{C}_{\mathcal{S}}$ and $\mathbb{R}^n \setminus \mathcal{C}_{\mathcal{S}}$ are sub-minimal in $\mathbb{R}^n$; finally we conclude with Proposition 9.4.2.

To prove that $E_k$ and $F_k$ are sub-minimal we consider the vector field

$$\xi := \frac{Df}{|Df|}$$

defined in $\Omega := \mathbb{R}^n \setminus \{0\}$. It holds

**Theorem 9.4.6.** *Let $m \geq 4$. Then $\xi$ is a sub-calibration of $E_k$ in $\Omega$, and $-\xi$ is a sub-calibration of $F_k^c$ in $\Omega$.*

*Proof.* Since $\partial E_k$ and $\partial F_k$ are $C^2$, we have that $\nu_{E_k}$ and $\nu_{F_k}$ are respectively the outer normal to the level sets $\{f(x, y) = -\frac{1}{k}\}$ and $\{f(x, y) = \frac{1}{k}\}$. Clearly

$$|\xi| = 1$$

$$\xi_{|\partial E_k} = \nu_{E_k}\,, \qquad \xi_{|F_k} = \nu_{F_k}$$

It remains to show that, for $m \geq 4$, $\xi$ has negative divergence. Since

$$\operatorname{div}(\xi) = \sum_{i=1}^{m} \left(\frac{f_{x_i}}{|Df|}\right)_{x_i} + \sum_{i=1}^{m} \left(\frac{f_{y_i}}{|Df|}\right)_{y_i}$$

we compute

$$\begin{aligned}
\sum_{i=1}^{m} \left(\frac{f_{x_i}}{|Df|}\right)_{x_i} &= \sum_{i=1}^{m} \frac{(2x_i^2 + |x|^2)(|x|^6 + |y|^6) - |x|^4 x_i \sum_{j=1}^{m}(2x_i x_j^2 + \delta_{ij} x_j |x|^2)}{|Df|^3} \\
&= \frac{2|x|^2|y|^6 + m|x|^8 + m|x|^2|y|^6 - |x|^8}{|Df|^3} \\
&= \frac{(m-1)|x|^8 + (m+2)|x|^2|y|^6}{|Df|^3}
\end{aligned}$$

For simmetry we also have

$$\sum_{i=1}^{m}\Big(\frac{f_{y_i}}{|Df|}\Big)_{y_i} = -\frac{(m-1)|y|^8+(m+2)|y|^2|x|^6}{|Df|^3}$$

Hence

$$\begin{aligned}
|Df|^3\mathrm{div}(\xi) &= (m-1)|x|^8+(m+2)|x^2|y|^6-(m-1)|y|^8-(m+2)|y|^2|x|^6\\
&= (|x|^4-|y|^4)((m-1)|x|^4-(m+2)|x|^2|y|^2+(m-1)|y|^4)\\
&= (|x|^4-|y|^4)\Big[\,|y|^4\Big((m-1)t^2-(m+2)t+(m-1)\Big)\Big]
\end{aligned}$$

where $t := \frac{|x|^2}{|y|^2}$.

We want show that $|Df|^3\mathrm{div}(\xi)$ has the same sign of $|x|^4-|y|^4$; to do this we prove that the quantity $(m-1)t^2-(m+2)t+(m-1)$ is non-negative:

$$\Delta = (m+2)^2-4(m-1)^2 = 3m(4-m) \le 0$$

for $m \ge 4$. Since $m-1>0$ we obtain that

$$\mathrm{div}(\xi) \le 0 \qquad \text{in } E_k$$

and

$$\mathrm{div}(-\xi) \le 0 \qquad \text{in } F_k^c$$

Hence $\xi$ and $-\xi$ are the sub-calibrations desired. □

So we obtain the following

**Theorem 9.4.7.** *The Simons cone $\mathcal{C}_{\mathcal{S}}$ is minimal in $\mathbb{R}^n$ for $n=2m$, $m \ge 4$.*

*Proof.* From the Theorem above, and Theorem 9.4.5 we obtain that $E_k$ and $F_k^c$ are sub-minimal in $\Omega$. Since if $E$ is measurable

$$P(E,A) = P(E, A\setminus\{0\})$$

we have that $E_k$ and $F_k^c$ are sub-minimal in $\mathbb{R}^n$. Since $E_k \subset \mathcal{C}_{\mathcal{S}}$, $F_k^c \subset \mathcal{C}_{\mathcal{S}}{}^c$ and $E_k \to \mathcal{C}_{\mathcal{S}}$, $F_k^c \to \mathcal{C}_{\mathcal{S}}{}^c$, from Proposition 9.4.3 we obtain that $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{C}_{\mathcal{S}}{}^c$ are sub-minimal in $\mathbb{R}^n$. Hence $\mathcal{C}_{\mathcal{S}}$ is minimal in $\mathbb{R}^n$. □

# Chapter 10

# Non-parametric minimal surfaces in $\mathbb{R}^n$

In Section 5.3 we have proved the existence of minimal surfaces. What we want to do in this chapter is to study the existence of minimal surface in a bounded open set $\Omega$ that are the graph of some function $u$ defined in $\Omega$. We call this surfaces *non-parametric* minimal surfaces, and hence call the others *parametric* minimal surfaces.
If we have a Lipschihtz function $u : \Omega \to \mathbb{R}$, from the Area Formula we have that the area of its graph is given by

$$\mathcal{A}(u;\Omega) := \int_\Omega \sqrt{1 + |Du|^2}\, \mathrm{d}x$$

In Section 10.1 we will study the existence of a minimum for the functional $\mathcal{A}$ in the class of the Lipschitz functions on $\Omega$ taking a prescribed value $\psi$ on $\partial\Omega$, showing that a solution of this problem exists if the mean curvature of $\partial\Omega$ is non-negative (Theorem 10.1.12). Moreover we will give an example of non existence of the solution in the case of positive curvature of $\partial\Omega$ (Example 10.1.3).

In Section 10.2 we will study the Dirichlet problem in the $BV$ space. The idea is the following one: first of all we extend the notion of "area" of the graph for functions in $BV$ (Definition 10.2.1); then we will give a weaker version of the problem: instead of minimize the functional $\mathcal{A}$ among all functions in $BV(\Omega)$ taking a prescribed value $\psi$ on $\partial\Omega$ (intended as the *trace* of the function), we will minimize the functional

$$\mathcal{I}(v,\Omega) := \int_\Omega \sqrt{1 + |Dv|^2} + \int_{\partial\Omega} |Tr(v) - \psi|\, \mathrm{d}\mathcal{H}^{n-1}$$

among all the function $v \in BV(\Omega)$. The last term is think as a "penalization" for not taking the boundary value $\psi$. We have to pass to this weaker formulation because, if we apply the direct method to the functional $\mathcal{A}$ obtaining a limit function $u$, we have no way to known which is the boundary value of $u$. The important fact is that there is always the existence of a minimun for the functional $\mathcal{I}$ (Theorem 10.2.5). To prove the regularity of this minimum we want to use the regularity results for parametric minimal surfaces, and to do this we have to find some connection between this two objects. First of all we will prove that the area of a graph in an open set $\Omega$ is the perimeter of its subgraph in $\Omega \times \mathbb{R}$ (Theorem 10.2.7), and hence we will prove that a function $u$ is a minimum of the area functional in $\Omega$ if and only if its subgraph locally minimize the perimeter in $\Omega \times \mathbb{R}$ (Theorem 10.2.10). This important result is due to Miranda.

Finally in Section 10.3 we will use Theorem 10.2.10 to extend the notion of functions that minimize the area of a graph (called *quasi-solutions*), and we will study some properties of this kind of functions. This extension is necessary because in proving the Bernstein problem we have to work with limits of non-parametric minimal surfaces, and we are not sure that this limits are thierselves non-parametric minimal surface. This problem is solved if we use quasi-solutions.

## 10.1 Classical solutions of the minimal surface equation

### 10.1.1 Existences results

In this section we study, in a classical framework, the Dirichlet problem for the area functional: let $u : \Omega \to \mathbb{R}$ ba a Lipschitz function, where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Then, from the Area Formula, we have that the area of the graph of $u$ is given by

$$\mathcal{A}(u, \Omega) := \int_\Omega \sqrt{1 + |Du|^2}\,\mathrm{d}x$$

Fix a Lipschitz function $\psi$ defined on $\partial\Omega$. We want to minimize the area functional among all the Lipschitz functions defined in $\Omega$ taking the value $\psi$ on $\partial\Omega$.

First of all we prove that the area functional is lower-semicontinuous with respect to the uniform convergence.

**Theorem 10.1.1.** *Let $(u_j)_j \subset C^{0,1}(\Omega)$ converging uniformly on $\Omega$ to a Lipschitz function $u$. Then*

$$\int_\Omega \sqrt{1 + |Du|^2}\,\mathrm{d}x \le \liminf_{j\to\infty} \int_\Omega \sqrt{1 + |Du_j|^2}\,\mathrm{d}x$$

*Proof.* Since $\Omega$ is a bounded set in $\mathbb{R}^n$ we have that $\mathcal{L}^n(\Omega) < \infty$. Then

$$\int_\Omega |u - u_j| \,\mathrm{d}x \leq \|u - u_j\|_{C^0(\Omega)} \mathcal{L}^n(\Omega)$$

Hence $u_j \to u$ in $L^1(\Omega)$. Moreover, since it is well note that $C^{0,1}(\Omega) = W^{1,\infty}(\Omega)$ we have that

$$\begin{aligned}
& \int_\Omega \sqrt{1 + |Du|^2} \,\mathrm{d}x = \int_\Omega |(Du, 1)| \,\mathrm{d}x \\
= \ & \sup \Big\{ \int_\Omega \langle (\varphi, \varphi_{n+1}), (|Du|, 1) \rangle \,\mathrm{d}x \mid \Phi = (\varphi, \varphi_{n+1}) \in C^1_c(\Omega; \mathbb{R}^{n+1}), |\Phi| \leq 1 \Big\} \\
= \ & \sup \Big\{ \int_\Omega (\varphi_{n+1} + u \mathrm{div}(\varphi)) \,\mathrm{d}x \mid \Phi = (\varphi, \varphi_{n+1}) \in C^1_c(\Omega; \mathbb{R}^{n+1}), |\Phi| \leq 1 \Big\}
\end{aligned}$$

Hence, if we fix $\Phi = (\varphi, \varphi_{n+1}) \in C^1_c(\Omega; \mathbb{R}^{n+1}), |\Phi| \leq 1$ we have that

$$\begin{aligned}
\int_\Omega (\varphi_{n+1} + u \mathrm{div}(\varphi)) \,\mathrm{d}x &= \lim_{j \to \infty} \int_\Omega (\varphi_{n+1} + u_j \mathrm{div}(\varphi)) \,\mathrm{d}x \\
&\leq \liminf_{j \to \infty} \int_\Omega \sqrt{1 + |Du_j|^2} \,\mathrm{d}x
\end{aligned}$$

□

Now our idea is to prove the existence of a minimum in the class of Lipschitz functions using the above semicountinuity result and using, as compactness theorem, the Ascoli-Arzelá Theorem. This theorem required as assumption that the minimizing sequence we will use to extract a subsequence converging to the minimum, consist of functions that are uniformly equicontinuous in the $C^0$ norm; an easy way to get this hypothesis is to required that these functions are uniformly bounded in the Lipschitz norm. In this way we can apply the Ascoli-Arzelá Theorem toghether with the above semicountinuity result to get the existence of a minimum in a subclass of the Lipschitz function. Then we will study when this minimum is a minimum in the whole class of Lipschitz function.

**Definition 10.1.2.** *Let $u$ be a Lipschitz function on $\Omega$. We denote by*

$$|u|_\Omega := \sup \Big\{ \frac{|u(x) - u(y)|}{|x - y|} \mid x \neq y \in \Omega \Big\} < \infty$$

*Now fix $\psi \in C^{0,1}(\Omega)$; for each $k > 0$ we define the spaces*

$$\begin{aligned}
L_k(\Omega) &:= \{ u \in C^{0,1}(\Omega) \mid |u|_\Omega \leq k \} \\
L_k(\Omega, \psi) &:= \{ u \in L_k(\Omega) \mid u = \psi \ \text{ on } \partial\Omega \} \\
L(\Omega, \psi) &:= \{ u \in C^{0,1}(\Omega) \mid u = \psi \ \text{ on } \partial\Omega \}
\end{aligned}$$

We have the following two results of existence

**Theorem 10.1.3.** *Let $\psi \in C^{0,1}(\partial\Omega)$, and suppose that $L_k(\Omega,\psi)$ is nonempty. Then the area functional $\mathcal{A}$ achives its minimum in $L_k(\Omega,\psi)$.*

*Proof.* Let $(u_j)_j \subset L_k(\Omega,\psi)$ be a minimizing sequence. Since the functions $u_j$ are uniformly Lipschitz we can apply the Ascoli-Arzelá Theorem, and hence find a subsequence converging uniformly on $\Omega$ to a continuous function $u$. Moreover $u \in L_k(\Omega,\psi)$: in fact if we fix $\varepsilon > 0$ we can find an integer $\overline{j}$ such that for each $j > \overline{j}$ it holds $\|u - u_j\|_{C^0} < \varepsilon$. Hence, for all $x \neq y \in \Omega$ and $j > \overline{j}$,

$$|u(x) - u(y)| \leq |u(x) - u_j(x)| + |u_j(x) - u_j(y)| + |u_j(y) - u_y| \leq k + 2\varepsilon$$

For the arbitrary of $\varepsilon$ we obtain that $u \in L_k(\Omega,\psi)$. Finally, from the semicountinuity of the functional area we obtain that $u$ minimize the area among all funcions in $L_k(\Omega,\psi)$. $\square$

The area functional is strictly convex. In fact if we take $u, v \in C^{0,1}(\Omega)$ and $t \in (0,1)$ we have that

$$\begin{aligned}
\mathcal{A}(tu + (1-t)v, \Omega) &= \int_\Omega \sqrt{1 + |tDu + (1-t)Dv|}\,\mathrm{d}x \\
&\leq \int_\Omega \sqrt{1 + t^2|Du|^2 + (1-t)^2|Dv|^2 + 2t(1-t)\langle |Du|, |Dv|\rangle}\,\mathrm{d}x \\
&= \int_\Omega \sqrt{1 + (t|Du| + (1-t)|Dv|)^2}\,\mathrm{d}x
\end{aligned}$$

So we need to prove the inequality:

$$\sqrt{1 + (ta + (1-t)b)^2} \leq t\sqrt{1 + a^2} + (1-t)\sqrt{1 + b^2}$$

where $a, b > 0$, $t \in (0,1)$. If we make the square of the two positive members, and make a little computation, we obtain $(a+b)^2 > 0$, and hence the desired result.
Hence, if $L_k(\Omega,\psi)$ is nonempty, there is a unique minimum of the area funcional. We denote it with $u^k$. Now from the existence of a minimum in $L_k(\Omega,\psi)$ we want to obtain a minimum in $L(\Omega,\psi)$.

**Theorem 10.1.4.** *Let $u^k$ be the point of minimum of $\mathcal{A}$ in $L_k(\Omega,\psi)$. Suppose $|u^k|_\Omega < k$. Then $u_k$ also minimize $\mathcal{A}$ in $L(\Omega,\psi)$.*

*Proof.* For $t \in [0,1]$ and $v \in L(\Omega,\psi)$ define

$$v_t := u^k + t(v - u^k)$$

Then $v_{t|\partial\Omega} = \psi$. Moreover we have that, for $x \neq y \in \Omega$

$$\frac{|v_t(x) - v_t(y)|}{|x - y|} \leq \|u^k\|_\Omega + t(\|v\|_\Omega + \|u^k\|_\Omega)$$

Since $\|u^k\|_\Omega < k$ if we choose $t$ such that $t < k - \|u^k\|_\Omega$ we obtain that $v_t \in L_k(\Omega, \psi)$. So

$$\mathcal{A}(u^k, \Omega) \leq \mathcal{A}(v_t, \Omega) \leq t\mathcal{A}(v, \Omega) + (1-t)\mathcal{A}(u^k, \Omega)$$

and hence

$$\mathcal{A}(u^k, \Omega) \leq \mathcal{A}(v, \Omega)$$

□

These two results tell us that, in order to obtain the existence of a minimum in $L(\Omega, \psi)$, we need to estimate the Lispschitz constant of a $u^k$. First of all we note that for each $\psi \in C^{0,1}(\partial\Omega)$ there exists a Lipschitz function $u_\psi \in L(\Omega, \psi)$. Hence we can apply Theorem 10.1.3. So, in order to obtain a minimum for the area functional in $L(\Omega, \psi)$, we only need to get an estimate of $\|u_\psi\|_\Omega$. Our aim is to find some conditions under which we can do it.

**Notation:** we will say that a function $u$ minimizes the area in $L_k(\Omega)$ to indent that $u \in L_k(\Omega)$ minimize the area among all functions in $L_k(\Omega)$ having the same boundary value on $\partial\Omega$.

First of all we note the following two facts:

- if $k \leq k'$ then $L_k(\Omega, \psi) \subset L_{k'}(\Omega, \psi)$
- if $u$ minimize $\mathcal{A}$ in $L_k(\Omega)$, then $u$ also minimize the area functional in $L_{\widetilde{k}}(\widetilde{\Omega})$ for each $\widetilde{\Omega} \subset \Omega$ and $\widetilde{k} \leq k$, if $\|u\|_{\widetilde{\Omega}} \leq \widetilde{k}$.
  Reasoning as follows: if for absurd there exists a function $v \in L_{\widetilde{k}}(\widetilde{\Omega})$ such that $v_{|\partial\widetilde{\Omega}} = u_{|\partial\widetilde{\Omega}}$ and $\mathcal{A}(v, \widetilde{\Omega}) < \mathcal{A}(u, \widetilde{\Omega})$, then the function

  $$f := \begin{cases} u(x) & , x \in \Omega \setminus \widetilde{\Omega} \\ v(x) & , x \in \widetilde{\Omega} \end{cases}$$

  is continuous, and belongs to $L_k(\Omega)$: take $x \neq y \in \Omega$

  - if $x, y \in \Omega \setminus \overline{\widetilde{\Omega}}$ then $|f(x) - f(y)| = |u(x) - u(y)| \leq k|x - y|$
  - if $x, y \in \overline{\widetilde{\Omega}}$ then $|f(x) - f(y)| = |v(x) - v(y)| \leq \widetilde{k}|x - y| \leq k|x - y|$

– if $x \in \widetilde{\Omega}$ and $y \in \Omega \setminus \overline{\widetilde{\Omega}}$ then, if we denote by $z$ a point of

$$\{\, x + t(y - x) \mid t \in [0,1] \,\} \cap \partial\widetilde{\Omega}$$

and recalling that $u$ and $v$ coincide on $\partial\widetilde{\Omega}$ and that $|x - y| = |x - z| + |z - y|$, we have that

$$\begin{aligned} u(y) &= v(x) + (u(y) - v(x)) = v(x) + (u(y) - u(z)) + (v(z) - v(v)) \\ &\leq v(x) + k|y - z| + \widetilde{k}|z - x| \leq v(x) + k|y - x| \end{aligned}$$

Moreover we have that $\mathcal{A}(f, \Omega) < \mathcal{A}(u, \Omega)$. Absurd.

**Definition 10.1.5.** *A function $w \in L_k(\Omega)$ is said to be*

- *a **supersolution** for $\mathcal{A}$ in $L_k(\Omega)$ if for all $v \in L_k(\Omega)$ such that $v \geq w$, we have $\mathcal{A}(v, \Omega) \geq \mathcal{A}(w, \Omega)$*
- *a **subsolution** for $\mathcal{A}$ in $L_k(\Omega)$ if for all $v \in L_k(\Omega)$ such that $v \leq w$, we have $\mathcal{A}(v, \Omega) \geq \mathcal{A}(w, \Omega)$*

It is clear that if $u$ minimize the area in $L_k(\Omega)$, then $u$ is both super and sub solution. It is clear that also the converse is true. An important tool is the following

**Lemma 10.1.6.** ***(Weak maximum principle)*** *Let $w$ be a supersolution and $z$ a subsolution in $L_k(\Omega)$. Suppose that $w \geq z$ in $\partial\Omega$. Then $w \geq z$ in $\overline{\Omega}$.*

*Proof.* Suppose the result does not hold. Then the set

$$K := \{\, x \in \Omega \mid w(x) < z(x) \,\}$$

is nonempty. Let $v := \max\{w, z\}$. Then $v \in L_k(\Omega)$ and $v \geq w$; hence, since $w$ is a supersolution, $\mathcal{A}(v, \Omega) \geq \mathcal{A}(w, \Omega)$; this imply that

$$\mathcal{A}(z, K) \geq \mathcal{A}(w, K)$$

Now, if we take $f := \min\{w, z\}$ we have that $f \in L_k(\Omega)$ and $f \leq v$; hence, since $v$ is a subsolution, $\mathcal{A}(f, \Omega) \geq \mathcal{A}(v, \Omega)$; this imply that

$$\mathcal{A}(w, K) \geq \mathcal{A}(z, K)$$

Hence we have obtain that

$$\mathcal{A}(w, K) = \mathcal{A}(z, K)$$

Since $z = w$ on $\partial K$ and $z > w$ in $K$ we must have $Dz \neq Dw$ in a set of positive measure. Hence

$$\mathcal{A}\Big(\frac{w+z}{2}, K\Big) < \frac{1}{2}\mathcal{A}(z, K) + \frac{1}{2}\mathcal{A}(w, K) = \mathcal{A}(w, K)$$

Absurd because $w$ is a supersolution in $L_k(K)$ and $\dfrac{w+z}{2} \geq w$ on $K$. □

As a consequence we have

**Lemma 10.1.7.** *Let $w$ be a supersolution and $z$ a subsolution in $L_k(\Omega)$. Then*

$$\sup_{x\in\Omega}(z(x) - w(x)) = \sup_{y\in\partial\Omega}(z(y) - w(y))$$

*Proof.* First of all we note that if $\alpha \in \mathbb{R}$, then $w+\alpha$ is again a supersolution. Now, let $x \in \partial\Omega$; then

$$z(x) \leq w(x) + \sup_{y\in\partial\Omega}[z(y) - w(y)]$$

The term on the right is finite because $z - w$ is a continuous function, and $\partial\Omega$ is a compact set. Then the function

$$f(x) := w(x) + \sup_{y\in\partial\Omega}(z(y) - w(y))$$

is a supersolution. Hence, for the previous lemma, we obtain the desired result. □

**Remark 10.1.8.** *In particular if $u$ and $v$ minimize the area in $L_k(\Omega)$, we obtain that*

$$\sup_{\Omega}|u - v| = \sup_{\partial\Omega}|u - v|$$

*In fact, since $u$ and $v$ minimize the area in $L_k(\Omega)$, then they are both super and sub-solution. So if we apply the previous lemma taking $u$ as super-solution and $v$ as sub-solution we obtain that*

$$\sup_{x\in\Omega}(v(x) - u(x)) = \sup_{y\in\partial\Omega}(v(y) - u(y))$$

*Moreover if we apply the previous lemma taking $v$ as super-solution and $u$ as sub-solution we obtain that*

$$\sup_{x\in\Omega}(u(x) - v(x)) = \sup_{y\in\partial\Omega}(u(y) - v(y))$$

*and so we obtain the desired result.*

Hence we obtain an important result due to von Neumann

**Lemma 10.1.9.** ***(Reduction to boundary estimate)*** *Suppose $u$ minime the area in $L_k(\Omega)$. Then*

$$\|u\|_\Omega = \sup\left\{ \frac{|u(x)-u(y)|}{|x-y|} \mid x \in \Omega, y \in \partial\Omega \right\}$$

*Proof.* Let $x_1 \neq x_2 \in \Omega$, $\tau := x_2 - x_1$. Then the function

$$u_\tau(x) := u(x+\tau)$$

minimize the area in $L_k(\Omega_\tau)$, where

$$\Omega_\tau := \{ z \in \mathbb{R}^n \mid z + \tau \in \Omega \}$$

We note that $\Omega \cap \Omega_\tau \neq \emptyset$ because it contains $x_1$; hence both $u$ and $u_\tau$ minimize the area in $\Omega \cap \Omega_\tau$. From the remark above we obtain that there exists $z \in \partial(\Omega \cap \Omega_\tau)$ such that

$$|u(x_1) - u_2| = |u(x_1) - u_\tau(x_1)| \leq |u(z) - u_\tau(z)| = |u(z) - u(z+\tau)|$$

Since $\partial(\Omega \cap \Omega_\tau) = (\partial\Omega \cap \overline{\Omega_\tau}) \cup (\partial\Omega_\tau \cap \overline{\Omega})$, at leat one of the points $z, z+\tau$ belongs to $\partial\Omega$. If we denote by $L$ the supremum of the thesis, we obtain that

$$|u(x_1) - u(x_2)| \leq L|x_1 - x_2|$$

and hence $L \leq \|u\|_\Omega$. Since

$$\|u\|_\Omega = \sup\left\{ \frac{|u(x)-u(y)|}{|x-y|} \mid x \neq y \in \overline{\Omega} \right\}$$

we obtain the desired result. □

Now we are in position to prove the existence of the solution of our minimum problem under some conditions. First we need some definitions

**Definition 10.1.10.** *Let $x \in \Omega$, and denote with $d(x)$ the distance of $x$ from $\partial\Omega$; for $t > 0$ we define the sets*

$$\Sigma_t := \{ x \in \Omega \mid d(x) < t \}, \qquad \Gamma_t := \{ x \in \Omega \mid d(x) = t \}$$

**Definition 10.1.11.** *Let $\psi \in C^{0,1}(\partial\Omega)$; an **upper barrier** $v^+$ relative to $\psi$ is a function $v^+ \in C^{0,1}(\Sigma_{t_0})$ for some $t_0 > 0$ such that*

- ${v^+}_{|\partial\Omega} = \psi$ *and* $v^+ \geq \sup_{\partial\Omega} \psi$ *on* $\Gamma_{t_0}$
- $v^+$ *is a supersolution in* $\Sigma_{t_0}$

*A* ***lower barrier*** *$v^-$ relative to $\psi$ is a function $v^- \in C^{0,1}(\Sigma_{t_0})$ for some $t_0 > 0$ such that*

- *${v^-}_{|\partial\Omega} = \psi$ and $v^- \leq \inf_{\partial\Omega} \psi$ on $\Gamma_{t_0}$*
- *$v^-$ is a subsolution in $\Sigma_{t_0}$*

We have the following

**Theorem 10.1.12.** *Let $\psi \in C^{0,1}(\partial\Omega)$, and suppose that there exist an upper barrier $v^+$ and a lower barrier $v^-$ relative to $\psi$. Then the area functional $\mathcal{A}$ achieves its minimum in $L(\Omega, \psi)$.*

*Proof.* Let $Q \geq \max\{\, \|v^+\|_{\Sigma_{t_0}}, \|v^-\|_{\Sigma_{t_0}} \,\}$ and let $k > Q$. As noted after the proof of Theorem 10.1.3, there exists a function $u \in L_k(\Omega)$ that minimize the area in $L_k(\Omega)$. Our aim is to prove that $\|u\|_\Omega < k$, and to do this we have to estimate $|u(x) - u(y)|$ when $x \in \Omega$ and $y \in \partial\Omega$.
First of all we note that $u$ also minimize the area in $L_k(\Sigma_{t_0})$, where $t_0 > 0$ is such that both $v^+$ and $v^-$ are defined in $\Sigma_{t_0}$. Moreover it is clear that for each $x \in \Omega$

$$\inf_{\partial\Omega} \psi \leq u(x) \leq \sup_{\partial\Omega} \psi$$

otherwise it is easy to find[1] a Lipschitz function with area less than the area of $u$. In particular we have that

$$v^-(x) \leq u(x) \leq v^+(x) \qquad \text{in } \Gamma_{t_0}$$

Hence, for the weak maximum principle, we have that

$$v^-(x) \leq u(x) \leq v^+(x) \qquad \text{in } \Sigma_{t_0}$$

Since $v^+ = u = v^-$ on $\partial\Omega$, for each $x \in \Gamma_{t_0}$ and each $y \in \partial\Omega$ we have that

$$v^-(x) - v^-(y) \leq u(x) - u(y) \leq v^+(x) - v^+(y)$$

and hence

$$|u(x) - u(y)| \leq \max\{\, |v^+(x) - v^+(y)|, |v^-(x) - v^-(y)| \,\} \leq Q|x - y|$$

[1] Suppose that the set $A := \{\, x \in \Omega \mid u(x) < \inf_{\partial\Omega} \,\}$ is non empty. Then if we define

$$f := \begin{cases} \inf_{\Omega} \psi & \text{, in } A \\ u & \text{, otherwise} \end{cases}$$

we have that $f \in L_k(\Omega)$ and $\mathcal{A}(f, \Omega) < \mathcal{A}(u, \Omega)$. In a similar way we thread the case $u > \sup_{\partial\Omega}$.

for each $x \in \Gamma_{t_0}$ and each $y \in \partial\Omega$. Now, if $y \in \partial\Omega$ and $x \in \Omega$ is such that $d(x) > t_0$ we have that

$$|u(x) - u(y)| \leq \max\{\sup_{\partial\Omega} \psi - u(y), u(y) - \inf_{\partial\Omega} \psi\} \leq Qt_0 \leq Q|x-y|$$

where we have used the fact that $v^+$ is a supersolution and $v^-$ is a subsolution, and hence

$$\sup_{\partial\Omega} \psi - u(y) \leq \sup_{\Gamma_{t_0}} v^+ - u(y) = \sup_{\Gamma_{t_0}} v^+ - u(y) \leq Qt_0$$

and, in the same way,

$$u(y) - \inf_{\partial\Omega} \psi \leq Qt_0$$

In conclusion we have obtained that, for each $x \in \Omega$ and $y \in \partial\Omega$

$$|u(x) - u(y)| \leq Q|x-y|$$

Now from Lemmma 10.1.9 we obtain that $\|u\|_\Omega \leq Q < k$, and hence from Theorem 10.1.4 we obtain that $u$ minimize the area in $L(\Omega, \psi)$. $\square$

### 10.1.2 Construction of barriers

Now our aim is to find some conditions under which the construction of upper and lower barriers is possible. First of all we note that we can restric our attection only on the existence of upper barrier, since if $v$ is an upper barrier relative to $-\psi$, then $-v$ is a lower barrier relative to $\psi$.

We start by findind a characterization of super-solution: let $v$ be a super-solution for the area functional in an open set $\Sigma$; hence for each $C_c^\infty(\Sigma)$ function $\eta \geq 0$ it holds that the function

$$g(t) := \mathcal{A}(v + t\eta, \Sigma) \qquad t \geq 0$$

has a minimum in $t = 0$, that is $g'(0) \geq 0$ (since it is defined only for $t \geq 0$). Calculating $g'(0)$, and integrating by parts we obtain that

$$\sum_{i=1}^n D_i\Big(\frac{D_i v}{\sqrt{1+|Dv|^2}}\Big) \leq 0 \qquad \text{in } \Sigma \tag{10.1}$$

Viceversa, if a function $v$ satisfied inequality (10.1) then, thanks to the strictly convexity of the area functional, we obtain that $g(0) \leq g(1)$, that is $v$ is a super-solution in $\Sigma$.

It is useful for later to write condition (10.1) in the following way: define the function $F(p) := \sqrt{1+|p|^2}$, where $p \in \mathbb{R}^n$; hence condition (10.1) becomes (using the formula for the derivation of composition of functions)

$$\sum_{i,j=1}^{n} a_{ij} D_i D_j v \leq 0 \quad ; \text{in } \Sigma$$

where

$$a_{ij} := D_i D_j F(Du) = \frac{\delta_{ij}(1+|Dv|^2) - D_i v D_j v}{(1+|Dv|^2)^{\frac{3}{2}}}$$

We suppose that $\partial\Omega$ is of class $C^2$, and hence that the diatance function $d$ defined by $d(x) := d(x, \partial\Omega)$ is of class $C^2$ in some set $\Sigma_{t_0}$ (see [Giu94]); moreover we suppose that $\psi \in C^2(\mathbb{R}^n)$. We are searching for upper barrier of the form

$$v(x) := \psi(x) + \varphi(d(x))$$

where $\varphi[0, R] \to \mathbb{R}$ is a $C^2$ function such that

$$\varphi(0) = 0\,, \quad \varphi'(t) \geq 1 \quad \varphi''(t) < 0$$

$$\varphi(R) \geq 2 \sup_{\Omega} |\psi|$$

where $R < t_0$ will be determined later.

In this case condition (10.1) becomes

$$\mathcal{L}(v) := a_{ij}(\psi_{ij} + \varphi' d_{ij}) + \varphi'' a_{ij} d_i d_j \leq 0 \tag{10.2}$$

where, for simplicity, we write $f_i$ instead of $D_i f$. Since $A(p) := (a_{ij}(z))_{ij}$ is the Hessian matrix of the strictly convex function $F$, we have that $A(p)$ is semi-definite positive; denoting by $\lambda(p)$ its minimum eigenvalue, and by $\Lambda(p)$ its maximum eigenvalue, we have that they are positive; more precisely

$$\lambda(p) = (1+|p|)^{-\frac{3}{2}}\,, \qquad \Lambda(p) = (1+|p|^2)^{-\frac{1}{2}}$$

Now we want to estimate $\mathcal{L}(v)$ under the assumption, that we will explain later,

$$a_{ij} d_{ij} \leq c_0(|Dv|^2 + 1)\lambda \tag{10.3}$$

**Note:** in what follows we will write $c_i$ to denote a constant, and we will write $\lambda$ and $\Lambda$ instead of $\lambda(p)$ and $\Lambda(p)$ respectively.

Since $a_{ij} d_{ij} \geq \lambda |Dd|^2 = \lambda$, and that $|\psi| < c_1$ in a neightborhood of $\Sigma_{t_0}$, we obtain that

$$\mathcal{L}(v) \leq c_1 \Lambda + \lambda[c_0 \varphi'(|Dv| + 1) + \varphi'']$$

Moreover
$$|Dv| \leq |D\psi| + \psi'|Dd| \leq c_2 + \psi'$$
$$|Dv| + 1 \leq c_2 + 1 + \psi' \leq (c_2+1)[1+\psi'] =: c_3[1+\psi']$$
that yelds to
$$\mathcal{L}(v) \leq \lambda\Big[\varphi'' + c_4\varphi'(1+\varphi') + c_1\frac{\Lambda}{\lambda}\Big]$$
Using the hypothesis $\psi' \geq 1$ we have the estimate
$$\frac{\Lambda}{\lambda} = (1+|Dv|^2) \leq 1 + c_2^2 + {\psi'}^2 + 2c_2\psi' \leq c_5{\psi'}^2$$
and hence we get
$$\mathcal{L}(v) \leq \lambda(\varphi'' + c{\varphi'}^2) \tag{10.4}$$
We want to explain the geometric meaning of condition (10.3):
$$a_{ij}(Dv)d_{ij} = \lambda[(1+|Dv|^2)\triangle d - v_iv_jd_{ij}] = \lambda[(1+|Dv|^2)\triangle d - \psi_i\psi_jd_{ij}]$$
Since $d$ and $\psi$ are $C^2$ functions, in a neighborhood of $\partial\Omega$ we can get an uppper estimate of the last term of the above equation: $\psi_i\psi_j \leq c$. Now, if we suppose that $\partial\Omega$ has non-negative mean curvature, that is $\triangle d \leq 0$, we can estimate
$$(1+|Dv|^2)\triangle d \leq (1-|Dv|^2)\triangle d \leq -2(1+|Dv|)\triangle d \leq c(1+|Dv|)$$
where in the last step we have used the fact that $\triangle d$ is lower bounded in a neighborhood of $\partial\Omega$. Putting all together we obtain
$$a_{ij}(Dv)d_{ij} \leq c(1+|Dv|)\lambda$$
under the assumption that $\triangle d \leq 0$.

Now that we have a simple estimate for $\mathcal{L}(v)$, we can easily prove that the function
$$\psi(d) := \frac{1}{c}\log(1+\sigma d)\,, \qquad \sigma > 0$$
is an upper barrier relative to $\psi$. In fact we have
$$\varphi' = \frac{\sigma}{c(1+\sigma d)}\,, \qquad \varphi'' = -\frac{\sigma^2}{c(1+\sigma d)^2}$$
Hence
$$\mathcal{L}(v) \leq \lambda\Big(-\frac{\sigma^2}{c(1+\sigma d)^2} + c\frac{\sigma^2}{c^2(1+\sigma d)^2}\Big) = 0$$
Moreover
$$\varphi' = \frac{\sigma}{c(1+\sigma d)} > \frac{\sigma}{c(1+\sigma R)}\,, \quad \varphi'' < 0$$

$$\varphi(R) = \frac{1}{c}\log(1+\sigma R)$$

So if we take $R$ sufficiently large, we obtain that $\varphi' > 1$ and $\varphi(R) > 2\sup_\Omega |\psi|$.

In conclusion we have proved the following

**Theorem 10.1.13.** *Let $\Omega$ be an open bounded set of $\mathbb{R}^n$ with $C^2$ boundary of non-negative mean curvature, and let $\psi$ be a $C^2$ function in $\mathbb{R}^n$. Then the Dirichlet problem for the area functional with boundary datum $\psi$ is uniquely solvable in $C^{0,1}(\Omega)$.*

### 10.1.3 Non existence of minimal surfaces

In this section we want to prove that the condition on the mean curvature of $\partial\Omega$ is necessary for the solvability od the Dirichlet problem. In fact we will prove in Theorem 10.1.16 that if in a point of $\partial\Omega$ the mean curvature is negative, then we can find a regular datum $\psi$ such that the Dirichlet problem for the area functional has no solution in $C^{0,1}$. We will also give a concrete example of such a situation, that will be useful for some observation we will do in Section 10.2.

First of all we need a variation of the maximum principle

**Lemma 10.1.14.** *Let $\Omega$ be a connected open set such that $\partial\Omega = \partial^0\Omega \cup \partial^1\Omega$, where $\partial^1\Omega$ is an open set, i.e. there exists an open set $A$ such that $A\cap\partial\Omega = \partial^1\Omega$, $\partial^0\Omega \neq \emptyset$, $\partial^0\Omega \cap \partial^1\Omega = \emptyset$. Let $u$ be a minimum for the area functional in $C^{0,1}(\Omega)$, and let $v \in C^1(\Omega)\cap C(\overline{\Omega})$ be a super-solution such that*

1. *$u \le v$ on $\partial^0\Omega$*
2. $\displaystyle\liminf_{t\to 0^+} \inf_{A\cap\Gamma_t} \frac{\partial v}{\partial \nu} > |u|_\Omega\,,$ *where $\nu$ is the outer normal to $\partial\Omega$.*

*Then $u \le v$ in $\Omega$.*

*Proof.* First of all we suppose that $u < v$ on $\partial^0\Omega$. For continuity there exists a $t_0 > 0$ such that for each $t < t_0$ we have

$$\frac{\partial v}{\partial \nu} > |u|_\Omega \quad \text{on } \Gamma_t \cap A \tag{10.5}$$

$$u \le v \qquad \text{on } \Gamma_t \setminus A \tag{10.6}$$

Suppose for absurd that there exists a point $x_0 \in \Omega$ such that $u(x_0) > v(x_0)$. Let $t < t_0$ such that $x_0 \in \Omega_t := \Omega \setminus \overline{\Gamma_t}$. From Lemma 10.1.7 the function

$w := u$ restrict to $\Omega_t$ must achives its positive maximum in a point $x_1 \in \Gamma_t \cap A$ (for condition (10.6)). Then we have

$$\liminf_{h\to 0^+} \frac{w(x_1 - h\nu) - w(x_1)}{h} \geq -|u|_\Omega + \frac{\partial v}{\partial \nu} > 0$$

where the last inequality follows from condition (10.5). But this is absurd, since in a point of maximum we would have a non-positive derivate. So we have prove the result under the assumption that $u < v$. For the general case consider the function $v_\varepsilon := v + \varepsilon$, $\varepsilon > 0$, and let $\varepsilon \to 0$. □

**Remark 10.1.15.** *In particular the above result holds if*

$$\frac{\partial v}{\partial \nu} = +\infty \qquad on\ \partial^1\Omega$$

We have the following non-existence result

**Theorem 10.1.16.** *Let $\Omega$ be a connected bounded open set in $\mathbb{R}^n$ of class $C^2$, and suppose that the mean curvature of $\partial\Omega$ is negative in a point $x_0 \in \partial\Omega$. Then there exists a regular function $\psi$ such that the area functional has no minimum in $C^{0,1}(\Omega)$.*

*Proof.* Suppose that $u$ minimize the area functional in $C^{0,1}$. We want to find a condition that the datum of the Dirichlet problem, i.e. the set $\Omega$ and the function $\psi$, must satisfied in order to have a solution. We start by estimate the solution $u$ in $\Omega \setminus B_r(x_0)$, for $r > 0$. For $x$ outside $B_r(x_0)$ we define the function

$$\delta(x) := d(x, B_r(x_0)) = |x - x_0| - r$$

Define the function

$$v(x) := A + \psi(\delta(x)) \qquad A > 0$$

In this case we obtain that

$$\mathcal{L}(v) = \lambda[(\varphi' + (\varphi')^3 \triangle\delta + \varphi'')]$$

So, if we choose $\varphi(\delta) := -B\sqrt{\delta}$ we obtain that

$$\mathcal{L}(v) = \lambda\Big[\Big( -\frac{B}{2\sqrt{\delta}} + (\varphi')^2\Big)\triangle\delta + \varphi''\Big] \leq \lambda[(\varphi')^2\triangle\delta + \varphi'']$$

Since

$$\triangle\delta = \frac{n-1}{|x - x_0|} \leq \frac{n-1}{\operatorname{diam}(\Omega)}$$

we obtain the estimate

$$\mathcal{L}(v) \leq \lambda\frac{B\delta^{-\frac{3}{2}}}{4}\Big[\frac{1-n}{2\operatorname{diam}(\Omega)}B^2 + 1\Big]$$

So, taking

$$B^2 := \frac{2\mathrm{diam}(\Omega)}{n-1}$$

we obtain $\mathcal{L}(v) \leq 0$, and hence $v$ is a super-solution.

Moreover, since

$$\frac{\partial v}{\partial \nu} = +\infty \qquad \text{on } \partial B_r(x_0)$$

we can apply Lemma 10.1.14, and choosing

$$A := \sup_{\partial\Omega \setminus B_r(x_0)} \psi + B\sqrt{\mathrm{diam}(\Omega)}$$

we obtain the estimate

$$\sup_{\Omega \setminus B_r(x_0)} u \leq \sup_{\Omega \setminus B_r(x_0)} v = A$$

In particular we obtain the estimate

$$\sup_{\Omega \cap \partial B_r(x_0)} \leq \sup_{\partial\Omega \setminus B_r(x_0)} \psi + B\sqrt{\mathrm{diam}(\Omega)} \tag{10.7}$$

Now we want to estimate $u$ in $\Omega \cap U_r(x_0)$. Since $\triangle d \geq 0$ and $\partial\Omega$ is of class $C^2$, there exist $\varepsilon, R > 0$ such that

$$\triangle d \geq \varepsilon \qquad \text{in } \Omega \cap U_R(x_0) \tag{10.8}$$

So we consider the ball $U_R(x_0)$; defined a function

$$v(x) := \alpha - \beta\sqrt{d}$$

With the same calculation as above we obtain

$$\mathcal{L}(v) \leq \lambda[(\varphi')^2 \triangle d + \varphi''] \leq \frac{\lambda\beta}{4d^{\frac{3}{2}}}(1 - \varepsilon\beta)$$

where in the last step we have use (10.8). So choosing $\beta$ such that $1 - \varepsilon\beta < 0$ we obtain $\mathcal{L}(v) \leq 0$, and hence $v$ is a super-solution. Now set

$$\alpha := \sup_{\partial U_R(x_0) \cap \Omega} u + \beta\sqrt{\mathrm{diam}(\Omega)}$$

and apply Lemma 10.1.14 we obtain

$$\sup_{\Omega \cap U_R(x_0)} \leq \sup_{\Omega \cap U_R(x_0)} = \sup_{\partial U_R(x_0) \cap \Omega} u \leq \sup_{\partial\Omega \setminus U_R(x_0)} \psi + (B + \beta)\sqrt{\mathrm{diam}(\Omega)}$$

where we have used estimate (10.7) with $r = R$. Using the fact that $u \equiv \psi$ on $\partial\Omega$ we obtain in particular that

$$\sup_{\partial\Omega \cap U_R(x_0)} \psi \leq \sup_{\partial\Omega \setminus U_R(x_0)} \psi + (B + \beta)\sqrt{\mathrm{diam}(\Omega)} \tag{10.9}$$

This condition is a **necessary** coondition on the datum for the existence of a solution of the Dirichlet problem. So if we take $\Omega$ and $\psi$ such that condition (10.9) is not satisfied, for example such that

$$\psi \equiv 0 \qquad \text{on } \partial\Omega \setminus U_R(x_0)$$

$$\psi(x_0) > (\beta + B)\sqrt{\operatorname{diam}(\Omega)}$$

we obtain that the Dirichlet problem **cannot** has solution in $C^{1,0}(\Omega)$. $\square$

**Example of non existence of minimal surface 10.1.3**: now we want to give an explicit example of $\Omega$ and $\psi$ such that the Dirichlet problem has no solution in the space $C^{1,0}$.

In $\mathbb{R}^2$ consider the set

$$\Omega := \{x \in \mathbb{R}^2 \mid \rho < |x| < R\}$$

where $0 < \rho < R$, and the function, for $M > 0$,

$$\psi := \begin{cases} 0 & , \text{on } \partial U_R \\ M & , \text{on } \partial U_\rho \end{cases}$$

Since the function $\psi$ and the set $\Omega$ are symmetric with respect to the origin, if there exists a minimum $u \in C^{1,0}(\Omega)$, this minimum must be symmetric itself; so we can suppose that $u = u(|x|) = u(r)$. We want to derive the minimal surface equation for such a function $u$. Since

$$\frac{\partial u}{\partial x_i} = \frac{x_i}{|x|}u'$$

Hence the minimal surface equation becomes

$$\sum_{i=1}^{2} D_i\Big(\frac{\frac{x_i}{|x|}u'}{\sqrt{1+(u')^2}}\Big) = 0$$

Making the explicit computation we get

$$u'' + \frac{1}{r}u'[1+(u')^2] = 0$$

To integrate this function we observe that it is an Eulero equation, and so

$$u'(x) = (\sqrt{kx^2 - 1})^{-1} \qquad k \in \mathbb{R}$$

Hence, integrating and imposing the condition $u(R) = 0$, we obtain that

$$u(x) = c\log\Big(\frac{R + \sqrt{R^2 - c^2}}{r + \sqrt{r^2 - c^2}}\Big)$$

where the constant $c$, $0 < c \leq \rho$, is such that $u(\rho) = M$. We have that

$$u(\rho) = c \log \Big( \frac{R + \sqrt{R^2 - c^2}}{\rho + \sqrt{\rho^2 - c^2}} \Big) \leq \rho \log \Big( \frac{R + \sqrt{R^2 - \rho^2}}{\rho} \Big) = M_0(R, \rho)$$

Hence we can solve the Dirichlet problem only if $M \leq M_0$.

Moreover if $M > M_0$ the minimal surface, that always exists for Theorem 5.3.3, is compose by the graph of the function $u$ corresponding to the limit value $M_0$ and by the part of the cilynder having has base $\partial U_\rho$ that lies between $M_0$ and $M$.

This example is very important because it tells us that the Dirichlet problem is not always solvable. This fact will motivate the introduction of a weaker form of the Dirichlet problem, setted in $BV$, where we do not impose the function $u$ to have $\psi$ as trace on $\partial\Omega$, but we introduce a penalization to not take the value $\psi$ on $\partial\Omega$.

### 10.1.4 The a priori estimate for the gradient

Now we present, without proof, two important results concerning the solutions of the minimal surface equation: the a priori estimate of the gradient, and an existence theorem for the Dirichlet problem with continous data.

We need some notation: we denote with $B^R(x_0)$ a ball in $\mathbb{R}^{n+1}$, and with $S$ the subgraph of a function $u$ defined in $B_R(x_0)$. moreover we define $S_R(x_0) := S \cap B^R(x_0)$.

**Theorem 10.1.17.** *Let $u$ be a solution of the minimal surface equation in $B_R(x_0)$. Then there exists a constant $c > 0$ such that*

$$\sup_{S_{R/6}(x_0)} |Du| \leq \exp \Big\{ c \Big( 1 + \frac{\sup_{B_R(x_0)} u - u(x_0)}{R} \Big) \Big\}$$

Theorem above is an important result in the theory of non-parametric minimal surfaces. As a first consequence we have the following

**Theorem 10.1.18.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with $C^2$ boundary of non-negative mean curvature, and let $\psi$ be a continous function on $\partial\Omega$. Then the Dirichlet problem for the minimal surface equation has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

We will see how the a priori estimate for the gradient will be useful for the solution of the Bernstein problem.

## 10.2 Dirichlet problem in the $BV$ space

In this section we will see how to face the problem of the existence of non-parametric minimal surfaces using the direct methods in the calculus of variations in the space $BV$. This method will allow us to solve the problem in a more general context: in fact we will deal with $L^1$ functions on the boundary, and we will not need limitations on the curvature of our domain. We will prove that this weaker form of the Dirichlet problem has always a solution (Theorem 10.2.5). In Section 10.2.2 we will find an important connection, due to Miranda (see [Mir64b]), between parametric and non-parametric minimal surfaces. This connection is useful because it allows us to get immediately regularity results for non-parametric minimal surfaces from the regularity results of the parametric one.

### 10.2.1 Weak formulation of Dirichlet problem

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary; our aim is to apply the direct method to minimize the area functional

$$\mathcal{A}(u,\Omega) := \int_\Omega \sqrt{1+|Du|^2}\,\mathrm{d}x$$

among all the functions $u$ taking presribed values $\psi \in L^1(\partial\Omega)$ on $\partial\Omega$. As for the parametric case we will use the space $BV(\Omega)$, and we intend the boundary value $\psi$ as the trace of $u$ on $\partial\Omega$. Now we need to define what is the area functional for a function $u \in BV(\Omega)$, in a way that extends the usual definition. The proof of Theorem 10.1.1 gives us an idea of how do it:

**Definition 10.2.1.** *Let $u \in BV(\Omega)$, where $\Omega$ is a bounded open set of $\mathbb{R}^n$. We define*

$$\int_\Omega \sqrt{1+|Du|^2} := |(Du,\mathcal{L}^n)|(\Omega)$$

*For the Riesz Representation Theorem we obtain that this number is equal to*

$$\sup\left\{ \int_\Omega (\varphi_{n+1} + \langle u, D\varphi\rangle)\,\mathrm{d}x \mid \Phi = (\varphi,\varphi_{n+1}) \in C^1_c(\Omega;\mathbb{R}^{n+1}),\ |\Phi| \le 1 \right\}$$

**Remark 10.2.2.** *We note the following two facts*

- *if $\Omega$ is a bounded open set, then*

$$|Du|(\Omega) \le \int_\Omega \sqrt{1+|Du|^2} \le |Du|(\Omega) + \mathcal{L}^n(\Omega) \tag{10.10}$$

*Moreover, from the regularity of the measure $|(Du,\mathcal{L}^n)|$, we obtain that these inequalities hold for each Borel set $B \subset \Omega$.*

- *if* $u \in W^{1,1}(\Omega)$, *then*

$$\int_\Omega \sqrt{1+|Du|^2} = \int_\Omega \sqrt{1+|Du|^2}\,\mathrm{d}x$$

A theorem of semi-continuity holds

**Theorem 10.2.3.** *Let* $(u_j)_j \subset BV(\Omega)$ *a sequence converging in* $L^1_{loc}(\Omega)$ *to a function* $u$*. Then*

$$\int_\Omega \sqrt{1+|Du|^2} \leq \liminf_{j\to\infty} \int_\Omega \sqrt{1+|Du_j|^2}$$

*Proof.* Let $\Phi = (\varphi, \varphi_{n+1}) \in C^1_c(\Omega;\mathbb{R}^{n+1})$, $|\Phi| \leq 1$; then

$$\int_\Omega (\varphi_{n+1}+\langle u, D\varphi\rangle)\,\mathrm{d}x = \lim_{j\to\infty}\int_\Omega (\varphi_{n+1}+\langle u_j, Dg\rangle)\,\mathrm{d}x \leq \liminf_{j\to\infty}\int_\Omega \sqrt{1+|Du_j|^2}$$

□

Now, let $(u_j)_j \in BV(\Omega)$ be a minimizing sequence; from Remark 10.2.2 we easily get that the sequence is bounded in the space $BV(\Omega)$; hence for the compactness theorem (see Theorem 5.3.2) the sequence is relative compact in $L^1(\Omega)$. Hence there exists a function $u \in L^1(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$; moreover, from the theorem above, we have that $u \in BV(\Omega)$ and that $u$ minimize the integral $\int_\Omega \sqrt{1+|Df|^2}$. The problem is that we do not known if $u$ has $\psi$ as trace on $\partial\Omega$. Moreover, from the example of non-existence of solution 10.1.3 we cannot exepect that our problem has always a solution. So we need to relax the condition on the trace, without changing the value of the minimum. Next proposition suggest us a good weak formulation of the Dirichlet problem.

**Proposition 10.2.4.** *Let* $\Omega$ *an open bounded subset of* $\mathbb{R}^n$ *with boundary of class* $C^1$*, and let* $\psi \in L^1(\partial\Omega)$*. Then*

$$\begin{aligned} & \inf\{\, \mathcal{A}(u,\Omega) \mid u \in BV(\Omega),\ Tr(u) = \psi \text{ on } \partial\Omega \,\} \\ = \; & \inf\{\, \mathcal{A}(u,\Omega) + \int_{\partial\Omega} |Tr(u)-\psi|\,\mathrm{d}\mathcal{H}^{n-1} \mid u \in BV(\Omega) \,\} \end{aligned}$$

*Proof.* The inequality $\geq$ is clear; to prove the other one let $u \in BV(\Omega)$ and fix $\varepsilon > 0$. From Theorem 7.3.4 we have that there exists a function $w \in W^{1,1}(\Omega)$ such that

- $w = u - \psi$ on $\partial\Omega$
- $\displaystyle\int_\Omega |Dw|\,\mathrm{d}x \leq (1+\varepsilon)\int_{\partial\Omega} |Tr(u)-\psi|\,\mathrm{d}\mathcal{H}^{n-1}$

So, if we take the function $v := w + u$ we obtain that $v \in BV(\Omega)$ and $v = \psi$ on $\partial\Omega$. Moreover

$$\begin{aligned}\int_\Omega \sqrt{1+|Dv|^2} &\leq \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega |Dw|\,\mathrm{d}x \\ &\leq \int_\Omega \sqrt{1+|Du|^2} + (1+\varepsilon)\int_{\partial\Omega} |Tr(u)-\psi|\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

Since $\varepsilon$ is arbitrary we obtain the desired result. □

Hence we can give a weaker formulation of our problem as follows: let $\Omega$ be a bounded open set with Lipschitz boundary, and let $\psi \in L^1(\partial\Omega)$; find a function $u \in BV(\Omega)$ that minimize the integral

$$\mathcal{I}(v,\Omega) := \int_\Omega \sqrt{1+|Dv|^2} + \int_{\partial\Omega} |Tr(v)-\psi|\,\mathrm{d}\mathcal{H}^{n-1}$$

among all the functions $v \in BV(\Omega)$.

Thuis weak formulation is very good because we have the following existence result

**Theorem 10.2.5.** *Let $\Omega$ be a bounded open set with Lipschitz boundary, and let $\psi \in L^1(\partial\Omega)$. Then the functional $\mathcal{I}$ achives its minimum in $BV(\Omega)$.*

*Proof.* First of all we prove that the functional $\mathcal{I}$ is lower semicontinous with respect to the $L^1$ convergence.

Let $\mathcal{B}$ be a ball such that $\mathcal{B}\setminus\overline{\Omega}$ has Lipschitz boundary. From Theorem 7.3.4 there exists a function $\widetilde{\psi} \in W^{1,1}(\mathcal{B}\setminus\overline{\Omega})$ such that $Tr(\widetilde{\psi}) = \psi$ on $\partial\Omega$ and $Tr(\widetilde{\psi}) = 0$ in $\partial\mathcal{B}$. For $v \in BV(\Omega)$ define the function

$$v^\psi := \begin{cases} v & \text{, in } \Omega \\ \widetilde{\psi} & \text{, in } \mathcal{B}\setminus\Omega \end{cases}$$

Then $v^\psi \in BV(\mathcal{B})$. Moreover if $(u_j)_j \subset BV(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$, then $u_j^\psi \to u^\psi$ in $L^1(\mathcal{B})$. Hence, from Theorem 10.2.3 we have that

$$\int_{\mathcal{B}} \sqrt{1+|Du^\psi|^2} \leq \liminf_{j\to\infty} \int_{\mathcal{B}} \sqrt{1+|Du_j^\psi|^2}$$

From this ineqaulity we want to obtain an inequality for the functional $\mathcal{I}$. Since ${u_j}_{|\mathcal{B}\setminus\Omega} = u_{|\mathcal{B}\setminus\Omega}$ inequality above can be written as

$$\int_{\overline{\Omega}} \sqrt{1+|Du^\psi|^2} \leq \liminf_{j\to\infty} \int_{\overline{\Omega}} \sqrt{1+|Du_j^\psi|^2} \tag{10.11}$$

Moreover, since $\partial\Omega$ is a Borel set such that $\mathcal{L}^n(\partial\Omega) = 0$, from Remark 10.2.2, we have that

$$\int_{\partial\Omega} \sqrt{1 + |Du^\psi|} = |Du^\psi|(\partial\Omega) = \int_{\partial\Omega} |Tr(u) - \psi| \, \mathrm{d}\mathcal{H}^{n-1}$$

Hence inequality (10.11) can be written as

$$\int_\Omega \sqrt{1 + |Du|^2} + \int_{\partial\Omega} |u^+ - \varphi| \, \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_j \Big( \int_\Omega \sqrt{1 + |Du_j|^2} + \int_{\partial\Omega} |u_j^+ - \varphi| \, \mathrm{d}\mathcal{H}^{n-1} \Big)$$

So let $(u_j)_j$ be a minimizing sequence for the functional $\mathcal{I}$; then from (10.10) we obtain that $\sup_j |Du_j|(\Omega) < \infty$; hence we can apply the compactness Theorem 5.3.2 to obtain the existence of a subsequence $(u_{j_k})_k$ and a function $u \in BV(\Omega)$ such that $u_{j_k} \to u$ in $L^1(\Omega)$. From the semi-continuity of $\mathcal{I}$ with respect to the $L^1$ convergence we obtain that the function $u$ minimize the functional $\mathcal{I}$. □

**Remark 10.2.6.** *The above theorem tells us that the weak formulation of the Dirichlet problem has always a solution, without requirement on the curvature of $\partial\Omega$. In particular, if we consider the problem in Example 10.1.3 with datum $M > M_0$, we obtain that the minimum of the functional $\mathcal{I}$ is take for the function $u$ corresponding to the limit value $M_0$. In particular we see that if $u$ is the minimum of the functional $\mathcal{I}$ it is not necessary that $u^+ = \psi$ on $\partial\Omega$.*

*Moreover if we consider the following Dirichlet problem*

$$\Omega := U_R \setminus (\partial U_\rho \cup U_\varepsilon)$$

$0 < \varepsilon < \rho < R$, *and the function*

$$\psi = \begin{cases} 0 & \text{, on } \partial U_R \\ M & \text{, on } \partial U_\rho \cup \partial U_\varepsilon \end{cases}$$

*with $M > M_0$, we find that the functional $\mathcal{I}$ is minimized by the function*

$$u(x) := \begin{cases} c \log\Big(\dfrac{R + \sqrt{R^2 - c^2}}{r + \sqrt{r^2 - c^2}}\Big) & \text{, in } U_R \setminus B_\rho \\ M & \text{, in } U_\rho \setminus B_\varepsilon \end{cases}$$

*where $c$ is such that $u(\rho) = M_0$. So we see that is foundamental that the function $u$ belongs to the space $BV(\Omega)$, instead of belonging to the space $W^{1,1}$ or $C^{0,1}$, because we need to allow the function $u$ to "jump" on a set of Lebesgue measure 0, in order to get the minimum of the functional $\mathcal{I}$.*

### 10.2.2 Connection between parametric and non-parametric surfaces

Now we have the problem of the regularity of non-parametric minimal surface. To solve it we want to connect parametric minimal surfaces with non parametric minimal surfaces, and hence using the regularity result of the previous chapter to get regularity theorems for our present case.
The idea is to prove that if a function $u$ minimize the area integral in $\Omega$, then its subgraph minimize the perimeter in $Q := \Omega \times \mathbb{R}$.

**Theorem 10.2.7.** *Let $u \in BV(\Omega)$ and let*

$$U := \{ (x,t) \in \Omega \times \mathbb{R} \mid t < u(x) \}$$

*Then*

$$\int_\omega \sqrt{1+|Du|^2} = |\partial U|(\Omega \times \mathbb{R})$$

*Proof.* First of all we note that the formula holds for $C^1$ functions, since each term represents the area of the graph of $u$. So, let $(u_j)_j \subset BV(\Omega) \cap C^\infty(\Omega)$ such that

$$u_j \to u \text{ in } L^1(\Omega)$$

and

$$\int_\Omega \sqrt{1+|Du_j|^2} \to \int_\Omega \sqrt{1+|Du|^2}$$

This can be done using the approximation sequence of the Anzellotti-Giaquinta Theorem. From the first condition we get

$$U_j \to U$$

where $U_j$ is the subgraph of the function $u_j$. Hence

$$|\partial U|(\Omega\times\mathbb{R}) \leq \liminf_{j\to\infty} |\partial U_j|(\Omega\times\mathbb{R}) = \lim_{j\to\infty} \int_\Omega \sqrt{1+|Du_j|^2} = \int_\Omega \sqrt{1+|Du|^2}$$

To prove the other inequality we will prove that, for any $\Phi = (\varphi_1, \cdots, \varphi_{n+1}) \in C^1_c(\Omega; \mathbb{R}^{n+1})$ with $|\Phi| \leq 1$ it holds

$$|\partial U|(\Omega \times \mathbb{R}) \geq \int_\Omega \left[ u \sum_{i=1}^n D_i\varphi_i + \varphi_{n+1} \right] \mathrm{d}x \tag{10.12}$$

So let $\eta : \mathbb{R} \to \mathbb{R}$ be a $C^\infty_c(\mathbb{R})$ function such that

$$\eta(x) = \eta(-x) \ \ \forall\, x \in \mathbb{R}, \quad \eta \equiv 1 \text{ in } [-1,1], \quad \operatorname{supp}(\eta) \subset [-2,2]$$

$$0 \leq \eta \leq 1, \qquad |\eta'| \leq 1$$

Then, for each $h \in \mathbb{N} \setminus \{0\}$ define the function

$$\eta_h(x) := \begin{cases} \eta\Big(\dfrac{x}{h}\Big) & , |x| \le h \\ \eta(x + h - 1) & , x < -h \\ \eta(x - h + 1) & , x > h \end{cases}$$

In particuolar we obtain that

$$\int_{-h-1}^{-h} \eta_h(x)\,\mathrm{d}x = \int_h^{h+1} \eta_h(x)\,\mathrm{d}x$$

is indipendent from $h$; hence we denote it by $c$. Moreover we note that

$$\int_{-\infty}^{u(x)} \eta_h(x)\,\mathrm{d}x = \begin{cases} c + u(x) + h & , \text{if } |u(x)| \le h \\ c + 2h + \displaystyle\int_h^{u(x)} \eta(x)\,\mathrm{d}x & , \text{if } u(x) > h \\ \displaystyle\int_{-h-1}^{u(x)} \eta(x)\,\mathrm{d}x & , \text{if } u(x) < -h \end{cases}$$

and that

$$\int_{-\infty}^{u(x)} \eta_h^{'}(x)\,\mathrm{d}x = \begin{cases} 1 & , \text{if } |u(x)| \le h \\ \eta_h(u(x)) & , \text{if } |u(x)| > h \end{cases}$$

Now fix $\Phi = (\varphi_1, \cdots, \varphi_{n+1}) \in C_c^1(\Omega; \mathbb{R}^{n+1})$ with $|\Phi| \le 1$, and define the function $\gamma_h : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ as

$$\gamma_h(x, x_{n+1}) := \Phi(x)\eta_h(x_{n+1})$$

Then $\gamma_h \in C_c^1(\Omega \times \mathbb{R}; \mathbb{R}^{n+1})$ and $|\gamma_h| \le 1$. Hence

$$\begin{aligned} |\partial U|(\Omega \times \mathbb{R}) &\ge \int_U \operatorname{div}(\gamma_h)\,\mathrm{d}x\,\mathrm{d}x_{n+1} = \int_U \sum_{i=1}^{n+1} D_i \gamma_h^i\,\mathrm{d}x\,\mathrm{d}x_{n+1} \\ &= \int_U \mathrm{d}x \int_{-\infty}^{u(x)} \Big[\varphi_{n+1}(x)\eta'(x_{n+1}) + \eta(x_{n+1}) \sum_{i=1}^{n} D_i\varphi_i(x)\Big]\,\mathrm{d}x_{n+1} \\ &= \int_\Omega \Big[\eta_h(u(x))\varphi_{n+1}(x) + \sum_{i=1}^{n} D_i\varphi_i(x) \int_{-\infty}^{u(x)} \eta_h(x_{n+1})\,\mathrm{d}x_{n+1}\Big]\,\mathrm{d}x \end{aligned} \tag{10.13}$$

Observ now that

$$\begin{aligned} \int_\Omega \eta_h(u(x))\varphi_{n+1}(x)\,\mathrm{d}x &= \int_{\{|u| \le h\}} \eta_h(u(x))\varphi_{n+1}(x)\,\mathrm{d}x + \int_{\{|u| > h\}} \eta_h(u(x))\varphi_{n+1}(x)\,\mathrm{d}x \\ &= \int_\Omega \varphi_{n+1}(x)\,\mathrm{d}x + \int_{\{|u| > h\}} (\eta_h(u(x)) - 1)\varphi_{n+1}(x)\,\mathrm{d}x \end{aligned} \tag{10.14}$$

and

$$
\begin{aligned}
&\int_\Omega \left(\sum_{i=1}^n D_i\varphi_i(x)\int_{-\infty}^{u(x)} \eta_h(t)\,\mathrm{d}t\right)\mathrm{d}x = \int_{\{|u|\le h\}}\left[\sum_{i=1}^n D_i\varphi_i(x)(c+u(x)+h)\right]\mathrm{d}x\\
&\quad+\int_{\{|u|>h\}}\left[\sum_{i=1}^n D_i\varphi_i(x)\left(c+2h+\int_h^{u(x)}\eta_h(t)\,\mathrm{d}t\right)\right]\mathrm{d}x\\
&\quad+\int_{\{u<-h\}}\left[\sum_{i=1}^n D_i\varphi_i(x)\int_{-h-1}^{u(x)}\eta_h(t)\,\mathrm{d}t\right]\\
=&\quad+\int_\Omega\left[\sum_{i=1}^n D_i\varphi_i(x)(c+u(x)+h)\right]\mathrm{d}x\\
&\quad+\int_{\{|u|>h\}}\left[\sum_{i=1}^n D_i\varphi_i(x)\left(h-u(x)+\int_h^{u(x)}\eta_h(t)\,\mathrm{d}t\right)\right]\mathrm{d}x\\
&\quad+\int_{\{u<-h\}}\left[\sum_{i=1}^n D_i\varphi_i(x)\left(\int_{-h-1}^{u(x)}\eta_h(t)\,\mathrm{d}t-c-u(x)-h\right)\right]
\end{aligned}
\tag{10.15}
$$

From (10.13), (10.14), (10.15) we obtain

$$
\begin{aligned}
&|\partial U|(\Omega\times\mathbb{R}) \ge \int_\Omega\left[\varphi_{n+1}(x)+\sum_{i=1}^n D_i\varphi_i(x)(c+u(x)+h)\right]\mathrm{d}x\\
&\quad+\int_{\{u>h\}}\left[(\eta_h(u(x))-1)\varphi_{n+1}(x)+\sum_{i=1}^n D_i\varphi_i(x)\left(\int_h^{u(x)}\eta_h(t)\,\mathrm{d}t+h-u(x)\right)\right]\mathrm{d}x\\
&\quad+\int_{\{u<-h\}}\left[(\eta_{n+1}(u(x))-1)\varphi_{n+1}(x)+\sum_{i=1}^n D_i\varphi_i(x)\left(\int_{-h-1}^{u(x)}\eta_h(t)\,\mathrm{d}t-c-u(x)-h\right)\right]\mathrm{d}x\\
=&\quad R_h+S_h+T_h
\end{aligned}
\tag{10.16}
$$

Since $\varphi_i \in C_c^\infty(\Omega)$ for all $i=1,\dots,n$ we have that

$$
R_h = \int_\Omega\left[\varphi_{n+1}(x)+\sum_{i=1}^n D_i\varphi_i(x)u(x)\right]\mathrm{d}x \qquad \text{for all } h
$$

Now we want to prove that

$$
\lim_{h\to\infty} S_h = \lim_{h\to\infty} T_h = 0
$$

If $u(x) > h$ the, since $\mathrm{supp}(\eta_h)\subset[-h-1,h+1]$ and $0\le\eta_h\le 1$ we have that

$$
\left|\int_h^{u(x)}\eta_h(t)\,\mathrm{d}t+h-u(x)\right| \le |u(x)-h|+\left|\int_h^{u(x)}\eta_h(t)\,\mathrm{d}t\right| \le |u(x)|+1 \le 2|u(x)|
$$

Then there exists a positive constant $c = c(\Omega, \Phi)$ such that

$$|S_h| \leq c \int_{\{u>h\}} |u(x)| \,\mathrm{d}x$$

Since $u \in L^1(\Omega)$ it follows that $\lim_{h\to\infty} S_h = 0$. A similar argument gives $\lim_{h\to\infty} T_h = 0$. Hence we have obtained the desired result. $\square$

Now we want to prove that, given a measurable set $F$ we can find a function $w$ whom area is less than the perimeter of $F$. We will do it in the following two results.

**Lemma 10.2.8.** *Let $F \subset Q := \Omega \times \mathbb{R}$ be a measurable set. Suppose that there exists a $T > 0$ such that*

$$\Omega \times (-\infty, -T) \subset F \subset \Omega \times (-\infty, T)$$

*For $x \in \Omega$ define the function*

$$w(x) := \lim_{k\to\infty} \Big[ \int_{-k}^{k} \chi_F(x,t) \,\mathrm{d}t - k \Big]$$

*Then*

$$\int_\Omega \sqrt{1 + |Dw|^2} \leq |\partial F|(Q)$$

*Proof.* We note that $\partial F \cap Q \subset \Omega \times [-T, T]$. For each $k$ set

$$w_k(x) := \int_{-k}^{k} \chi_F(x,t) \,\mathrm{d}t - k$$

for $x \in \Omega$. Then, for $k, h \geq T$ we obtain that $w_k = w_h$, and hence $w(x) = \int_{-T}^{T} \chi_F(x,t) \,\mathrm{d}t$. Hence $w$ is a bounded measurable function, in particular

$$-T \leq w(x) \leq T$$

Now let $\Phi \in C_c^1(\Omega; \mathbb{R}^{n+1})$, $|\Phi| \leq 1$, and let $\eta : \mathbb{R} \to \mathbb{R}$ a smooth function such that

$$0 \leq \eta \leq 1$$

$$\eta(t) = 0 \text{ if } |t| \geq T + 1\,, \qquad \eta(t) = 1 \text{ if } |t| \leq T$$

We have that

$$\int_{-\infty}^{\infty} \eta'(t) \chi_F(x,t) \,\mathrm{d}t = 1$$

for each $x \in \Omega$, and

$$\begin{aligned}\int_{-\infty}^{\infty} \eta(t)\chi_F(x,t)\,\mathrm{d}t &= \int_{-T-1}^{-T} \eta(t)\,\mathrm{d}t + \int_{-T}^{T} \chi_F(x,t)\,\mathrm{d}t \\ &= \int_{-T-1} \eta(t)\,\mathrm{d}t + w(x) + T = w(x) + \alpha\end{aligned}$$

where $\alpha \geq 0$. Then, if we set $\gamma(x, x_{n+1}) := \Phi(x)\eta(x_{n+1})$, we have that $\gamma \in C_c^1(Q\ \mathbb{R}^{n+1})$ and $|\gamma| \leq 1$. Hence

$$\begin{aligned}|\partial F|(Q) &\geq \int_Q \chi_F(x, x_{n+1}) \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i}[\eta(x_{n+1})\varphi_i(x)]\,\mathrm{d}x\,\mathrm{d}x_{n+1} \\ &= \int_\Omega \Big[ (w+\alpha) \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_i} + \varphi_{n+1} \Big]\,\mathrm{d}x \\ &\geq \int_\Omega \Big( \varphi_{n+1} + w \sum_{i=1}^{n} D_i\varphi_i \Big)\,\mathrm{d}x\end{aligned}$$

Now, taking the supremum over all $\Phi$, we obtain the desired result. $\square$

Now we want to remove the assumption that $\partial\Omega \cap Q$ is bounded.

**Theorem 10.2.9.** *Let $F$ be a measurable set in $Q := \Omega \times \mathbb{R}$, where $\Omega$ is a bounded open set with Lipschitz boundary. Suppose that, for a.e. $x \in \Omega$ it hold*

1. $\lim_{t\to+\infty} \chi_F(x,t) = 0\,, \qquad \lim_{t\to-\infty} \chi_F(x,t) = 1$
2. *the set $F_0 := F \triangle Q^-$, where $Q^- := \{\,(x,t) \in Q \mid t < 0\,\}$, is such that $\mathcal{L}^{n+1}(F_0) < \infty$*

*Then the function*

$$w(x) := \lim_{k\to\infty} \Big[ \int_{-k}^{k} \chi_F(x,t)\,\mathrm{d}t - k \Big]$$

*belongs to $L^1(\Omega)$ and*

$$\int_\Omega \sqrt{1 + |Dw|^2} \leq |\partial F|(\Omega \times \mathbb{R})$$

*Proof. Step 1*: for each $k \in \mathbb{N}$ define the function

$$w_k(x) := \int_{-k}^{k} \chi_F(x,t)\,\mathrm{d}t - k$$

Then from hypothesis (1) it follows that for almost every $x \in \Omega$

$$k(x) := \inf \Big\{ s > 0 \mid \chi_F(x,t) = 0\,, \forall\, t > s\,, \chi_F(x,t) = 1\,, \forall\, t < -s \Big\}$$

Hence

$$w_k(x) = \int_{-k(x)}^{k(x)} \chi_F(x,t)\,\mathrm{d}t - k(x) \quad \text{for each } k \geq k(x)$$

and so the function $w$ is well-defined. Moreover it holds

$$w_k(x) \to w(x)$$

for each $x \in \Omega$.

*Step 2*: for each $x \in \Omega$ define

$$M_x := \{t \in \mathbb{R} \mid (x,t) \in F_0\} \subset \mathbb{R}$$

and consider the function $g : \Omega \to \mathbb{R}$ defined as

$$g(x) := \mathcal{L}^1(M_x) = \int_{\mathbb{R}} \chi_{F_0}(x,t)\,\mathrm{d}t$$

From the Fubini's Theorem we obtain that $g$ is $\mathcal{L}^n$-measurable, and that

$$\int_\Omega |g(x)|\,\mathrm{d}x = \int_\Omega \mathcal{L}^1(M_x)\,\mathrm{d}x = \int_\Omega \mathrm{d}x \int_{\mathbb{R}} \chi_{F_0}(x,t)\,\mathrm{d}t = \mathcal{L}^{n+1}(F_0) < \infty$$

Hence $g \in L^1(\Omega)$. Moreover

$$\begin{aligned}
|w_k(x)| &= \Big| \int_{-k}^{k} \chi_F(x,t)\,\mathrm{d}t - k \Big| = \Big| \int_{-k}^{0} (\chi_F(x,t) - 1)\,\mathrm{d}t + \int_0^k \chi_F(x,t)\,\mathrm{d}t \Big| \\
&\leq \int_{-k}^{0} |\chi_F(x,t) - \chi_{Q^-}(x,t)|\,\mathrm{d}t + \int_0^k |\chi_F(x,t) - \chi_{Q^-}(x,t)|\,\mathrm{d}t \\
&= \int_{-k}^{k} \chi_{F_0}(x,t)\,\mathrm{d}t \leq g(x)
\end{aligned}$$

That is $|w_k| \leq |g|$ in $\Omega$. Hence from the Lebesgue's dominate convergence Theorem it follows that $w_k \to w$ in $L^1(\Omega)$.

*Step 3*: for $k \in \mathbb{N}$ consider the set

$$F_k := F \cup [\Omega \times (-\infty, -k)] \setminus [\Omega \times (k, \infty)]$$

From the previous lemma it follows that

$$\int_\Omega \sqrt{1 + |Dw_k|^2} \leq |\partial F_k|(Q)$$

Hence, takin into account that $\Omega$ has Lipschitz boundary, and the definition of $F_k$, we have that

$$\begin{aligned}\int_\Omega \sqrt{1+|Dw_k|^2} &\leq |\partial F_k|(Q)\\ &= |\partial F_k|(\,(\Omega\times(-\infty,k))\cap\partial F_k\,)+|\partial F_k|(\,(\Omega\times[k,\infty])\cap\partial F_k\,)\\ &\quad +|\partial F_k|(Q\setminus[(\Omega\times(-\infty,k))\cup(\Omega\times[k,\infty])])\\ &= |\partial F_k|(\Omega\times\{-k\})+|\partial F_k|(\Omega\times\{k\})\\ &\quad +|\partial F|(Q\setminus[(\Omega\times(-\infty,k))\cup(\Omega\times[k,\infty])])\\ &\leq |\partial F|(Q)+\int_{\Omega\times\{-k\}}(1-\chi_F)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\Omega\times\{k\}}\chi_F\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

Hence, letting $k\to\infty$, we obtain that

$$\begin{aligned}\int_\Omega\sqrt{1+|Dw|^2} &\leq \lim_{k\to\infty}\int_\Omega\sqrt{1+|Dw_k|^2}\\ &\leq \lim_{k\to\infty}\left[|\partial F|(Q)+\int_{\Omega\times\{-k\}}(1-\chi_F)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\Omega\times\{k\}}\chi_F\,\mathrm{d}\mathcal{H}^{n-1}\right]\\ &= |\partial F|(Q)\end{aligned}$$

where in the first step we have use the fact that $w_k\to w$ and then Theorem 10.2.3, and in the last step we have used hypothesis (1). □

Now we can connect parametric and non-parametric surfaces

**Theorem 10.2.10.** *Let $u\in BV_{loc}(\Omega)$ be a local minimum of the area. Then the set*

$$U := \{\,(x,t)\in\Omega\times\mathbb{R}\mid t<u(x)\,\}$$

*minimizes locally the perimeter in $\Omega\times\mathbb{R}$.*

*Proof.* Let $A\Subset\Omega$ e $F$ be a Caccioppoli set in $Q$ coinciding with $U$ outside a compact set $K\subset A\times\mathbb{R}$. We want to apply the previous theorem to the set $F$; hence we need to prove that $F$ satisfied the required hypothesis. First of all we prove that $U$ satisfied the hypothesis of Theorem 10.2.9:

1. since $u\in L^1(\Omega)$ we have that, up to a set of measure 0, $|u(x)|<\infty$ for each $x\in\Omega$, and hence

$$\lim_{t\to\infty}\chi_U(x,t)=0\,,\qquad \lim_{t\to-\infty}\chi_U(x,t)=1$$

for each $x\in\Omega$.

2. $\mathcal{L}^{n+1}(U_0)=\int_\Omega|u|\,\mathrm{d}x<\infty$ since $u\in L^1(\Omega)$

Now we can pprove that $F$ satisfied the hypothesis

1. since $K$ is compact there exists a $T > 0$ such that $K \subset \Omega \times [-T, T]$; hence, since $F \equiv U$ outside $K$ we have that
$$\lim_{t\to\infty} \chi_F(x,t) = \lim_{t\to\infty} \chi_U(x,t) = 0$$
$$\lim_{t\to-\infty} \chi_F(x,t) = \lim_{t\to-\infty} \chi_U(x,t) = 1$$
for each $x \in \Omega$

2. $\mathcal{L}^{n+1}(F_0) \leq \mathcal{L}^{n+1}(K) + \mathcal{L}^{n+1}(U_0) < \infty$

Then we can apply Theorem 10.2.9 obtaining a function $w$ such that
$$\int_\Omega \sqrt{1 + |Dw|^2} \leq |\partial F|(\Omega \times \mathbb{R})$$
Since the function $w$ defined coincide with $u$ outside $A$, we have that
$$|\partial U|(A \times \mathbb{R}) = \int_A \sqrt{1 + |Du|^2} \leq \int_A \sqrt{1 + |Dw|^2} \leq |\partial F|(A \times \mathbb{R})$$
So we have obtained the desired result. □

Now that we have connect the non parametric minimal surfaces with the parametric minimal surfaces, we can use the regularity results of the previous chapter to get regularity results for our case. We will state the results without proof.

**Theorem 10.2.11.** *Let $u \in BV_{loc}(\Omega)$, where $\Omega$ is a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary, a function that minimize*
$$\int_\Omega \sqrt{1 + |Dv|^2}$$
*among all the function $v \in BV_{loc}(\Omega)$ having trace $\psi$ on $\partial\Omega$, where $\psi \in L^1(\partial\Omega)$ is a fixed funcion. Then $u$ is Lipschitz continous, and hence analitic, in $\Omega$.*

For the boundary regularity it holds

**Theorem 10.2.12.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary, and let $u$ be a minimum of the functional*
$$\mathcal{I}(v, \Omega) := \int_\Omega \sqrt{1 + |Dv|^2} + \int_{\partial\Omega} |Tr(u) - \psi| \, \mathrm{d}\mathcal{H}^{n-1}$$
*Suppose that $\partial\Omega$ has non-negative mean curvature near a point $x_0$, and that $\psi$ is continous at $x_0$. Then*
$$\lim_{x\to x_0} u(x) = \psi(x_0)$$

## 10.3 Quasi-solutions

Non-parametric minimal surfaces are not so good if we pass to the limit of a sequence: in fact in we have a sequence of hyperplanes converging to a vertical hyperplanes. The problem of hyperplanes is that they are not graph, and hence we can not use none of the results of the previous section, when we deal with them. But Theorem 10.2.10 gives us a method to extend the notion of a non-parametric minimal surface, just requiring that the subgraph minimize the perimeter. This idea lead to the definition of *quasi-solutions*, that clearly extend the notion of non-parametric solutions. In this section we will show two important properties of quasi-solution: they have a good behaviour when we pass to the limit of a sequence (Proposition 10.3.5), and if they not take the value $+\infty$ then they are locally bounded above (Proposition 10.3.8).

**Definition 10.3.1.** *Let $u : \Omega \to [-\infty, +\infty]$ be a measurable function. We say that $u$ is a **quasi-solution** of the minimal surface equation in $\Omega$ if its subgraph locally minimize the primeter in $\Omega \times \mathbb{R}$.*

We note, thanks to the results of the previous section, that every non-parametric minimal surface is a quasi-solution. Moreover a result similar to Proposition 9.2.8 holds.

**Proposition 10.3.2.** *Let $E \subset \Omega$ be a measurable set. Define the function*

$$u(x) := \begin{cases} +\infty & , x \in E \\ -\infty & , x \notin E \end{cases}$$

*Then $u$ is a quasi-solution in $\Omega$ if and only if $E$ has least perimeter in $\Omega$.*

*Proof.* First suppose that $E$ has least perimeter in $\Omega$. Let $V$ be a Caccioppoli set coinciding with $U := E \times \mathbb{R}$ outside a compact set $K \subset \Omega \times \mathbb{R}$. Let $A \Subset \Omega$ and $T > 0$ such that

$$K \subset A_T := A \times (-T, T)$$

For $-T < t < T$ set

$$V_t := \{ x \in \Omega \mid (x, t) \in V \}$$

We have that $V_t = E$ outside $A$, and hence, from the minimality of $E$, we get

$$|\partial E|(A) \leq |\partial V_t|(A)$$

Hence, since $\chi_U$ is indipendent from the last coordinate, we have

$$|\partial U|(A_T) = \int_{-T}^{T} \mathrm{d}t \int_A \mathrm{d}|\partial E| \leq \int_{-T}^{T} \mathrm{d}t \int_A \mathrm{d}|\partial V_t| \leq |\partial V|(A_T)$$

Then $U$ has least perimeter in $\Omega\times\mathbb{R}$; since $U$ is the subgraph of $u$ we obtain that $u$ is a quasi-solution in $\Omega$.

Now suppose that $u$ is a quasi-solution in $\Omega$, and suppose for absurd that $E$ has not least perimeter in $\Omega\times\mathbb{R}$. Then there exists a compact set $K\subset\Omega$, $\delta>0$ and a Caccioppoli set $F$ coinciding with $E$ outside $K$ such that

$$|\partial F|(K)\leq|\partial E|(K)-\delta$$

We can suppose that $K$ is smooth. For $T>0$ define

$$F_T:=\begin{cases} F\times\mathbb{R} & ,\text{in } K_T:=K\times[-T,T] \\ E\times\mathbb{R} & ,\text{outside } K_T \end{cases}$$

Hence

$$\begin{aligned}
|\partial F_T|(K_T) &= |\partial F_T|(K\times(-T,T))+|\partial F_T|(K\times\{-T\}\cup K\times\{T\}) \\
&= |\partial(F\times\mathbb{R})|(K\times(-T,T))+\int_{K\times\{-T\}}|\chi^+_{F_T}-\chi^-_{F_T}|\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad +\int_{K\times\{T\}}|\chi^+_{F_T}-\chi^-_{F_T}|\,\mathrm{d}\mathcal{H}^{n-1} \\
&\leq |\partial(F\times\mathbb{R})|(K\times(-T,T))+2\mathcal{L}^n(K)=\int_{-T}^{T}|\partial F|(K)\,\mathrm{d}t+2\mathcal{L}^n(K) \\
&\leq \int_{-T}^{T}[\,|\partial E|(K)-\delta\,]\,\mathrm{d}t+2\mathcal{L}^n(K) \\
&= \int_{-T}^{T}|\partial E|(K)\,\mathrm{d}t-2T\delta+2\mathcal{L}^n(K) \\
&= |\partial(E\times\mathbb{R})|(K\times(-T,T))-2T\delta+2\mathcal{L}^n(K) \\
&\leq |\partial(E\times\mathbb{R})|(K_T)-2T\delta+2\mathcal{L}^n(K)
\end{aligned}$$

So, if we take $T\delta>\mathcal{L}^n(K)$ we obtain a contraddition to the minimality of the subgraph of $u$ (that is $E\times\mathbb{R}$) in $\Omega\times\mathbb{R}$. □

**Definition 10.3.3.** *Let $u$ be a qusi-solution in $\Omega$; we define the sets*

$$P:=\{\,x\in\Omega\mid u(x)=+\infty\,\},\quad N:=\{\,x\in\Omega\mid u(x)=-\infty\,\}$$

Quasi-solutions allow to get existence results for the Dirichlet problem in unbounded domains of infinite measure or in bounded domains with infinite data. We are not intersted in it; we will only prove some results useful for the solution of the Bernstein problem.

**Lemma 10.3.4.** *Let $(u_k)_k$ be a sequence of measurable functions in $\Omega$, and let $U_k$ be the subgraph of $u_k$. Suppose $U_k \to U$ in $Q := \Omega \times \mathbb{R}$. Then $U$ is a subgraph of a measurable function $u := \Omega \to [-\infty, +\infty]$, and there exists a subsequence of $(u_k)_k$ that converges almost everywhere to $u$.*

*Proof.* Let $x \in \Omega$ and $V \subset Q$. Define

$$V^x := \{\, t \in \mathbb{R} \mid (x,t) \in V \,\}$$

Since $\chi_{U_k} \to \chi_U$ in $L^1_{loc}(Q)$, for every compact set $K \subset \Omega$ and every $T > 0$ we have that

$$\lim_{k\to\infty} \int_K \mathrm{d}x \int_{-T}^{T} |\chi_{U_k^x} - \chi_{U^x}| \,\mathrm{d}t = 0$$

Hence, possibly passing to a subsequence, we have that

$$\lim_{k\to\infty} \int_{-T}^{T} |\chi_{U_k^x} - \chi_{U^x}| \,\mathrm{d}t = 0$$

for each $T > 0$ and almost every $x \in K$, that is $U_k^x \to U^x$ for almost every $x \in \Omega$. Since $U_k^x = (-\infty, u_k(x))$, the set $U^x$ must be an half line (possibly $\emptyset$ or $\mathbb{R}$) for almost every $x \in \Omega$. So if we define

$$u(x) := \sup U^x$$

we have the desired result. □

Now we state a compactness result for quasi-solutions.

**Proposition 10.3.5.** *Every sequence of quasi-solutions $(u_k)_k$ in $\Omega$ has a subsequence converging almost everywhere to a quasi-solution.*

*Proof.* Let $K \subset Q := \Omega \times \mathbb{R}$ be a compact set. We can suppose that $K$ has smooth boundary. Let $U_k$ be the subgraph of $u_k$; hence from the minimality of $U_j$ in $Q$ we get

$$|\partial U_j|(K) \le |\partial(U_j \setminus K)|(K) \le \mathcal{H}^{n-1}(\partial K)$$

Hence there exists a subsequence of $(\chi_{U_k})_k$, still denoted with $(\chi_{U_k})_k$, and a function $u \in L^1(K)$ such that $\chi_{U_k} \to u$ in $L^1(K)$; moreover we can suppose that $u$ is the characteristic function of some set in $Q$. Covering $Q$ with compact sets and using a diagonal procedure we can select a subsequence $(U_k)_k$ converging to a set $U$ in $L^1_{loc}(Q)$; from the above lemma we obtain that $U$ is a subgraph of a measurable function $u$, and that $u_k \to u$ almost everywhere; hence $u_k \to u$ in $L^1(\Omega)$. Finally, from Lemma 9.2.1 we obtain that $U$ is a minimal set in $Q$, and hence $u$ is a quasi-solution. □

Finally we want to prove two results concerning a quasi-solution $u$ and the sets $P$ and $N$. Since if $u$ is a quasi-solution, then also $-u$ is a quasi-solution and the sets $P$ and $N$ are interchange, we will only prove the results for $P$.

**Theorem 10.3.6.** *Let $u$ be a qusi-solution in $\Omega$. Then $P$ has locally least perimeter in $Q := \Omega \times \mathbb{R}$.*

*Proof.* For $j$ define the functions

$$u_j(x) := u(x) - j$$

Obviously the functions $u_j$ are quasi-solutions in $\Omega$. For $j \to \infty$ the sequence $u_j$ converges almost everywhere to the function

$$v(x) := \begin{cases} +\infty & , x \in P \\ -\infty & , x \notin P \end{cases}$$

From the previous proposition we have that $v$ is a quasi-solution, and hence, from Proposition 10.3.2 we obtain that $P$ minimize the perimeter in $Q$. □

**Remark 10.3.7.** *Since $P$ is minimal we have that there exists a constant $c > 0$ such that for every $x \in \Omega$ and every $0 < R < d(x, \partial\Omega)$*

$$\mathcal{L}^n(P \cap B_R(x)) > cR^n$$

*This imply that if $A \subset \Omega$ is an open set such that $\mathcal{L}^n(P \cap A) = 0$, then $P \cap A = \emptyset$. Moreover if $P \neq \emptyset$, then*

$$\mathcal{L}^n(P) > c\delta^n$$

*where $\delta := \sup_{x \in P} d(x, \partial\Omega)$.*

**Proposition 10.3.8.** *Let $u$ be a quasi-solution in $\Omega$ and let $P = \emptyset$. Then $u$ is locally bounded above in $\Omega$.*

*Proof.* Suppose the thesis is not true. Then there exists a compact set $K \subset \Omega$ and sequence $(x_j)_j \subset K$ converging to a point $x_0 \in K$ such that

$$u(x_j) > j$$

Let $2R < d(x_0, \partial\Omega)$, and suppose that $|x_j - x_0| < R$ for each $j$. Let $U_j$ be the subgraph of the funcion $u_j(x) := u(x) - j$, that is a quasi-solution. Then $u(x_j) > 0$ for each $j$, and hence the point $z_j := (x_j, 0) \in U_j$. Since $U_j$ is a minimal set in $Q$ we have that

$$\mathcal{L}^n(U_j \cap B_R(z_j)) > cR^{n+1}$$

and hence

$$\mathcal{L}^n(U_j \cap B_{2R}(z_0)) > cR^{n+1} \tag{10.17}$$

Since $U_j \to P \times \mathbb{R}$, from (10.17) we obtain that $\mathcal{L}^n(P \times \mathbb{R}) > 0$, and hence $P$ is not empty. Absurd. □

# Chapter 11

# The Bernstein Problem in $\mathbb{R}^n$

In this chapter we will solve the Bernstein Problem in the Euclidean case, showing its validity in dimension $n \leq 7$ (Theorem 11.0.15). The fact that the theorem is false in hygher dimensions pass throught a counterexample due to De Giorgi, Giusti e Bombieri. Since the calculations under this counterexample are very hard, we will only state that Bernstein Theorem is false in dimension higher than 7 (Theorem 11.0.16).

In the introduction we proved the Bernstein Theorem in dimension $n = 2$ with a technique a hoc for this dimension. An idea suitable for all dimensions was given by Fleming: if we have a minimal set $U$ in $\mathbb{R}^n$ and we blow-in it, what we obtain will be an half-space if we are in dimension $n \leq 8$ (because we have proved that no singular minimal cones exist in these dimensions). So, using the estimate for minimal sets proved in Section 8.2, we will find out that also $U$ must be a cone, and hence an half-space (Theorem 11.0.9). We will apply this idea in Theorem 11.0.15 when $U$ is the subgraph of a function $u$. First of all we will prove in Proposition 11.0.12, using the calibration method, that if $u$ satisfied the minimal surface equation, then its subgraph is a minimal set in $\mathbb{R}^n \times \mathbb{R}$. Then we will blow-in the set $U$, and hence consider the sets $U_j$, that are themselves subgraphs of some function $u_j$. The sets $U_j$ converges to some set $C$, that is itself the subgraph of a suitable function $v$, thanks to Proposition 10.3.5. The foundamental fact is that, if we are in dimension $n \leq 7$, the function $v$ cannot assume the value $+\infty$ or $-\infty$, and hence it turns out that the gradient of the function $u$ is bounded in $\mathbb{R}^n$. Finally, using standard results of the theory of elliptic equations of second order, we will obtain that $u$ is an affine function (Theorem 11.0.13).

We start by proving the foundamental brick of our idea

**Theorem 11.0.9.** *Let $U$ be a minimal set in $\mathbb{R}^n$. Then $n \geq 8$ or $\partial U$ is an hyperplane.*

*Proof.* For $j \in \mathbb{N} \setminus \{0\}$ define the "blow-in" of $U$

$$U_j := \{x \in \mathbb{R}^n \mid jx \in U\}$$

The first part of the proof is similar to the proof of Theorem 9.2.2. First of all we prove that $U_j$ is a minimal set in $\mathbb{R}^n$. To prove this fix $R > 0$, and let $F$ Caccioppoli set such that $F \triangle U_j \Subset B_R$; hence $F_{\frac{1}{j}} \triangle U \Subset B_{jR}$, and from the minimality of $U$ in $B_{jR}$ we obtain that

$$|\partial F|(B_R) = j^{1-n} |\partial F_{\frac{1}{j}}|(B_{jR}) \geq j^{1-n} |\partial U|(B_{jR}) = |\partial U_j|(B_R)$$

Now we want to prove that there exists a minimal set $C$ such that $U_j \to C$ in $\mathbb{R}^n$. Fix $R > 0$; since each $U_j$ is minimal in $B_R$, from the estimate (8.8) we obtain that

$$|\partial U_j|(B_R) \leq \frac{1}{2} n \omega_n R^{n-1}$$

Hence from the Compactness Theorem 5.3.2 we obtain that there exists a set $C_R$ such that $U_j \to C_R$ in $B_R$. Moreover from Lemma 9.2.1 we obtain that $C_R$ is minimal in $B_R$. Finally, using a diagonal process we obtain that there exists a subsequence $(r_j)_j$ and a minimal set $C$ such that $U_{r_j} \to C$ in $\mathbb{R}^n$; moreover, also using Lemma 9.2.1 we obtain that for almost every $R > 0$ it holds

$$|\partial U_{r_j}|(B_R) \to |\partial C|(B_R) \tag{11.1}$$

Now we want to prove that $C$ is a cone, and we will prove it showing that the function

$$r \mapsto r^{1-n} |\partial C|(B_r)$$

is indipendent from $r$; hence using (8.5) we obtain that, up to a set of measure 0, $C$ is a cone with vertex at the origin. To do this consider the function

$$p(r) := r^{1-n} |\partial U_{r_j}|(B_r)$$

From (11.1) we have that for a.e. $R > 0$

$$\lim_{j \to \infty} p(r_j R) = R^{1-n} |\partial C|(B_R) \tag{11.2}$$

Fix $\rho < R$; then for each $r_j$ there exists an integer $m_{r_j}$ such that $(r_j + m_{r_j})\rho > jR$. Using the mononicity of the function $p$ (see (8.6)) we obtain that

$$p(r_j \rho) \leq p(r_j R) \leq p((r_j + m_{r_j})\rho)$$

Hence from (11.2) we obtain that for a.e. $\rho < R$

$$\rho^{1-n}|\partial C|(B_\rho) = R^{1-n}|\partial C|(B_R)$$

It follows that $C$ is a minimal cone in $\mathbb{R}^n$.

Now suppose that $n \le 7$; in this case we have proved that all minimal cones must be half-spaces, and hence $C$ is an half-space. So we have that $R^{1-n}|\partial C|(B_R) = \omega_{n-1}$ for each $R > 0$. Moreover from (8.9) we have that $\omega_{n-1} \le R^{1-n}|\partial U|(B_R)$. So we obtain that

$$\omega_{n-1} \le (Rr_j)^{1-n}|\partial U|(B_{r_jR}) = R^{1-n}|\partial U_{r_j}|(B_R) \to R^{1-n}|\partial C|(B_R) = \omega_{n-1} \tag{11.3}$$

Since the function $R \mapsto R^{1-n}|\partial U|(B_R)$ is a non-decreasing function, and $r_j \to \infty$ for $j \to \infty$, from (11.3) we obtain that

$$R^{1-n}|\partial U|(B_R) = \omega_{n-1}$$

for all $R > 0$, and hence from (8.5) we obtain that $U$ is a cone itself. Hence $U$ is a minimal cone in $\mathbb{R}^n$, and since $n \le 7$ we obtain that $U$ is an half-space, and hence $\partial U$ is an hyperplane. □

Now we want to prove that if $u$ satisfied the minimal surface equation then its subgraph is a minimal set in $\mathbb{R}^n \times \mathbb{R}$. To do this we need a stronger version of Theorem 9.4.5.

**Definition 11.0.10.** *Let $E \subset \Omega$ be a measurable set with $C^2$ boundary. We say that a vector field $\xi \in C^1_c(\Omega;\mathbb{R}^n)$ is a* ***calibration*** *for $E$ in $\Omega$ if*

- $\mathrm{div}(\xi) = 0$ *in* $\Omega \cap E$
- $|\xi| \le 1$
- $\xi \equiv \nu_E$ *on* $\partial E \cap \Omega$

**Lemma 11.0.11 (Calibration method).** *If a Caccioppoli set $E$ with $C^2$ boundary has a calibration in $\Omega$, then $E$ is minimal in $\Omega$*

*Proof.* We can repeat the proof of Theorem 9.4.5 to obtain that $\xi$ is a sub-calibration for $E$ and for $\Omega \setminus E$. Then we obtain that $E$ and $\Omega \setminus E$ are sub-minimal in $\Omega$, and hence $E$ is minimal in $\Omega$. □

**Proposition 11.0.12.** *If a function $u : \Omega \to \mathbb{R}$ is a solution of the minimal surface equation in an open set $\Omega \subset \mathbb{R}^n$, then its subgraph $U$ is a minimal set in $Q := \Omega \times \mathbb{R}$.*

*Proof.* The boundary of the set $U$ is of class $C^2$, since it is the graph of $u$. Moreover the vector field

$$\xi := \frac{(Du, -1)}{\sqrt{1+|Du|^2}}$$

is a calibration for $U$ in $Q$. Then from the previous lemma we obtain that $U$ is minimal in $Q$. $\square$

Before going on we note that the minimal surface equation

$$\sum_{i=1}^{n} D_i\left(\frac{D_i u}{\sqrt{1+|Du|^2}}\right) = 0 \tag{11.4}$$

is an elliptic equation of second order in divergence form. In fact if we define

$$T(p) := \frac{p}{\sqrt{1+|p|^2}}\,, \qquad p \in \mathbb{R}^n$$

we have that equation (11.4) can be write as

$$\mathrm{div}(T(Du)) = 0$$

Since

$$\frac{\partial T_i}{\partial p_j} = \frac{\varepsilon_{ij}(1+|p^2|) - p_i p_j}{1+|p|^2}$$

we have that the matrix $A := \left(\frac{\partial T_i}{\partial p_j}\right)_{ij}$ is a symmetric matrix, and hence it is diagonalizable. Denoting with $\nu$ and $\Lambda$ the minimum and the maximum eigenvalue of $A$ respectively, we have that

$$\nu|x|^2 \le \sum_{i,j=1}^{n} \frac{\partial T_i}{\partial p_j} x_i x_j \le \Lambda |x|^2 \quad \text{ for each } x \in \mathbb{R}^n$$

Finally it is clear that $|T(p)| \le 1$, and hence we have obtained that the minimal surface equation in $\mathbb{R}^n$ is an elliptic equation of second order in divergence form.

Now we want to prove that an entire solution of the minimal surface equation with bounded gradient is an affine function.

**Theorem 11.0.13.** *Let $u$ be a solution of the minimal surface equation in $\mathbb{R}^n$. Suppose that $u$ has bounded gradient in $\mathbb{R}^n$. Then $u$ is an affine function.*

*Proof.* Recalling the calculation at the end of Section 10.1, when we proved the existence of upper barrier, we have that the function $u$ satisfied the integral equation

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} D_i F(Du) D_i \varphi \,\mathrm{d}x = 0 \qquad \forall\, \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$$

where $F(p) := \sqrt{1+|p|^2}$. Now if we take as $\varphi$ the function $D_s\psi$, where $1 \le s \le n$ is a fixed index, and $\psi \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$, we obtain, integrating two times by parts

$$0 = \int_{\mathbb{R}^n} \sum_{i=1}^{n} D_i F(Du) D_i \varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} \sum_{i=1}^{n} D_i(D_s D_i F(Du)) \psi \,\mathrm{d}x$$

Since this equation holds for every $\psi \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$, we must have that

$$\sum_{i=1}^{n} D_i(D_s D_i F(Du)) = 0$$

that is the function $w := D_s u$ satisfied the equation

$$\sum_{i=1}^{n} D_i(a_{ij}(x) D_j w) = 0 \tag{11.5}$$

where

$$a_{ij}(x) := D_i D_j F(Du) = \frac{\varepsilon_{ij}(1+|Du|^2) - D_i u D_j u}{(1+|Du|^2)^{\frac{3}{2}}}$$

Since $|Du|$ is bounded in $\mathbb{R}^n$, and hence the coefficients $a_{ij}$ are bounded, we obtain a lower bounded for the minimum eigenvalue of the matrix $A := (a_{ij})_{ij}$. Recalling that the function $F$ is strictly convex, and hence the matrix $A$ is definite positive, we obtain that there exists $\nu > 0$ such that

$$a_{ij}\xi_i\xi_j \ge \nu|\xi|^2$$

for all $\xi \in \mathbb{R}^n$. So we have obtained that the equation (11.5) is uniformly elliptic. Since $w$ is bounded, because $Du$ is, we obtain that $\inf w > -\infty$; hence the function $z := w - \inf w$ satisfied themself equation (11.5). From the Harnack's inequality (see [Mos61]) we obtain that there exists a constant $c > 0$ such that, for all $R > 0$,

$$\sup_{B_R} z \le c \inf z$$

Letting $R \to \infty$ we get $\sup_{\mathbb{R}^n} z = 0$ and hence $w$ is constant. So we obtain that for each $s = 1, \ldots, n$ $D_s u$ is constant, and hence $u$ is an affine function as desired. □

Next technical result says that if a sequence of quasi-solutions converges to a function that does not assume the value $+\infty$, then the quasi-solutions of the sequence are uniformly locally bounded above.

**Lemma 11.0.14.** *Let $\Omega$ be an open bounded set in $\mathbb{R}^n$, and let $(u_j)_j$ be a sequence of quasi-solutions in $\Omega$ converging almost everywhere to a quasi-solution $v$. Supposet that*

$$P := \{ x \in \Omega \mid v(x) = +\infty \} = \emptyset$$

*Then for every compact set $K \subset \Omega$ there exists a constant $c(K) > 0$ such that*

$$\sup_j \sup_{x \in K} u_j(x) \leq c(K)$$

*That is $(u_j)_j$ is uniformly locally bounded above.*

*Proof.* From Proposition 10.3.8 we have that $v$ is locally bounded above in $\Omega$. Let $K \subset \Omega$ be a compact set, and let $2d := d(K, \partial\Omega)$ (if $\Omega = \mathbb{R}$ we set $d = 1$). Set

$$c(K) := \sup_{x \in K_d} v(x)$$

where

$$K_d := \{ x \in \mathbb{R}^n \mid d(x, K) \leq d \}$$

We note that $c(K) < \infty$ because $P = \emptyset$. Then it holds

$$\sup_j \sup_{x \in K} u_j(x) \leq c(K)$$

Otherwise there would exists $0 < \varepsilon < d$, a subsequence $(u_j^*)_j$ and a sequence of points $(x_j)_j \subset K$ such that

$$z_j := (x_j, c(K) + \varepsilon) \in U_j^*$$

Since $U_j^*$ has leat perimeter in $\Omega \times \mathbb{R}$ we have that

$$\mathcal{L}^n(U_j^* \cap B_\varepsilon(z_j)) \geq \alpha(n)\varepsilon^{n+1}$$

Since, from the definition of the point $z_j$, $K_\varepsilon \times (c(K), c(K) + 2\varepsilon) \supset B_\varepsilon(z_j)$, we have that

$$\mathcal{L}^n(U_j^* \cap (K_\varepsilon \times (c(K), c(K) + 2\varepsilon))) \geq \alpha(n)\varepsilon^{n+1}$$

Since $u_j \to v$ almost everywhere, we have that $U_j \to V$, where $V$ is the subgraph of $v$. Hence

$$\mathcal{L}^n(V \cap (K_\varepsilon \times (c(K), c(K) + 2\varepsilon))) \geq \alpha(n)\varepsilon^{n+1} > 0$$

That is

$$V \cap (K_\varepsilon \times (c(K), c(K) + 2\varepsilon)) \neq \emptyset$$

This is absurd for the definition of $c(K)$. $\square$

Now we have all the results to extend the Bernstein Theorem in higher dimensions

**Theorem 11.0.15.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be an entire solution of the minimal surface equation*

$$\sum_{i=1}^{n} D_i\Big(\frac{D_i u}{\sqrt{1+|Du|^2}}\Big) = 0$$

*Then $n \geq 8$ or the graph of $u$ is an hyperplane.*

*Proof.* Let $U$ be the subgraph of $u$; define, for each $j$,

$$U_j := \{\, x \in \mathbb{R}^n \mid jx \in U \,\}$$

Then $U_j$ is the subgraph of the function

$$u_j(x) := \frac{1}{j} u(jx)$$

Moreover we already known that there exists a subsequence $U_{r_j} \to C$, where $C$ is a minimal cone. Then, from Lemma 10.3.4 and Proposition 10.3.5 we have that $C$ is the subgraph of a quasi-solution $v$. Let

$$P := \{\, x \in \mathbb{R}^n \mid v(x) = +\infty \,\}$$

$$N := \{\, x \in \mathbb{R}^n \mid v(x) = -\infty \,\}$$

First suppose that $P = \emptyset$. Since $U_{r_j} \to U$ imply $u_j \to v$ almost everywhere, from the previous lemma we have that the functions $u_{r_j}$ are uniformly bounded above in $B_1$, and hence

$$\sup_{x \in B_{r_j}} u(x) \leq c(B_1) r_j$$

From the a priori estimate of the gradient we have that

$$\sup_{S_{r_j/6}} |Du| \leq \exp\Big\{\, c\Big(1 + c(B_1) - \frac{u(0)}{r_j}\Big) \,\Big\}$$

Letting $j \to \infty$ we obtain

$$\sup |Du| \leq \gamma$$

where $\gamma > 0$ is a constant. Hence we can apply Theorem 11.0.13 to conclude that $u$ is an affine function. Note that the same conclusion can be obtained also if we suppose $N = \emptyset$.

Now we prove that if $n \leq 7$, then one of $P$ or $N$ must be empty. Otherwise they are both non-empty, and since $v$ is a quasi-solution, from Theorem

10.3.6 we get that they are minimal sets in $\mathbb{R}^n$. Moreover since $C$ is a cone, we have that $P$ and $N$ are cones in $\mathbb{R}^n$ with vertex at the origin: in fact if $x \in P$, then $v(x) = +\infty$ and hence, for each $t > 0$, $(x,t) \in C$; since $\{\lambda(x,t) \mid \lambda \geq 0\} \in C$ we obtain that $\lambda x \in P$ for each $\lambda \geq 0$, that is $P$ is a cone. Same argument for $N$.

Now, since $n \leq 7$, the regularity result for minimal cones in these dimensions, tells use $P$ and $N$ must be half-spaces. So we obtain that

$$v(x) = \begin{cases} +\infty & , x \in P \\ -\infty & , x \in N = \mathbb{R}^n \setminus P \end{cases}$$

Hence $C$ is an half-space, and $\partial C$ is a vertical hyperplane. Arguing as in Theorem 11.0.9 we obtain that $U = C$. But this is impossible since $\partial U$ is the graph of the function $u : \mathbb{R}^n \to \mathbb{R}$ and hence it cannot be a vertical hyperplane. So both $P$ and $N$ must be empty if $n \leq 7$. □

As noted at the beginning of this chapter, if we denote by $\mathcal{C}_S$ the Simons cone in $\mathbb{R}^8$, we have that the function

$$f(x) := \begin{cases} +\infty & , x \in \mathcal{C}_S \\ -\infty & , x \notin \mathcal{C}_S \end{cases}$$

we obtain that $f$ is a quasi-solution in $\mathbb{R}^8$. Moreover using the Simons cone, Bombieri, De Giorgi e Giusti in [BDGG69], can be able to construct a suitable super and sub-solution of the minimal surface equation that make possible an estimate of the solution of the Dirichlet problem for the area functional for a suitable boundary datum that make possible to conclude that the solution cannot be an hyperplane. So they proved the following foundamental

**Theorem 11.0.16.** *Let $n \geq 8$. Then there exists entire solutions of the minimal surface equation*

$$\sum_{i=1}^{n} D_i\Big(\frac{D_i u}{\sqrt{1 + |Du|^2}}\Big) = 0$$

*which are not hyperplane.*

This results tells us that in dimension higher than 7, there exists solution of the minimal surface equation which are not hyperplanes. This result, together with Theorem 11.0.15 solve the Bernstein Problem in the Euclidean space.

# Chapter 12

# The sub-Riemannian Heisenberg group $\mathbb{H}^n$

The aim of this chapter is to introduce the Heisenberg group $\mathbb{H}^n$ and all the notions and results we need to state the Bernstein Problem in the Heisenberg group. We begin with Section 12.1 where we recall the basic results on Lie algebras and Lie group; in particular we point out that a Carnot group $\mathbb{G}$ is diffeomorphic to some $\mathbb{R}^n$; so we can represent $\mathbb{G}$ by the so called graded coordinates. Then, in Section 12.2 we introduce the (representation of the) Heisenberg group $\mathbb{H}^n$ as a Carnot group of step 2 (see Definition 12.2.1). In Section 12.3 we introduce the Carnot-Carathèodory spaces, i.e. an $\mathbb{R}^n$ endowed with a family $X$ of vector fields defined on it. In particular we define the Carnot-Carathèodory distance $d_c$ that arise from the family $X$, and we see that $(\mathbb{R}^n, d_c)$ is actually a metric space, i.e. $d_c(x, y)$ is finite for each $x, y \in \mathbb{R}^n$, if the Lie algebra generates by the family $X$ has dimension $n$ (the so called Chow-Hörmander's condition, see Definition 12.3.4 and Theorem 12.3.5). In Section 12.4 we see the Heisenberg group $\mathbb{H}^n$ as a Carnot-Carathèodory space with the distance $d_c$; we will introduce an equivalent distance $d_\infty$ that has the property of being explicity, differently from $d_c$. We have that, despite $\mathbb{H}^n$ and $\mathbb{R}^{2n+1}$ are topologically equivalent (and hence they have the same topological dimension $2n+1$), they are not metrically equivalent, and the Hausdorff dimension of $\mathbb{H}^n$ with respect to $d_\infty$ is $2n+2$. Section 12.6 is dedicated to the notion of $\mathbb{H}$-perimeter (see Definition 12.6.4), defined in the same way as in the Euclidean case. Moreover we can define the inward normal $\nu_E$ to a set $E$ (see Theorem 12.6.5) and the notion of $\mathbb{H}$-reduced boundary (see Definition 12.6.8). To state the analogous of Theorem 6.3.2 for $\mathbb{H}$-Caccioppoli sets in $\mathbb{H}^n$ we need a suitable definition of regular surface in $\mathbb{H}^n$; we will give one in Section 12.7 that seems to be the correct generalization of $C^1$ hypersurfaces in $\mathbb{R}^n$, because it can be prove some important properties concerned $\mathbb{H}$-regular hypersurface, as for example an Implicit Function Theorem (see Theorem 12.7.8). Moreover to state the Bernstein Problem in $\mathbb{H}^n$ we need a notion of graph in $\mathbb{H}$ that takes into account the geometry of our space: the (intrinsic) notion of $X_1$-graph of a function $\omega : \mathbb{R}^{2n} \to \mathbb{R}$ is give in Definition 12.7.7, and a complete characterization of those function $\omega$ whose $X_1$-graph turns out to be an $\mathbb{H}$-regular hypersurface is give in Theorem 12.7.11. This characterization makes use of the differential operator $W^\phi \phi$, that seems to be the right counterpart of the Euclidean gradient. Finally in Section 12.8 we state the Rectificabilty Theorem for the $\mathbb{H}$-reduced boundary of a $\mathbb{H}$-Caccioppoli set in $\mathbb{H}^n$ (see Theorem 12.8.3).

For a more satisfied traetment of all the questions presented in this chapter, we adrees the reader to [Vit08].

## 12.1 Carnot groups

The aim of this section is to recall some basic results on Lie groups and Lie algebras. In particular we state that the set of left invariant vector fields on a Lie group $\mathbb{G}$ (see Definition 12.1.7) endowed with the operation $[\cdot,\cdot] : (X,Y) \mapsto [X,Y] := XY - YX$ forms a Lie algebra, that turns out, under some assumptions, to be diffeomorphic to $\mathbb{G}$ (see Theorem 12.1.9). Then we introduce Carnot groups (see Definition 12.1.10) and define dilatations on them (see Definition 12.1.13). In particular, thans to Theorem 12.1.9, we see that, given a Carnot group $\mathbb{G}$ we can find a group structure on some $\mathbb{R}^n$ in a way that $\mathbb{G}$ turns out to be isomorphic to this $\mathbb{R}^n$, and the Lie algebra of $\mathbb{G}$ turns out to be isomorphic to those of $\mathbb{R}^n$; so we can represent Carnot groups in $\mathbb{R}^n$. Finally we introduce the notion of homogeneous dimension on a Carnot group, that turns out to be the Hausdorff dimension of the group with respect to any homogeneous distance defined on it (see Theorem 12.1.18), and we see that the Lebesgue measure is the Haar measure of (the representation of) a Carnot group.

### 12.1.1 Lie groups and Lie algebras

**Definition 12.1.1.** *A **Lie group** $\mathbb{G}$ is a manifold endowed with the structure of differential group, i.e. a group such that the maps*

$$\begin{array}{rcl} \mathbb{G}\times\mathbb{G} & \to & \mathbb{G} \\ (x,y) & \longmapsto & xy \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbb{G} & \to & \mathbb{G} \\ x & \longmapsto & x^{-1} \end{array}$$

*are of class $C^\infty$.*

**Notation**: we denote by $e$ the identity of the group $\mathbb{G}$.

**Definition 12.1.2.** *If $\mathbb{G}$ is a Lie group we define and $x \in \mathbb{G}$ we define the* ***left translation*** *by $x$, $l_x$ as the $C^\infty$-map*

$$y \mapsto xy$$

**Definition 12.1.3.** *A vector space $\mathfrak{g}$ is a* ***Lie algebra*** *if there is a bilinear and anti-symmetric map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfied the Jacobi's identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

*for all $X, Y, Z \in \mathfrak{g}$.*

**Remark 12.1.4.** *We give an important example of Lie algebra. Let $M \subset \mathbb{R}^n$ be a differential manifold, and let $\Gamma(TM)$ be the space of vector fields on $M$. We recall that the* ***commutator*** *of two vector fields $X, Y \in \Gamma(TM)$ is defined as*

$$[X,Y] := XY - YX$$

*We recall that we identify vector fields as first order operators. So we can write $[X,Y]$ in coordinates as*

$$[X,Y] = \sum_{i=1,j}^{n} \Big(a_j(x)\partial_j b_i(x) - b_j(x)\partial_j a_i(x)\Big)\partial_i$$

*where we write $\partial_i$ instead of $\frac{\partial}{\partial x_i}$, and the vector fields $X$ and $Y$ are written as*

$$X = \sum_{i=1}^{n} a_i(x)\partial_i\,, \qquad Y = \sum_{i=1}^{n} b_i(x)\partial_i$$

*It is quite easy to prove that the bilinear and anti-symmetric map $(X,Y) \mapsto [X,Y]$ satisfied the Jacobi's identity. So the space $\Gamma(TM)$ of vector fields on $M$ endowed with the product $[\cdot,\cdot]$ is a Lie algebra.*

**Notation**: if $\mathfrak{a}, \mathfrak{b}$ are subalgebras of a Lie albegra $\mathfrak{g}$, we denote by $[a,b]$ the vector subspace generated by the elements of

$$\{\,[X,Y] \mid X \in \mathfrak{a},\ Y \in \mathfrak{b}\,\}$$

**Definition 12.1.5.** *Given a Lie algebra $\mathfrak{g}$ we define $\mathfrak{g}^1 := \mathfrak{g}$ and for $k \geq 1$, $\mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k]$. We say that $\mathfrak{g}$ is* ***nilpotent of step*** *$i$ if $\mathfrak{g}^i \neq \{0\}$ and $\mathfrak{g}^{i+1} = \{0\}$.*

**Definition 12.1.6.** *We say that a Lie algebra $\mathfrak{g}$ is* ***stratified*** *if it admits linear subspaces $\mathfrak{g}_1, \dots, \mathfrak{g}_i$ such that*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_i$$

$$\mathfrak{g}_k = [\mathfrak{g}_1, \mathfrak{g}_{k-1}] \qquad \text{for all } k = 2, \dots, k$$

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \{0\}$$

Now, given a Lie group $\mathbb{G}$ we want to associate to it a Lie algebra in a natural way.

**Definition 12.1.7.** *We say that a vector field $X \in \Gamma(T\mathbb{G})$ is* ***left invariant*** *if, for each $x \in \mathbb{G}$ it holds*

$$X(x) = \mathrm{d}l_x(X(e))$$

*We denote by $\mathfrak{g}$ the set of left invariant vector fields of $\Gamma(T\mathbb{G})$.*

It holds that $\mathfrak{g}$ is a Lie algebra, endowed with the product $[X, Y] := XY - YX$. Moreover it is clear that we can canonically identify the algebra $\mathfrak{g}$ with the tangent space $T_e\mathbb{G}$ via the isomorphism

$$X \longleftrightarrow v$$

where $v \in T_e\mathbb{G}$ is such that $X(x) = \mathrm{d}l_x(v)$ for each $x \in \mathbb{G}$.
We will say that a Lie group $\mathbb{G}$ is nilpotent of step $k$, or that it is stratified if its associate Lie algebra is.

The importance of the associate Lie algebra $\mathfrak{g}$ of a Lie group $\mathbb{G}$ is that, under some assumptions, they are diffeomorphic. To state this result, let $X \in \mathfrak{g}$, $x \in \mathbb{G}$ and consider the solution $\dot{\gamma}_x^X$ of the Cauchy problem

$$\begin{cases} \dot{\gamma}_x^X(t) &= X(\gamma_x^X(t)) \\ \gamma_x^X(0) &= x \end{cases}$$

Since left invariant vector fields are complete, the curve $\gamma_x^X$ is defined for each time $t$. We denote by $\exp(X)(x) := \gamma_x^X(1)$.

**Definition 12.1.8.** *We define the* ***exponential map*** $\exp : \mathfrak{g} \to \mathbb{G}$ *as follows*

$$\exp(X) := \exp(X)(e)$$

Explain in words $\exp(X)$ is a translation of "lenght" 1 along the trajectory of $X$.

The following result is very important because it states a connection between the Lie algebra $\mathfrak{g}$ and the Lie group $\mathbb{G}$.

**Theorem 12.1.9.** *Let $\mathbb{G}$ be a nilpotent, connected and simply connected Lie group. Then the exponential map* $\exp : \mathfrak{g} \to \mathbb{G}$ *is a diffeomorphism.*

Now we want to define on the Lie algebra $\mathfrak{g}$ an opeartion $C : \mathfrak{g} \to \mathfrak{g}$ that makes exp a group isomorphism. Suppose the hypothesis of the above theorem hold. So, given two elements $X, Y \in \mathfrak{g}$, we define $C(X,Y)$ as the element that satisfied

$$\exp(C(X,Y)) = \exp(X) \cdot \exp(Y)$$

We can compute explicity $C(X,Y)$ thanks to the *Baker-Campbell-Hausdorff formula*: let $\alpha = (\alpha_1, \dots, \alpha_m)$ a multi-index of non-negative index, and define

$$|\alpha| := \alpha_1 + \cdots + \alpha_m$$

$$\alpha! := \alpha_1! \dots \alpha_m!$$

and we will say that $m$ is the lenght of $\alpha$. Now, if $\beta = (\beta_1, \dots, \beta_m)$ is another multi-index of lenght $m$ such that $\alpha_m + \beta_m \geq 1$, we define

$$C_{\alpha\beta}(X,Y) := \begin{cases} (\mathrm{X})^{\alpha_1}(\mathrm{Y})^{\beta_1} \dots (\mathrm{X})^{\alpha_m}(\mathrm{Y})^{\beta_m - 1}Y & ,\text{if } \beta_m > 0 \\ (\mathrm{X})^{\alpha_1}(\mathrm{Y})^{\beta_1} \dots (\mathrm{X})^{\alpha_m - 1}X & ,\text{if } \beta_m = 0 \end{cases}$$

where $X, Y \in \mathfrak{g}$. We recall that the adjoint operator $\mathrm{ad}X : \mathfrak{g} \to \mathfrak{g}$ is defined by $(\mathrm{ad}X)(Y) := [X,Y]$, and we set $(\mathrm{ad}X)^0$ as the identity map. Finally we define

$$C(X,Y) := \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{\substack{\alpha=(\alpha_1 \dots, \alpha_m) \\ \beta=(\beta_1 \dots \beta_m) \\ \alpha_i+\beta_i \geq 1 \, \forall i}} \frac{1}{\alpha! \beta! |\alpha + \beta|} C_{\alpha\beta}(X,Y) \tag{12.1}$$

We note that we can write

$$C(X,Y) = X + Y + \frac{1}{2}[X,Y] + \mathcal{R}_3(X,Y)$$

where $\mathcal{R}_3(X,Y)$ is a series of commutators of lenght more than 3.

### 12.1.2 Carnot groups

Now we have all the elements to define what a Carnot group is

**Definition 12.1.10.** *A **Carnot group** $\mathbb{G}$ is a finite dimensional, connected, simply connected and stratified Lie group. We say that a Carnot group $\mathbb{G}$ is **of step** $i$ if the stratification of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is $\mathfrak{g}_1, \dots \mathfrak{g}_i$. Note that such a group, since it is finite dimensional, is also nilpotent of step $i$.*

**Remark 12.1.11.** *For Carnot groups Theorem 12.1.9 holds.*

Now we want to define the notion of dilatation in Carnot groups.

**Definition 12.1.12.** *Let $\mathbb{G}$ be a stratified Lie group, and let $\mathfrak{g}_1, \dots, \mathfrak{g}_i$ a stratification. Fix $r > 0$ we define the **dilatation** $\delta_r$ of the algebra as follows: if $X \in \mathfrak{g}_k$ then $\delta_r(X) := r^k X$, and we extend this map to all the algebra $\mathfrak{g}$ by linearity.*

The following properties hold for all $X, Y \in g$ and $r, s > 0$

- $\delta_{rs} = \delta_r \circ \delta_s$
- $\delta_r([X, Y]) = [\delta_r X, \delta_r Y]$
- $\delta_r(C(X, Y)) = C(\delta_r X, \delta_r Y)$

Since for carnot groups Theorem 12.1.9 holds, we have that the map $\exp : g \to \mathbb{G}$ is a diffeomorphism. So we can define on $\mathbb{G}$ a one-parameter group of automorphisms starting from the dilatations of its Lie algebra $\mathfrak{g}$.

**Definition 12.1.13.** *Let $\mathbb{G}$ be a Carnot group, and let $\delta_r$ be the dilatation of $r$ defined on its Lie algebra $\mathfrak{g}$. We define the dilatation of $r$ on $\mathbb{G}$, denoted again with $\delta_r$ as follows*

$$\delta_r(x) := \exp(\delta_r(\exp^{-1}(x)))$$

*The map $\delta_r$ turns out to be an automorphism of $\mathbb{G}$.*

Using the properties of the dilatations defined on $\mathfrak{g}$ it is easy to prove that for the dilatations $\delta_r$ defined on $\mathbb{G}$ the following two properties hold

- $\delta_{rs} = \delta_r \circ \delta_s$
- $\delta_r(x \cdot y) = \delta_r(x) \cdot \delta_r(y)$

Now we want to find a convenient way to represent Carnot groups.

**Definition 12.1.14.** *Let $\mathbb{G}$ be a Lie algebra, and let $X_1, \ldots, X_n$ be a basis of its Lie algebra $\mathfrak{g}$. We define the system of* ***exponential coordinates*** *associate with the basis $X_1, \ldots, X_n$ as the map*

$$\begin{array}{rccc} F & : & \mathbb{R}^n & \longrightarrow & \mathbb{G} \\ & & x & \longmapsto & \exp\Big(\sum_{i=1}^n x_i X_i\Big) \end{array}$$

**Definition 12.1.15.** *Let $\mathbb{G}$ be a Carnot group and let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_i$ be its statification. Define for each $k = 1, \ldots, i$*

$$m_k := \dim \mathfrak{g}_k$$

*and*

$$n_k := m_1 + \cdots + m_k$$

*and $n_0 := 0$. If the basis $X_1, \ldots, X_n$ is such that $X_{n_{k-1}+1}, \ldots X_{n_k}$ is a basis for $\mathfrak{g}_k$ we say that the basis $X_1, \ldots, X_n$ is* ***adapted*** *to the stratification, and we called the system of coordinates associate with this basis* ***graded coordinates****.*

Now we want to complete the identification of $\mathbb{G}$ with $\mathbb{R}^n$. To do this we need to put on $\mathbb{R}^n$ a group law that makes $F$ a group isomorphism. So let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, and define

$$x \cdot y := z = (z_1, \ldots, z_n) \in \mathbb{R}^n$$

if and only if

$$\sum_{i=1}^n z_i X_i = C\Big(\sum_{i=1}^n x_i X_i, \sum_{i=1}^n y_i X_i\Big)$$

In this representation the group identity is the origin, and $x^{-1} = -x$. So we have obtain that

**Theorem 12.1.16.** *So $(\mathbb{R}^n, \cdot)$ is a Lie group isomorphic to $\mathbb{G}$, whose Lie algebra is isomorphic to $\mathfrak{g}$.*

Moreover we can read the dilatation in coordinates:

$$\delta_r(x) = (rx_1, \ldots, rx_{n_1}, r^2 x_{n_1+1}, \ldots, r^3 x_{n_2}, \ldots, r^i x_{n_{i-1}+1}, \ldots, r^i x_n)$$

### 12.1.3 Homogeneous dimension and Haar measure

Now we want to introduce a suitable dimension on a Carnot group $\mathbb{G}$.

**Definition 12.1.17.** *Let $\mathbb{G}$ be a Carnot group with stratified algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. We define the **homogeneous dimension** $Q$ of $\mathbb{G}$ as*

$$Q := \sum_{i=1}^{k} i \dim \mathfrak{g}_i$$

We have the following

**Theorem 12.1.18.** *The homegeneous dimension $Q$ coincide with the Hausdorff dimension of the group $\mathbb{G}$ with respect to any homogeneous metric $\rho$ defined on it.*

In particular if $\rho$ is a metric defined on $\mathbb{G}$ we denote by $\mathcal{H}^m_\rho$ the $m$-th dimensional Hausdorff measure associate with $\rho$. It hold

$$\mathcal{H}^m_\rho(x \cdot E) = \mathcal{H}^m_\rho(E)\,, \qquad \mathcal{H}^m_\rho(\delta_r E) = r^d \mathcal{H}^m_\rho(E)$$

for any measurable set $E \subset \mathbb{G}$ and any $x \in \mathbb{G}$ and $r > 0$.

Now suppose to represent a Carnot group $\mathbb{G}$ with $\mathbb{R}^n$ via graded coordinates. Then it holds

**Theorem 12.1.19.** *For any measurable set $E \subset \mathbb{R}^n$ and any $x \in \mathbb{R}^n$ it holds*

$$\mathcal{L}^n(x \cdot E) = \mathcal{L}^n(E \cdot x) = \mathcal{L}^n(E)$$

*that is $\mathcal{L}^n$ is both left and right inveriant, and so $\mathcal{L}^n$ is the Haar measure of the group $\mathbb{G}$.*
*Moreover for each $x \in \mathbb{R}^n$ and $r > 0$ it holds*

$$\mathcal{L}^n(U^c_r(x)) = r^Q \mathcal{L}^n(U^c_1(x)) = r^Q \mathcal{L}^n(U^c_1(0))$$

A diffuculty in studying Carnot groups is the following

**Theorem 12.1.20.** *If the Carnot groups $\mathbb{G}$ is not abelian, then the metric (Hausdorff) dimension $Q$ is strictly greater than the topological dimenasion $n$.*

## 12.2 The Heisenberg group $\mathbb{H}^n$

In this section we want to present the simplest example of Carnot groups: the Heisenberg group $\mathbb{H}^n$. We will give the representation in graded coordinates of it and we calculate the representation of the generators of its Lie algebra.

**Definition 12.2.1.** *The* n*-th* ***Heisenberg group*** $\mathbb{H}^n$ *is the* $2n+1$*-dimensional Carnot group with stratified algebra*

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$$

*where*

$$\mathfrak{h}_1 = \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$$

*and*

$$\mathfrak{h}_2 = \operatorname{span}\{T\}$$

*The only non-vanishing commutation relationship among the generators are*

$$[X_i, Y_i] = -4T$$

*for all* $i = 1, \dots, n$.

Since the Lie algebra $h$ is nilpotent of step 2, the Baker-Campbell-Hausdorff formula (12.1) become very easy

$$C(X,Y) = X + Y + \frac{1}{2}[X,Y]$$

Then, if $X = \sum_{i=1}^n x_i X_i + \sum_{i=1}^n y_i Y_i + tT$ and $Y = \sum_{i=1}^n x_i' X_i + \sum_{i=1}^n y_i' Y_i + t'T$, we have

$$C(X,Y) = X = \sum_{i=1}^n (x_i + x_i')X_i + \sum_{i=1}^n (y_i + y_i')Y_i + \sum_{i=1}^n (t + t' + 2x_i' y_i - 2x_i y_i')T$$

So we can represent the Heisenberg group $\mathbb{H}^n$ throught graded coordinates as $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group law

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \times \begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ t + t' + 2\langle x', y\rangle - 2\langle x, y'\rangle \end{pmatrix}$$

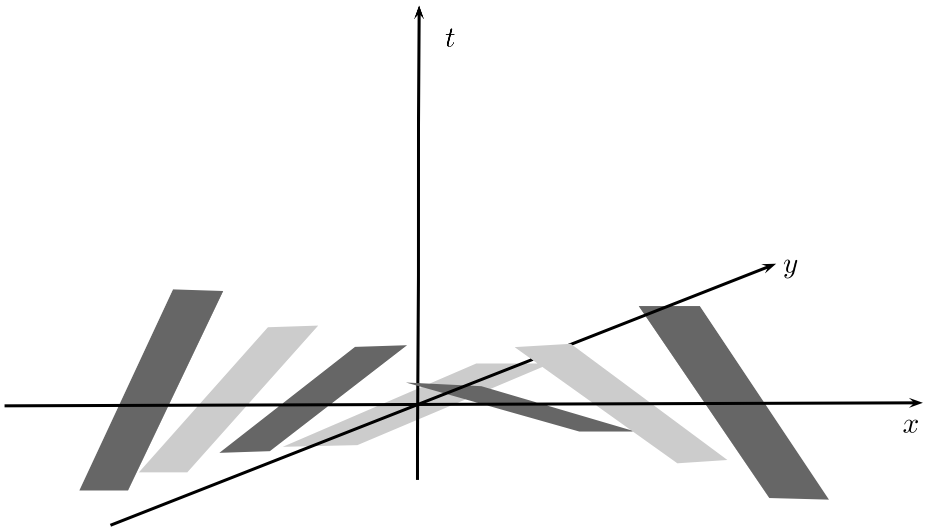



Figure 12.1: Example of horizontal planes in $\mathbb{H}^1$ at different points

Finally we want to represent the left invariant vector fields $X_i, Y_i, T$. Since if $X$ is a left invariant vector field it holds

$$X(g) = \mathrm{d}l_g(X(e))$$

and $X_j(0) = \partial_j$, $Y_j(0) = \partial_{n+j}$, $T = \partial_{2n+1}$, and since

$$\mathrm{d}l_{(x,y,t)}(0) = \begin{pmatrix} Id_n & 0 & 0 \\ 0 & Id_n & 0 \\ 2y & -2x & 1 \end{pmatrix}$$

where $Id_n$ denotes the $n \times n$ identity matrix, we have that

$$X_j(x, y, t) = \mathrm{d}l_{(x,y,t)}(\partial_j) = \partial_j + 2_{y_j}\partial_{2n+1}$$

$$Y_j(x, y, t) = \mathrm{d}l_{(x,y,t)}(\partial_{n+j}) = \partial_{n+j} - 2_{x_j}\partial_{2n+1}$$

$$T(x, y, t) = \mathrm{d}l_{(x,y,t)}(\partial_j) = \partial_{2n+1}$$

We will always use this representation when we work with the Heisengerg group $\mathbb{H}^n$.

## 12.3 Carnot-Carathèodory spaces

In this section we introduce the Carnot-Carathèodory spaces, i.e. $\mathbb{R}^n$ endowed with a family $X = (X_1, \dots, X_m)$ of Lipschitz vector field defined on it. We introduce the Carnot-Carathèodory distance (see Definition 12.3.3) and we see that $\mathbb{R}^n$ endowed with this distance is actually a metric space if the family $X$ satisfied the Chow-Hörmander's condition (see Definition 12.3.4 and Theorem 12.3.5).

### 12.3.1 Definition and properties of $d_c$

**Definition 12.3.1.** *Let $(X_1, \dots, X_m)$ be a family of Lipschitz continuous vector fields on $\mathbb{R}^n$, i.e.*

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\partial_i \,, \qquad j = 1, \dots, n$$

*where the functions $a_{ij}$ are Lipschitz. The subspace of $T_x\mathbb{R}^n \equiv \mathbb{R}^n$ generated by $X_1(x), \dots, X_m(x)$ is called **horizontal subspace**, and it is denoted by $H_x\mathbb{R}^n$. The collection of all horizontal fibres $H_x\mathbb{R}^n$ forms what we called the **horizontal subboundle** $H\mathbb{R}^n$ of $T\mathbb{R}^n$.*

**Notation**: if $X_1, \dots, X_m$ are of class $C^\infty$, we denote by $\mathcal{L}(X_1, \dots, X_m)$ the Lie algebra generates by them, i.e. the subspace generates by $X_1, \dots, X_m$ and by the vectors given by the iterated operation of $[\cdot, \cdot]$.

**Definition 12.3.2.** *Let $\gamma : [0, T] \to \mathbb{R}^n$ be a Lipschitz continous curve. We say that $\gamma$ is a **subunit** if there exist measurable functions $h_1, \dots, h_m$ such that for almost every $t \in [0, T]$*

$$\dot\gamma(t) = \sum_{i=1}^{m} h_i(t) X_i(\gamma(t)) \,, \quad \sum_{i=1}^{m} h_i^2(t) \le 1$$

**Definition 12.3.3.** *We define the **Carnot-Carathèodory distance** $d_c$ between the points $x, y \in \mathbb{R}^n$ as*

$$d_c(x, y) := \inf \Big\{ T \ge 0 \mid \exists\, \gamma : [0, T] \to \mathbb{R}^n \text{ subunit path s.t. } \gamma(0) = x,\, \gamma(T) = y \Big\}$$

*If the above set is empty we set $d_c(x, y) := +\infty$.*

If the distance $d_c$ is finite for every $x, y \in \mathbb{R}^n$ then $d_c$ is a distance, and hence $(\mathbb{R}^n, d_c)$ becomes a metric space, and we called it a **Carnot-Carathèodory space**.

The problem is to understand when we can say that $d_c(x, y) < \infty$ for each $x, y, \in \mathbb{R}^n$.

**Example**: we give an example of non existence of subunit path between two points: in $\mathbb{R}^2$ let $m = 1$ and $X_1 := \partial_1$; then if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are such that $x_2 \neq y_2$, then clearly there is no subunit path from $x$ to $y$, and hence $d_c(x, y) = +\infty$.

A sufficient condition to ensure that there exists always a subunit path from any pair of points, and hence that $d_c$ is a distance, is the so called Chow-Hörmander's condition

**Definition 12.3.4** (**Chow-Hörmander's condition**). *A $C^\infty$ family of vector fields on $\mathbb{R}^n$, $X_1, \dots, X_m$ is said to satisfied* ***Chow-Hörmander's condition*** *in $\mathbb{R}^n$ if*

$$\dim \mathcal{L}(X_1(x) \dots, X_m(x)) = n$$

*for each $x \in \mathbb{R}^n$.*

We have the following result

**Theorem 12.3.5** (**Chow-Hörmander**). *Let $X_1, \dots, X_m$ be a family of $C^\infty$ vector fields on $\mathbb{R}^n$ that satisfied Chow-Hörmander's condition on $\mathbb{R}^n$. Then for each pair of points $x \neq y \in \mathbb{R}^n$ there exists a subunit path from $x$ to $y$.*

**Note**: the CC-space satisfied Chow-Hörmander's condition are called **sub-Riemannian spaces**.

Now we want to study the connection between the $d_c$ distance and the usual Euclidean distance. First of all we see that $\mathbb{R}^n$ with the standard basis of its tangent is a Carnot-Carathèodory space

**Theorem 12.3.6.** *In $\mathbb{R}^n$ consider the vector fields $X_1 := \partial_1, \dots, X_m := \partial_m$. Then $\mathbb{R}^n$ endowed with this vector fields is a Carnot-Carathèodory space, and in particular*

$$d_c(x, y) = |x - y|$$

*for each $x, y \in \mathbb{R}^n$.*

Moreover we have that

**Theorem 12.3.7.** *Let $(\mathbb{R}^n, d_c)$ be a Carnot-Carathèodory space. Then the identity map*

$$id : (\mathbb{R}^n, d_c) \to (\mathbb{R}^n, |\cdot|)$$

*is continuous.*

**Remark 12.3.8.** *It is easy to show that, in general, $(\mathbb{R}^n, d_c)$ is not homeomorphic to $(\mathbb{R}^n, |\cdot|)$. We give an example of vector fields such that $d_c$ is not continuous with respect to $|\cdot|$: in $\mathbb{R}^2$ consider the vector fields $X_1 := \partial_1$ and $X_2(x_1, x_2) := f(x_1)\partial_2$ where $f$ is a $C^\infty(\mathbb{R})$ function that is negative when $x_1 > 0$ and null otherwise. Hence consider two points*

$$A := (x, y_a)\,, \qquad B = (x, y_b)$$

*with $x < 0$ and $y_a \neq y_b$. It is clear that if we want to joint $A$ and $B$ with a subunit path, we need to join $A$ to a point $B := (x_c, y_a)$ with $x_c > 0$, then $C$ to a point $D := (x_c, y_b)$, and finally $D$ with $B$. Hence if we let $|a - b| \to 0$, the $d_c$ distance from $A$ and $B$ remains great or equal to $2|x|$.*

To conclude that a general Carnot-Carathèodory space $(\mathbb{R}^n, d_c)$ is homeomorphic to $(\mathbb{R}^n, |\cdot|)$ a sufficient condition that ensure it is once again Chow-Hörmander's condition.

**Theorem 12.3.9.** *Let $X_1, \ldots, X_m$ be a family of $C^\infty$ vector fields on $\mathbb{R}^n$, and suppose that the vector space generates by $X_1, \ldots, X_m$ and by iterated operation of at most $p \geq 1$ commutators has dimension $n$. Then for each compact set $K$ there exists a constant $c(K) > 0$ such that*

$$d_c(x, y) \leq c(K)|x - y|^{\frac{1}{p}}$$

*for each $x, y \in K$.*

In particular we obtain that, if the $C^\infty$ family of vector fields $X_1, \ldots, X_m$ satisfied Chow-Hörmander's condition, then $(\mathbb{R}^n, d_c)$ and $(\mathbb{R}^n, |\cdot|)$ are topologically equivalent. Moreover if in the previous theorem $p = 1$, then the two metric spaces are also metrically equivalent, but if $p > 1$ they are not.

## 12.4 $\mathbb{H}^n$ as a Carnot-Carathèodory space

Let $(\mathbb{H}^n, \cdot)$ be the $n$-th Heisenberg group represented in graded coordinates associated with a basis adapted to the stratification $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. The stratification assumption ensures that the subspace $\mathfrak{h}_1$ Lie generate the whole algebra $\mathfrak{h}$, and hence the family $X = (X_1, \dots X_n, Y_1, \dots, Y_n)$ satisfied Chow-Hörmander's condition; then the function $d_c$ defined with the family $X$ is actually a distance on $\mathbb{H}^n$. So we can see $\mathbb{H}^n$ as a Carnot-Carathèodory space. Moreover the distance $d_c$ have good properties with respect to translations and dilatations

**Proposition 12.4.1.** *For each $x, y, z \in \mathbb{H}^n$ and $r > 0$ we have*

1. $d_c(z \cdot x, z \cdot y) = d_c(x, y)$
2. $d_c(\delta_r x, \delta_r y) = r d_c(x, y)$

These properties makes $d_c$ what we called an **homogeneous distance** on the Carnot group $(\mathbb{H}^n, \cdot)$. Moreover we have that

- $l_y(U_r^c(x)) = U_r^c(l_x(y))$
- $\delta_\lambda(U_r^c(x)) = U_{\lambda r}^c(\delta_r x)$

for each $x, y \in \mathbb{H}^n$ and $r, \lambda > 0$.

The problem of the distance $d_c$ is that it is not explicit, and hence it is difficult to estimate. To avoid this disadvantage we introduce in $\mathbb{H}^n$ a new homogeneous distance eqauivalent to $d_c$

**Definition 12.4.2.** *Let $p = (z, t), q \in \mathbb{H}^n$ and define the **infinity norm***

$$\|p\|_\infty := \max\{\, |z|_{\mathbb{R}^{2n}}, |t|^{\frac{1}{2}} \,\}$$

*and the associate distance*

$$d_\infty(p, q) := \|p^{-1} \cdot q\|_\infty$$

It turns out that $d_\infty$ is actually an homogeneous distance, that is equivalent to $d_c$.

**Theorem 12.4.3.** *Let $\Omega$ a bounded set in $\mathbb{H}^n$. Then there exist constants $C_1, C_2 > 0$ such that for each $x, y \in \Omega$*

$$C_1|x - y| \leq d_\infty(x, y) \leq C_2\sqrt{|x - y|}$$

**Note**: the distance $d_\infty$ is **not** a Riemannian distance.

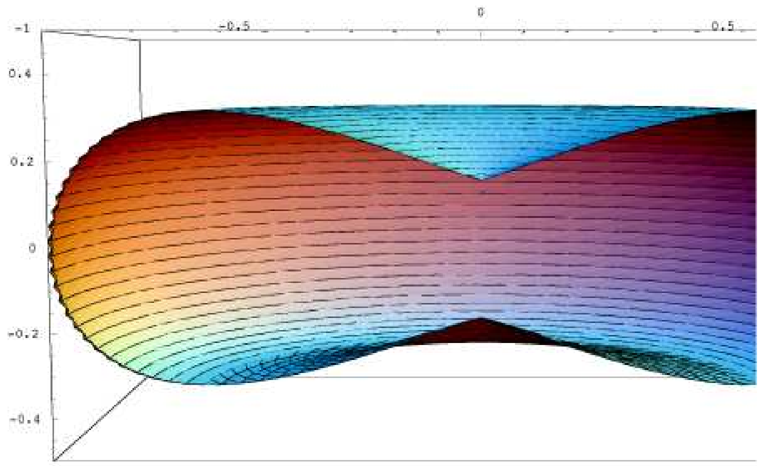



Figure 12.2: Section of the unit ball with the $d_C$ distance in $\mathbb{H}^1$

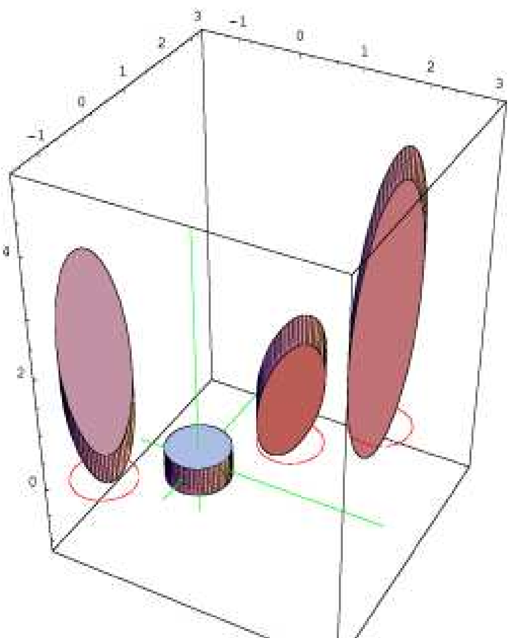



Figure 12.3: Example in $\mathbb{H}^1$ of unit balls with the $d_\infty$ distance

**Remark 12.4.4.** *It can be proved (see [Rig04]) that the Heisenberg group $\mathbb{H}^n$ endowed with the distance $d_C$ is not directionally limited. The same is true for the distance $d_\infty$.*

Another difficulty in studying the Heisenberg group is the following one

**Theorem 12.4.5.** *The topological dimension of $\mathbb{H}^n$ is $2n+1$ while the metric (Hausdorff) dimension of $\mathbb{H}^n$ is $2n+2$.*

## 12.5 Pansu Theorem

In this section we want to state the analogous of the Radameacher Theorem for Lipschitz functions from $\mathbb{H}^n$ to $\mathbb{R}$, where in the Heisemberg group $\mathbb{H}^n$ we consider the distance $d_\infty$.

**Definition 12.5.1.** *We say that a map $L : \mathbb{H}^n \to \mathbb{R}$ is* ***linear***

1. *$L(x \cdot y) = L(x) + L(y)$ for each $x, y \in \mathbb{H}^n$*
2. *for each $x \in \mathbb{H}^n$ and $\lambda > 0$ it holds*

$$L(\delta_\lambda(x)) = \lambda L(x)$$

**Remark 12.5.2.** *We note that a linear map $L : \mathbb{H}^n \to \mathbb{R}$ must be of the form*

$$L(x, y, t) = \langle a, x\rangle_{\mathbb{R}^n} + \langle b, y\rangle_{\mathbb{R}^n}$$

*for some $a, b \in \mathbb{R}^n$.*

**Definition 12.5.3.** *We say that a function $f : \mathbb{H}^n \to \mathbb{R}$ is* ***Pansu differentiable*** *in a point $p_0 \in \mathbb{H}^n$ if there exists a unique linear function $L : \mathbb{H}^n \to \mathbb{R}$ such that*

$$\lim_{p \to p_0} \frac{f(p) - f(p_0) - L(p_0^{-1} \cdot p)}{d_\infty(p, p_0)} = 0$$

*In this case we denote the function $L$ by $d_{\mathbb{H}} f(p_0)$.*

**Definition 12.5.4.** *We say that a function $f : \mathbb{G} \to \mathbb{R}$ is* ***Lipschitz continuous*** *if there exists a constant $C > 0$ such that for each $x, y \in \mathbb{H}^n$ it holds*

$$|f(x) - f(y)| \leq C d_\infty(x, y)$$

*We denote by $Lip(\mathbb{H}^n, \mathbb{R})$ the space of all Lipschitz continuous functions from $\mathbb{H}^n$ to $\mathbb{R}$.*

The following result holds

**Theorem 12.5.5 (Pansu's Theorem).** *Let $f : \mathbb{H}^n \to \mathbb{R}$ be a Lipschitz function. Then $f$ is Pansu differentiable in $p$ for $\mathcal{L}^{2n+1}$-a.e. $p \in \mathbb{H}^n$.*

Thanks to the previous result we can define a notion of "vertical plane" in the Heisenberg group $\mathbb{H}^n$ as the level set of a linear function. Since from Remark 12.5.2 we know that a linear map $L : \mathbb{H}^n \to \mathbb{R}$ has to be of the form

$$L(x, y, t) = \langle a, x\rangle_{\mathbb{R}^n} + \langle b, y\rangle_{\mathbb{R}^n}$$

for some $a, b, \in \mathbb{R}^n$, a "vertical plane" $V$ in $\mathbb{H}^n$ is a set of the form

$$V = \{(x, y, t) \in \mathbb{H}^n \mid \langle a, x\rangle_{\mathbb{R}^n} + \langle b, y\rangle_{\mathbb{R}^n} = c\}$$

for some $a, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We note that we can see a vertical plane $V$ as the left translation of the maximal subgroup $V_0$ of $\mathbb{H}^n$, $V = P \cdot V_0$, where $P \in V$ and

$$V_0 = \{(x, y, t) \in \mathbb{H}^n \mid \langle a, x\rangle_{\mathbb{R}^n} + \langle b, y\rangle_{\mathbb{R}^n} = 0\}$$

## 12.6 $\mathbb{H}$-perimeter in $\mathbb{H}^n$

In this section we want to introduce the notion of $\mathbb{H}$-perimeter in the same way we have done it in the Euclidean case. First of all we need to define a notion of divergence that takes into account of the geometry of our space $\mathbb{H}$ (see Definition 12.6.3). Then we can define the $\mathbb{H}$-perimeter in a open set $\Omega$ of a measurable set $E \subset \mathbb{H}^n$ as the variation of its characteristic function in $\Omega$ (see Definition 12.6.4); also in this case, thanks to th Riesz Representation Theorem, we can introduce an horizontal normal $\nu_E$. Finally we introduce the notion of $\mathbb{H}$-reduced boundary of an $\mathbb{H}$-Caccioppoli set in the same way we have done for the Euclidean case. We will underline the problems that arise when we try to state a structure theorem for the $\mathbb{H}$-reduced boundary of a $\mathbb{H}$-Caccioppoli set, and that motivate the notions we will introduce in the following sections.

### 12.6.1 Differential structure of $\mathbb{H}^n$

As usual we will identify the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ as first order operators.

**Definition 12.6.1.** *We define the* ***horizontal subbundle*** *$H\mathbb{H}^n$ as the vector subbundle of $T\mathbb{H}^n$, the tangent boundel of $\mathbb{H}^n$, generates by the vectors $X_1, \ldots, X_n, Y_1, \ldots, Y_n$.*

Now, since each fiber of $H\mathbb{H}^n$ can be canonically identified with a $2n$-dimensional subspace of $\mathbb{R}^{2n+1}$, we can identify each section $\varphi$ of $H\mathbb{H}^n$ with a map $\varphi : \mathbb{H}^n \to \mathbb{R}^{2n}$. Moreover, for each point $P \in \mathbb{H}^n$, we can endowed its horizontal fiber $H_P\mathbb{H}^n$ with a scalar product $\langle\cdot,\cdot\rangle_P$ and the associate norm $|\cdot|_P$, that makes the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$ orthonormal. Hence we can identify each section $\varphi$ with the function

$$\varphi = (\varphi_1, \dots, \varphi_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$$

such that

$$\varphi = \sum_{i=1}^{n} \varphi_i X_i + \sum_{i=1}^{n} \varphi_{n+i} Y_i$$

**Definition 12.6.2.** *We denote by $C^k(\mathbb{H}^n, H\mathbb{H}^n)$ the space of all $C^k$ continuous section of $H\mathbb{H}^n$, where the $C^k$ regularity is understood as regularity between smooth manifolds.*

**Definition 12.6.3.** *Let $\Omega$ be an open set of $\mathbb{H}^n$, and $\varphi = (\varphi_1, \dots, \varphi_{2n}) \in C^1(\mathbb{H}^n, H\mathbb{H}^n)$. We define the* ***horizontal divergence*** $\mathrm{div}_{\mathbb{H}}(\varphi)$ *as*

$$\mathrm{div}_{\mathbb{H}}(\varphi) := \sum_{i=1}^{n} \Big( X_i\varphi_i + Y_i\varphi_{n+i} \Big)$$

### 12.6.2 $\mathbb{H}$-perimeter

In this section we introduce the notions of $\mathbb{H}$-perimeter and of $\mathbb{H}$-reduced boundary of a measurable set $E \subset \mathbb{H}^n$ in the same way we have done for the Euclidean case.

**Notation**: we will use the symbols $\mathcal{H}^m_\infty$ and $\mathcal{S}^m_\infty$ to denote the $m$-dimensional Hausforff measure in $\mathbb{H}^n$ with respect to the distance $d_\infty$.

**Definition 12.6.4.** *Let $E \subset \mathbb{H}^n$ be a measurable set, and let $\Omega \subset \mathbb{H}^n$ be an open set. We define the $\mathbb{H}$-**perimeter** of $E$ in $\Omega$, $|\partial E|_{\mathbb{H}}(\Omega)$ as the $\mathbb{H}$-total variation of its characteristic function in $\Omega$, i.e.*

$$|\partial E|_{\mathbb{H}}(\Omega) := \sup\Big\{ \int_E \mathrm{div}_{\mathbb{H}}(\varphi)\, \mathrm{d}\mathcal{L}^{2n+1} \mid \varphi \in C^1_c(\Omega; H\mathbb{H}^n)\,,\ |\varphi|_P \le 1\ \forall\, P \in \mathbb{H}^n \Big\}$$

*We say that a set $E$ is a $\mathbb{H}$-**Caccioppoli set** in $\Omega$ if $|\partial E|_{\mathbb{H}}(\Omega) < \infty$.*

Using the Riesz Representation Theorem, we can obtain the following

**Theorem 12.6.5.** *Let $E$ be an ℍ-Caccioppoli set in $\Omega$. Then there exists a unique $|\partial E|_{\mathbb{H}}$-measurable section $\nu_E : \Omega \to H\mathbb{H}$ such that*

- $|\nu_E(x)|_P = 1$ *for* $|\partial E|$*-a.e.* $P \in \mathbb{H}^n$
- $\displaystyle\int_E \mathrm{div}_{\mathbb{H}}(\varphi)\,\mathrm{d}\mathcal{L}^{2n+1} = -\int_{\mathbb{H}^n} \langle \varphi, \nu_E \rangle \,\mathrm{d}|\partial E|_{\mathbb{H}}$ *for all* $\varphi \in C_c^1(\Omega; H\mathbb{H}^n)$

*Here the measurability of $\nu_E$ means that its coordinates $\nu_1, \dots, \nu_{2n}$ are $|\partial E|_{\mathbb{H}}$-measurable functions. We will call $\nu_E$ the* ***horizontal inward normal*** *to $E$.*

We have the following representation result

**Proposition 12.6.6.** *Let $E \subset \mathbb{H}^n$ be an Euclidean Lipschiz open bounded set. Then*

$$|\partial E|_{\mathbb{H}} = \sqrt{\sum_{i=1}^{n} \left( \langle X_i, \nu \rangle_{\mathbb{R}^n}^2 + \langle Y_{n+i}, \nu \rangle_{\mathbb{R}^n}^2 \right)} \mathcal{H}^{2n} \llcorner \partial E$$

*where $\nu$ denotes the Euclidean normal to $\partial E$.*
*Moreover any Euclidean Caccippoli set $E$ in $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ is an ℍ-Caccioppoli set, and the $|\partial E|_{\mathbb{H}}$ is absolutely continuous with respect to the Euclidean surface measure on $\partial E$.*

**Remark 12.6.7.** *The above result is strict, in the sense that there are ℍ-Caccioppoli sets that are not Euclidean Caccioppoli sets (see [Vit08], Example* 3.8*).*

We have the following two properties for the ℍ-perimeter: let $E \subset \mathbb{H}^n$ be a measurable set, $\Omega$ an open set of $\mathbb{H}^n$, $x \in \mathbb{H}^n$ and $r > 0$; then we have

- $|\partial(x \cdot E)|_{\mathbb{H}}(x \cdot \Omega) = |\partial E|_{\mathbb{H}}(\Omega)$
- $|\partial(\delta_{rE})|_{\mathbb{H}}(\delta_r(\Omega)) = r^{(2n+2)-1} |\partial E|_{\mathbb{H}}(\Omega)$

Now we want to define the ℍ-reduced boundary of an ℍ-Caccioppoli set.

**Definition 12.6.8.** *Let $E \subset \mathbb{H}^n$ be an ℍ-Caccioppoli set. We define the* ℍ***-reduced bounday*** *$\partial_{\mathbb{H}}^* E$ of $E$ as the set of points $P \in \mathbb{H}^n$ such that*

- $|\partial E|_{\mathbb{H}}(U_r^c(P)) > 0$ *for all* $r > 0$
- $|\nu_E(P)|_P = 1$
- $\displaystyle\lim_{r \to 0} \fint_{U_r^c(P)} \nu_E \,\mathrm{d}|\partial E|_{\mathbb{H}} = \nu_E(P)$

**Problem**: can I say that $\partial^*_{\mathbb{H}} E$ is not empty? We recall that in the Euclidean case we can conclude that $|\partial E|_{Eucli}$-a.e. point $P \in \mathbb{R}^n$ belongs to the reduced boundary $\partial^* E$ thanks to the Lebesgue's point Theorem (see Theorem 2.7.10) which state that, if $\mu$ is a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n; \mu)$, then

$$\lim_{r \to 0} \fint_{B_r(x)} f \,\mathrm{d}\mu = f(x) \quad \text{for} \mu - \text{a.e.} x \in \mathbb{R}^n$$

To prove this result we have used the Differentiation Theorem for Radon measures in $\mathbb{R}^n$ (Theorem 2.7.4) that used the Besicovitch's covering Theorem in $\mathbb{R}^n$ (Theorem 2.6.6). Since $\mathbb{H}^n$ is a metric space we would like to use the results of Chapter 4: but the Radon measure $|\partial E|_{\mathbb{H}}$ is not doubling, as we can see using Proposition 4, and the space $\mathbb{H}^n$ is not directionally limited (see Remark 12.4.4). Nethertheless it has been proved in [Amb01] the following result

**Theorem 12.6.9.** *Let $E$ be an $\mathbb{H}$-Caccioppoli set. Then*

$$\lim_{r \to 0} \fint_{U^c_r(P)} \nu_E \,\mathrm{d}|\partial E|_{\mathbb{H}} = \nu_E(P) \qquad \textit{for } |\partial E|_{\mathbb{H}} - \textit{a.e. } P \in \mathbb{H}^n$$

This result allows us to conclude that $|\partial E|_{\mathbb{H}}$-a.e. point $P \in \mathbb{H}^n$ belongs to $\partial^*_{\mathbb{H}} E$.

**Note**: it is still an open problem if the result holds also for generic Radon measure on $\mathbb{H}^n$: let $\mu$ be a Radon measure on $\mathbb{H}^n$, and let $f \in L^1_{loc}(\mathbb{H}^n, \mu)$; it is true that

$$\lim_{r \to 0} \fint_{U^c_r(P)} f \,\mathrm{d}\mu = f(P)$$

for $\mu$-a.e. $P \in \mathbb{H}^n$?

Now we want to state a rectificability theorem for the $\mathbb{H}$-reduced boundary of the same spirit of those of de Giorgi in $\mathbb{R}^n$. To do this we need to define a suitable notion of regular surface in $\mathbb{H}^n$.

## 12.7 $\mathbb{H}$-regular surfaces and Implicit Function Theorem

In this section we define a notion of regular surface in $\mathbb{H}^n$ that seems to be the correct one.

**Definition 12.7.1.** *Let $\Omega$ be an open set in $\mathbb{H}^n$. We denote by $C^1_{\mathbb{H}}(\Omega)$ the set of continuous real functions $f$ in $\Omega$ such that the distributional derivate*

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$$

*is represented by a $C^1$ section of $H\mathbb{H}^n$.*
*We will denote by $C^k_{\mathbb{H}}(\Omega; H\mathbb{H}^n)$ the set of all sections $\varphi$ of $H\mathbb{H}^n$ whose canonical coordinates $\varphi_i$ belong to $C^k_{\mathbb{H}}(\Omega)$ for all $i = 1, \dots, 2n$.*

**Note**: the inclusion $C^1(\Omega) \subset C^1_{\mathbb{H}}(\Omega)$ is **strict**.

**Definition 12.7.2.** *Let $P = (x, y, t) \in \mathbb{H}^n$ and $P_0 \in \mathbb{H}^n$. We define*

$$\pi_{P_0}(P) := \sum_{i=1}^{n} x_i X_i + \sum_{i=1}^{n} y_i Y_i$$

*Then the map $P_0 \mapsto \pi_{P_0}(P)$ is a smooth section of $H\mathbb{H}^n$.*

**Definition 12.7.3.** *We say that $S \subset \mathbb{H}^n$ is a $\mathbb{H}$-**regular hypersurface** if for every $P \in S$ there exists an open ball $U^c_r(P)$ and a function $f \in C^1_{\mathbb{H}}(U^c_r(P))$ such that*

- $\nabla_{\mathbb{H}} f \neq 0$
- $S \cap U^c_r(P) = \{ Q \in U^c_r(P) \mid f(Q) = 0 \}$

*We will also denote by $\nu_S(P)$ the **horizontal normal** to $S$ at the point $P$, i.e. the vector*

$$\nu_S(P) := -\frac{\nabla_{\mathbb{H}}(P)}{|\nabla_{\mathbb{H}}(P)|}$$

*In what follows we will assume, and it is not restrictive, that $X_1 f \neq 0$.*

**Note**: it can be proved that $\nu_S(P)$ is continuous and does not depend on the choise of the function $f$.

**Remark 12.7.4.** *In [KSC04] it has been shown an example of $\mathbb{H}$-regular hypersurface in $S \subset \mathbb{H}^1$ such that $S$ has (Euclidean) Hausdorff dimension 2.5.*

Toghether with the notion of $\mathbb{H}$-regular hypersurface we give a notion of "tangent hyperplane " to an $\mathbb{H}$-regular hypersurface.

**Definition 12.7.5.** *Let $S \subset \mathbb{H}^n$ be an $\mathbb{H}$-regular hypersurface. Define the* ***tangent group*** *$T^g_{\mathbb{H}}S(P)$ to $S$ in $P$ as*

$$T^g_{\mathbb{H}}S(P) := \{\, Q \in \mathbb{H}^n \mid \langle\, \nabla_{\mathbb{H}}(f \circ l_P)(0), \pi_0(Q)\,\rangle = 0 \,\}$$

*where $f$ is any function that define $S$ near $P$.*

**Note**: the above definition does not depend on the choise of the function $f$. Moreover one can equivalently define the tangent group to $S$ in $P$ as

$$T^g_{\mathbb{H}}S(P) := \{\, Q \in \mathbb{H}^n \mid \langle\, \nu_{P^{-1}\cdot S}(0), \pi_0(Q)\,\rangle = 0 \,\}$$

**Definition 12.7.6.** *The* ***tangent plane*** *to $S$ in $P$ is the lateral*

$$T_{\mathbb{H}}S(P) := P \cdot T^g_{\mathbb{H}}S(P)$$

The definition of $\mathbb{H}$-regular hypersurfaces seems to be a good one because it produced some important results. One of the most important is an Implicit Function Theorem for $\mathbb{H}$-regular surfaces. In the Euclidean setting the implicit Function Theorem tells us that we can locally see a $C^1$ regular surface $S$ as the graph of $C^1$ functions defined on hyperplanes. Here the role of hyperplanes (see Section 12.5) is played by maximal subgroups of $\mathbb{H}^n$, that are sets of the type

$$V_\omega := \Big\{\, Q \in \mathbb{H}^n \mid \Big\langle\, \sum_{i=1}^{n} \Big(\omega_i X_i + \omega_{n+i} Y_i\Big), \pi_0(Q)\,\Big\rangle = 0 \,\Big\}$$

for some $\omega \in \mathbb{R}^{2n}$. Note that for an $\mathbb{H}$-regular hypersurface we have that $T^g_{\mathbb{H}}S(P) = V_{\nu_{P^{-1}\cdot Q}(0)}$. In what follows we will focus our attenction on intrinsic graph over the hyperplane

$$V_1 := V_{(1,0,\dots,0)} = \{\, Q \in \mathbb{H}^n \mid x_1 = 0 \,\}$$

We want to identify $V_1$ with $\mathbb{R}^{2n}$. To do this we define the map $\mathfrak{i}$ as follows: if $n = 1$

$$\begin{aligned} \iota \;\; : \;\; & \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau \to V_1 \\ & (\eta,\tau) \mapsto (0,\eta,\tau) \end{aligned}$$

and for $n \geq 1$

$$\begin{aligned} \iota \;\; : \;\; & \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}^{2n-2}_{\nu=(\nu_2,\dots,\nu_n,\nu_{n+2},\dots,\nu_{2n})} \times \mathbb{R}_\tau \to V_1 \\ & (\eta,\nu,\tau) \mapsto (0,\nu_2,\dots,\nu_n,\eta,\nu_{n+2},\dots,\nu_{2n},\tau) \end{aligned}$$

Moreover we use the notation, if $s \in \mathbb{R}$, $se_1 := \exp(sX_1) = (s,0,\dots,0)$.

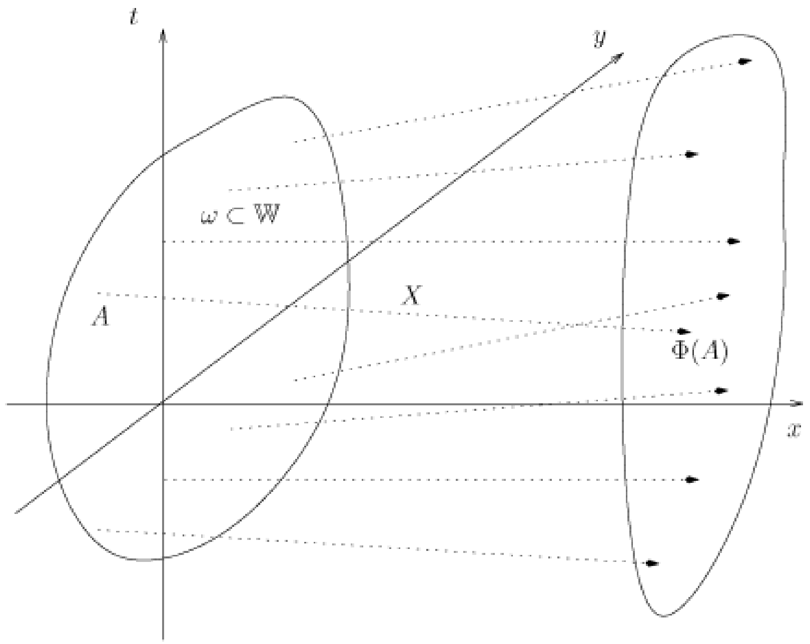



Figure 12.4: Intrinsic graph

**Definition 12.7.7.** *Let $\omega$ be an open subset of $\mathbb{R}^{2n}$, and let $\phi$ be a real function defined on it. The **intrinsic** $X_1$**-graph** of $\phi$ is the map*

$$\begin{array}{cccc} \Phi: & \omega & \to & \mathbb{H}^n \\ & A & \mapsto & \iota(A) \cdot \phi(A) e_1 \end{array}$$

In coordinates we have that, if $n \geq 1$,

$$\Phi(\eta, \nu, \tau) = (\phi(\eta, \nu, \tau), \nu_2, \ldots, \nu_n, \eta, \nu_{n+2}, \ldots, \tau + 2\eta\phi(\eta, \nu, \tau))$$

and if $n = 1$

$$\Phi(\eta, \nu, \tau) = (\phi(\eta, \nu, \tau), \eta, \tau + 2\eta\phi(\eta, \nu, \tau))$$

One could also interpret the notion of intrinsic $X_1$-graph in this way: start from the point $\iota(A) \in V_1 \subset \mathbb{H}^n$ and follow the flux of the field $X_1$ (which is a sort of "normal direction" to $V_1$) for a time $\phi(A)$, then the point one reaches is exactly $\Phi(A)$. Observe that this is exactly what happens for Euclidean graphs: one starts from a point of the hyperplane and follows the flux of the normal for a length given by the function itself, thus reaching the graph.

Then we have the following important result (see [FSSC01])

**Theorem 12.7.8 (Implicit Function Theorem).** *Let $\Omega$ be an open set in $\mathbb{H}^n$, $0 \in \Omega$, and let $f \in C^1_{\mathbb{H}}(\Omega)$ be such that $X_1 f(0) > 0$ and $f(0) = 0$. Let*

$$E := \{P \in \Omega \mid f(P) < 0\} \quad \textit{and} \quad S := \{P \in \Omega \mid f(P) = 0\}$$

*Then there exist $\delta, h > 0$ such that if we put $I := [-\delta, \delta] \times [-\delta, \delta]^{2n-2} \times [-\delta^2, \delta^2] \subset \mathbb{R}^{2n}_{\eta,\nu,\tau}$, $J := \{(s, 0, \dots, 0) \in \mathbb{H}^n \mid s \in [-h, h]\}$ and $\mathcal{U} := \iota \cdot J$ we have that*

$$\begin{aligned} &E \textit{ has finite } \mathbb{H}-\textit{perimeter in } \mathcal{U} \\ &\partial E \cap \mathcal{U} = S \cap \mathcal{U} \\ &\nu_E(P) = \nu_S(P) \quad \textit{for all } P \in S \cap \mathcal{U} \end{aligned}$$

*Moreover there exists a unique function $\phi : I \to [-h, h]$ such that $S \cap \overline{\mathcal{U}} = \Phi(I)$ where $\Phi : I \to \mathbb{H}^n$ is the $\Phi$ is the $X_1$ graph of $\phi$ in $I$, and the $\mathbb{H}$-perimeter has the integral representation*

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = \int_I \frac{|\nabla_{\mathbb{H}} f|}{X_1 f}(\Phi(A))\, \mathrm{d}\mathcal{L}^{2n}(A)$$

*Finally the $\mathbb{H}$-perimeter measure $|\partial E|_{\mathbb{H}}$ coincides with $c(n)\mathcal{S}^{Q-1}_{\infty} \llcorner S$, where the constant $c(n)$ depends only on $n$.*

**Note**: it can be shown that it is not restrictive to consider only $X_1$-graphs, because similar results can be obtained if we consired $X_i$-graphs with $i \geq 2$ or $Y_i$-graphs.

Now we want to answer this question: given a function $\phi : \omega \to \mathbb{R}$, where $\omega$ is an open set of $\mathbb{R}^{2n}$, set $S := \Phi(\omega)$. There is a characterization of all the functions $\phi$ for whom $S$ is a $\mathbb{H}$-regular hypersurface? This problem has been solved in [ASCV06].

**Definition 12.7.9.** *Given a function $\phi : \omega \to \mathbb{R}$, where $\omega$ is an open set of $\mathbb{R}^{2n}$, we define the family of first order operators*

$$\begin{aligned} &\widetilde{X}_i\phi := \frac{\partial \phi}{\partial \nu_i} + 2\nu_{n+i}\frac{\partial \phi}{\partial \tau}, \quad \widetilde{Y}_i\phi := \frac{\partial \phi}{\partial \nu_{n+i}} - 2\nu_i\frac{\partial \phi}{\partial \tau}, \quad i = 2, \dots, n \\ &\widetilde{Y}_1\phi := \frac{\partial \phi}{\partial \eta}, \quad \widetilde{T} := \frac{\partial \phi}{\partial \tau} \\ &W^{\phi}_{n+1}\phi := \frac{\partial \phi}{\partial \eta} - 2\frac{\partial \phi^2}{\partial \tau} \end{aligned}$$

*and*

$$W^{\phi}\phi := \begin{cases} \Big(\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, W^{\phi}_{n+1}\phi, \widetilde{Y}_1\phi, \dots, \widetilde{Y}_n\phi\Big) & \text{,if } n \geq 2 \\ W^{\phi}_2\phi & \text{,if } n = 1 \end{cases}$$

*all intended in distributional sense.*

**Remark 12.7.10.** *The operator $W^{\phi}\phi$ is the* projection *of the gradient $\nabla_{\mathbb{H}}\phi$ on $T\mathbb{R}^{2n} \equiv \mathbb{R}^{2n}$.*

We have the following result: in particular we are interested in the second part of the theorem.

**Note**: for the notion of $W^{\phi}$-differentiability we adress the reader to [ASCV06].

**Theorem 12.7.11.** *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. Let $\Phi$ the $X_1$-graph of $\phi$ and define $S := \Phi(\omega)$. Then the following two conditions are equivalent*

- *$S$ is an $\mathbb{H}$-regular hypersurface and $\nu_S^1(P) < 0$ for all $P \in S$, where $\nu_S(P) = (\nu_S^1(P), \dots, \nu_S^{2n}(P))$ is the horizzonatal normal to $S$ in $P$*
- *the distribution $W^{\phi}\phi$ is represented by a continuous function and there exists a family $(\phi_\varepsilon)_{\varepsilon>0} \subset C^1(\omega)$ such that, for any open set $\omega' \Subset \omega$ we have*

  $$\phi_\varepsilon \to \phi \quad \textit{and} \quad W^{\phi_\varepsilon}\phi_\varepsilon \to W^{\phi}\phi$$

  *uniformly on $\omega'$*

*Moreover for all $P \in S$ we have*

$$\nu_S(P) = \Big( -\frac{1}{\sqrt{1 + |W^{\phi}\phi|^2}}, \frac{W^{\phi}\phi}{\sqrt{1 + |W^{\phi}\phi|^2}} \Big)(\Phi^{-1}(P))$$

*and*

$$c(n)\mathcal{S}_\infty^{Q-1}(S) = \int_\omega \sqrt{1 + |W^{\phi}\phi|^2}\, \mathrm{d}\mathcal{L}^{2n}$$

*where $\mathcal{L}^{2n}$ denotes the Lebesgue measure on $\mathbb{R}^{2n}$ and $c(n)$ is as in Theorem 12.7.8.*

Thanks to this result we can say that $W^{\phi}\phi$ seems to be the right counterpart of the Euclidean gradient for $C^1$ surfaces.

## 12.8 Rectifiability in $\mathbb{H}^n$

In this section we state the results for the rectifiability of the $\mathbb{H}$-reduced boundary, that are analogous of those for the Euclidean case. All the results in this section has been obtained in [FSSC01].

First of a all a blow-up Theorem holds: let $E$ be an $\mathbb{H}$-Caccioppoli set in an open set $\Omega$, and define, for $r > 0$ and $P_0 \in \partial^*_{\mathbb{H}} E$

$$E_{r,P_0} := \delta_{\frac{1}{r}}(l_{P_0^{-1}} E) = \{P \in \mathbb{H}^n \mid P_0 \cdot \delta_r(P) \in E\}$$

and for $\nu \in H_{P_0 \mathbb{H}^n}$ define the half-spaces $S^+_{\mathbb{H}}(\nu)$ and $S^-_{\mathbb{H}}(\nu)$ "ortoghonal" to $\nu$ as

$$S^+_{\mathbb{H}}(\nu) := \{\, P \in \mathbb{H}^n \mid \langle \pi_{P_0}(P), \nu \rangle \ \geq 0 \,\}$$
$$S^-_{\mathbb{H}}(\nu) := \{\, P \in \mathbb{H}^n \mid \langle \pi_{P_0}(P), \nu \rangle \ \leq 0 \,\}$$

Then it holds

**Theorem 12.8.1.** *Let $E$ be an $\mathbb{H}$-Caccioppoli set and let $P_0 \in \partial^*_{\mathbb{H}} E$. Then*

$$\lim_{r \to 0} \chi_{E_{r,P_0}} = \chi_{S^+_{\mathbb{H}}(\nu_E(P_0))} \quad \text{in } L^1_{loc}(\mathbb{H}^n)$$

*Moreover*

$$\lim_{r \to} |\partial E|_{\mathbb{H}}(U^c_R(P_0)) = |\partial S^+_{\mathbb{H}}(\nu_E(P_0))|(U^c_R(P_0)) = 2\omega_{2n-1} R^{2n+1}$$

*for any $R > 0$.*

**Definition 12.8.2.** *We say that a set $\Gamma \subset \mathbb{H}^n$ is $\mathbb{H}$-**rectificable** if*

$$\Gamma \subset N \cup \bigcup_{i=0}^{\infty} K_i$$

*where $\mathcal{H}^{Q-1}_\infty(N) = 0$ and each $K_i$ is a compact subset of an $\mathbb{H}$-regular hypersurface $S_i$.*

Then we have

**Theorem 12.8.3.** *If $E \subset \mathbb{H}^n$ is an $\mathbb{H}$-Caccioppoli set then its $\mathbb{H}$-reduced boundary is $\mathbb{H}$-rectificable. More precisely it is possible to find a decomposition*

$$\partial^*_{\mathbb{H}} E = N \cup \bigcup_{i=0}^{\infty} K_i$$

*such that $\mathcal{H}_\infty^{Q-1}(N) = 0$ and each $K_i$ is a compact subset of an $\mathbb{H}$-regular hypersurface $S_i$ with the property that*

$$\nu_E(P) = \nu_S(P) \quad \textit{for each } P \in K_i$$

*Finally one has*

$$|\partial E|_{\mathbb{H}} = \frac{2\omega_{n-1}}{\omega_{n+1}} \mathcal{S}_\infty^{Q-1} \llcorner \partial_{\mathbb{H}}^* E$$

**Corollary 12.8.4.** *If $E \subset \mathbb{H}^n$ is an $\mathbb{H}$-Caccioppoli set in a open $\Omega$ then*

$$\int_E \operatorname{div}_{\mathbb{H}}(\varphi)\, \mathrm{d}\mathcal{L}^{2n+1} = -\frac{2\omega_{2n-1}}{\omega_{2n+1}} \int_{\partial_{\mathbb{H}}^* E} \langle \nu_E, \varphi \rangle \, \mathrm{d}\mathcal{S}_\infty^{Q-1}$$

*for all the sections $\varphi \in C_c^1(\Omega; H\mathbb{H}^n)$.*

# Chapter 13

# The Bernstein Problem in $\mathbb{H}^n$

In this chapter we want to present the Bernstein Problem in the Heisenberg group $\mathbb{H}^n$. We have to find a counterpart of the Euclidean objects involved in the Euclidean Bernstein Problem. The notion of intrinsic vertical planes aries from the Pansu's Theorem, while the notion of subgraph in the Heisenberg group can be defined in two different ways: t-subgraphs, and $X_1$-subgraphs. We will present the Bernstein Problem with the notion of $X_1$-subgraphs. First of all in Section 13.1 we derive the minimal surface equation for $X_1$-graphs; then in Section 13.2 we will give two counterparts in $\mathbb{H}^n$ of the classical Bernstein Problem: we stress the fact that in the Euclidean case if $u : \mathbb{R}^n \to \mathbb{R}$ is an entire $C^2$ solution of the minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

then its subgraph is a minimizer for the perimeter in $\mathbb{R}^n$. In the Heisenberg group $\mathbb{H}^n$ an unexpected phenomena arises: there are examples of solutions of the minimal surface equation for $X_1$-graphs (13.4) whose $X_1$-subgraph is not a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^n$. This fact motivate us to give two formulations of the Bernstein Problem in the Heisenberg group (see Section 13.2). Moreover in Section 13.3 we prove a calibration method in the Heisenberg group that allows us to prove the minimality for the $\mathbb{H}$-perimeter of some important classes of $X_1$-subgraphs. Finally in Section 13.4 we state the important result obtained in [BASCV07] and we give counterexample of the validity of the Bernstein Problem in $\mathbb{H}^n$ in dimension $n \geq 5$. We remember that the Bernstein Problem in the Heisenberg group is still open in the cases $n = 2, 3, 4$.

## 13.1 Minimal surface equation for $X_1$-graphs

In this section we want to derive the minimal surface equation for $X_1$-graphs. Consider a $C^1$ function $\varphi : \omega \to \mathbb{R}$, where $\omega$ is an open subset of $\mathbb{R}^{2n}$, and let

$$E_\phi := \{\iota(A) \cdot (s, 0, \dots, 0) \mid A \in \omega,\ s < \phi(A)\}$$

be the $X_1$-subgraph of $\phi$, and let

$$C_\omega := \iota(\omega) \cdot \{(s, 0, \dots, 0) \in \mathbb{H}^n \mid s \in \mathbb{R}\}$$

be the cylinder of base $\iota(\omega)$ along $X_1$. Thanks to Theorem 12.7.11 we know that

$$|\partial E_\phi|_{\mathbb{H}}(C_{X_1})(\omega) = \int_\omega \sqrt{1 + |W^\phi \phi|^2}\, \mathrm{d}\mathcal{L}^{2n}$$

Now suppose that $E_\phi$ is a minimizer fot the $\mathbb{H}$-perimeter in $C_{X_1}(\omega)$; so if we fix $\psi \in C_c^\infty(\omega)$ and set $\phi_s := \phi + s\psi$, we have that, if we also assume that $\omega$ is compact (and it is not restrictive),

$$E_\phi \triangle E_{\phi_s} \Subset C_{X_1}(\omega)$$

and hence the sets $E_{\phi_s}$ are competitors with $E_\phi$ for the $\mathbb{H}$-perimeter. So if we define the function

$$g(s) := |\partial E_{\phi_s}|(C_{X_1}(\omega)) = \int_\omega \sqrt{1 + |W^{\phi_s} \phi_s|^2}\, \mathrm{d}\mathcal{L}^{2n} \tag{13.1}$$

we obtain that it must holds $g'(0) = 0$, since $E_\phi$ is an $\mathbb{H}$-minimizer. Now we want to compute explicitely $g'(0)$. In the following we will write $\widetilde{X}_j := \widetilde{Y}_{j-n}$ for $j = n+1, \dots, 2n$. Let us recall that, in order to integrate by parts, we have

$$\widetilde{X}_j^* = -\widetilde{X}_j \quad \forall\, 2 \le j \le 2n\,, \qquad \widetilde{T}^* = -\widetilde{T} \tag{13.2}$$

while

$$(W^\phi_{n+1})^* \psi = -W^\phi_{n+1}\psi + 4\psi \widetilde{T}\psi\,, \quad \forall\, \psi \in C^\infty \tag{13.3}$$

In fact

$$\begin{aligned}
\int_\omega (W^\phi_{n+1} f) g\, \mathrm{d}\mathcal{L}^{2n} &= \int_\omega \Big(\frac{\partial f}{\partial \nu} - 4\phi \frac{\partial f}{\partial \tau}\Big) g\, \mathrm{d}\mathcal{L}^{2n} \\
&= \int_\omega \Big(-g\frac{\partial g}{\partial \nu} + 4fg\frac{\partial \phi}{\partial \tau} + 4f\phi\frac{\partial g}{\partial \tau}\Big)\, \mathrm{d}\mathcal{L}^{2n} \\
&= \int_\omega -f\Big(-W^\phi_{n+1} g + 4g\frac{\partial \phi}{\partial \tau}\Big)\, \mathrm{d}\mathcal{L}^{2n}
\end{aligned}$$

Hence we have

$$\begin{aligned}
W^{\phi_s}_{n+1}\phi_s &= \widetilde{Y}_1\phi + s\widetilde{Y}_1\psi - 4(\phi + s\psi)(\widetilde{T}\phi + s\widetilde{T}\psi) \\
&= W^\phi_{n+1}\phi - s(W^\phi_{n+1})^*\psi - 4s^2\psi\widetilde{T}\psi
\end{aligned}$$

So we can rewrite (13.1) as

$$g(s)=\int_\omega\left[1+\sum_{\substack{j=2\\ j\neq n+1}}^{2n}(\widetilde{X}_j\phi+s\widetilde{X}_j\psi)^2(W^\phi_{n+1}\phi-s(W^\phi_{n+1})^*\psi-4s^2\psi\widetilde{T}\psi)^2\right]^{\frac{1}{2}}\mathrm{d}\mathcal{L}^{2n}$$

and hence, writing $\sum_j$ for $\displaystyle\sum_{\substack{j=2\\ j\neq n+1}}^{2n}$, we obtain

$$g'(s)=\int_\omega\frac{\sum_j\widetilde{X}_j\phi_s\widetilde{X}_j\psi+W^{\phi_s}_{n+1}\phi_s\Big(-(W^\phi_{n+1})^*\psi-8s\psi\widetilde{T}\psi\Big)}{\sqrt{1+|W^{\phi_s}\phi_s|^2}}\mathrm{d}\mathcal{L}^{2n}$$

in particular

$$g'(0)=\int_\omega\frac{\sum_j\widetilde{X}_j\phi\widetilde{X}_j\psi-W^{\phi_s}_{n+1}\phi_s(W^\phi_{n+1})^*\psi}{\sqrt{1+|W^\phi_\phi|^2}}\mathrm{d}\mathcal{L}^{2n}$$

Finally, integrating the previous equation by parts using (13.2) and (13.3), we obtain

$$g'(0)=\int_\omega\left[-\sum_j\widetilde{X}_j\Big(\frac{\widetilde{X}_j\phi}{\sqrt{1+|W^\phi\phi|^2}}\Big)-W^\phi_{n+1}\Big(\frac{W^\phi_{n+1}\phi}{\sqrt{1+|W^\phi\phi|}}\Big)\right]\psi\,\mathrm{d}\mathcal{L}^{2n}$$

for each $\psi\in C^\infty$. Hence the Euler equation for the area functional in $\mathbb{H}^n$ is

$$W^\phi\cdot\frac{W^\phi\phi}{\sqrt{1+|W^\phi\phi|^2}}=0\qquad\text{on }\omega\tag{13.4}$$

## 13.2 Formulations of the Bernstein Problem in $\mathbb{H}^n$ for intrinsic graphs

Now we want give some formulations of the Bernstein problem in the Heisenberg group $\mathbb{H}^n$. To do this we recall the classical formulation of the Bernstein Problem in the Euclidean setting

**The Bernstein Problem in $\mathbb{R}^n$ - version I**: are there entire $C^2$ solutions $u:\mathbb{R}^{n-1}\to\mathbb{R}$ of the minimal surface equation

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0$$

which do not parametrize hyperplanes?

This formulation, thanks to Proposition 11.0.12, is equivalent to the following

**The Bernstein Problem in $\mathbb{R}^n$ - version II**: let $u : \mathbb{R}^{n-1} \to \mathbb{R}$ be such that the subgraph $U$ of $u$ is a minimal set in $\mathbb{R}^n$. It is true that $\partial U$ is an hyperplane?

In the Heisenberg group $\mathbb{H}^n$ the notion of hyperplanes (maximal subgroups of $\mathbb{R}^n$) are replaced by the notion of *vertical planes*, i.e. sets $V \in \mathbb{H}^n$ such that

$$V = \{(x, y, t) \in \mathbb{H}^n \mid \langle a, x\rangle_{\mathbb{R}^n} + \langle b, y\rangle_{\mathbb{R}^n} = c\}$$

for some $a, b, \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We recall that the notion of subgraph of a function $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ can be replace with the notion of t-subgraph

$$E^t_\phi := \{(x, y, t) \in \mathbb{H}^n \mid t < \phi(x, y)\}$$

or with the notion of $X_1$-*subgraph*

$$E_\phi := \{(x, y, t) \in C_{X_1}(\omega) \mid x_1 < \phi \circ \iota^{-1}\big((x, y, t) \cdot (-x_1 e_1)\big)\}$$

Here we want to consider the notion of $X_1$-subgraphs in $\mathbb{H}^n$. First of all we note that the functions $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ of the form

$$\phi(\eta, \nu, \tau) := c + \langle(\eta, \nu), w\rangle_{\mathbb{R}^{2n-1}} \tag{13.5}$$

for some $w \in \mathbb{R}^{2n-1}$ and $c \in \mathbb{R}$ if $n \geq 2$, and

$$\phi(\eta, \tau) := c + \eta w$$

with $w \in \mathbb{R}$ if $n = 1$, parametrize exactly the vertical planes in $\mathbb{H}^n$. It is clear that such a functions satisfied the minimal surface equation for $X_1$-graphs (13.4) in $\mathbb{H}^n$. Moreover we will also prove in Section 13.3 that vertical planes are minimizers for the $\mathbb{H}$-perimeter.

So with this notions of hyperplanes and subgraphs we can give this two counterpart in $\mathbb{H}^n$ of the two formulations of the Bernstein Problem in $\mathbb{R}^n$:

**(B1) - Bernstein Problem in $\mathbb{H}^n$ - version I**: are there entire $C^2$ solutions of the minimal surface equation (13.4) wich do not parametrize vertical planes?

**(B2) - Bernstein Problem in $\mathbb{H}^n$ - version II**: let $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ be such that its $X_1$-subgraph $E_\phi$ is a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^n$. It is true that $\partial E_\phi$ is a vertical plane?

A main difference from the Euclidean case is that this two formulations **are not equivalent**! In fact there exists a $C^2$ functions $\phi : \mathbb{R}^2 \to \mathbb{R}$ that is a solution of the minimal surface equation (13.4), but such that whose subgraph $E_\phi$ is not a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^1$ and it is not a vertical plane. Such a function provided a positive answer to Problem ($B1$). The function $\phi$ is defined as, for $\alpha > 0$,

$$\phi(\eta, \tau) := -\frac{\alpha\eta\tau}{1 + 2\alpha\eta^2}$$

and has been obtained in [DGN08], where we adress the reader for the proof of the non-minimality of the subgraph of $\phi$. Here we prove that $\phi$ is a solution of the minimal surface equation (13.4). Since we are in $\mathbb{H}^1$ the operator $W^\phi\phi$ just become

$$W_2^\phi\phi := \frac{\partial\phi}{\partial\tau} - 4\phi\frac{\partial\phi}{\partial\tau}$$

and hence the minimal surface equation (13.4) becomes

$$W_2^\phi\left(\frac{W_2^\phi\phi}{\sqrt{1 + |W_2^\phi\phi|^2}}\right) = 0$$

Since

$$W_2^\phi\phi = -\frac{\alpha\tau}{1 + 2\alpha\eta^2}$$

we obtain that

$$W_2^\phi\left(\frac{W_2^\phi\phi}{\sqrt{1 + |W_2^\phi\phi|^2}}\right) = W_2^\phi\left(\frac{-\alpha\tau}{\sqrt{(1 + 2\alpha\eta^2)^2 + \alpha^2\tau^2}}\right)$$

$$= \frac{4\alpha^2\eta\tau}{((1 + 2\alpha\eta^2) + \alpha^2\tau^2)^{\frac{3}{2}}}(1 + 2\alpha\eta^2) + \frac{4\alpha\eta\tau}{1 + 2\alpha\eta^2}\frac{-\alpha(1 + 2\alpha\eta\tau)^2}{((1 + 2\alpha\eta^2)^2 + \alpha^2\tau^2)^{\frac{3}{2}}} = 0$$

and hence $\phi$ satisfied the minimal surface equation (13.4). This counterexample tells us that the area functional for $X_1$-graphs is **not** convex. Moreover $\partial E_\phi$ is not a vertical plane; in fact

$$\begin{aligned} E_\phi &= \{\phi(\eta, \tau), \eta, \tau + 2\eta\phi(\eta, \tau) \in \mathbb{H}^1 \mid (\eta, \tau) \in \mathbb{R}^2\} \\ &= \Big\{\Big(-\frac{\alpha\eta\tau}{1 + 2\alpha\eta^2}, \eta, \frac{\tau}{1 + 2\alpha\eta^2} \in \mathbb{H}^1 \mid (\eta, \tau) \in \mathbb{R}^2\Big)\Big\} \\ &= \{(x, y, t) \in \mathbb{H}^1 \mid x = -\alpha yt\} \end{aligned}$$

which is clearly not a vertical plane.

## 13.3 Calibration method for the $\mathbb{H}$-perimeter

In this section we want to prove an useful tool we will use to prove the $\mathbb{H}$-minimality of some sets. We will use the following

**Lemma 13.3.1.** *Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Carnot group, and et $\varrho \in C^\infty(\mathbb{R}^n)$ be such that $0 \le \varrho \le 1$, $\int_{\mathbb{R}^n} \varrho \, \mathrm{d}\mathcal{L}^n = 1$, $\mathrm{supp}(\varrho) \subset B_1(0)$ and $\varrho(x^{-1}) = \varrho(x)$ for all $x \in \mathbb{R}^n$. Let us denote*

$$\varrho_\varepsilon(x) := \varepsilon^{-Q} \varrho(\delta_{\frac{1}{\varepsilon}}(x)) \,, \qquad x \in \mathbb{R}^n$$

$$(\varrho_\varepsilon \star f)(x) := \int_{\mathbb{R}^n} \varrho_\varepsilon(y) f(y^{-1} \cdot x) \, \mathrm{d}\mathcal{L}^n(y) = \int_{\mathbb{R}^n} \varrho_\varepsilon(x \cdot y^{-1}) f(y) \, \mathrm{d}\mathcal{L}^n(y)$$

*Then*

- *if $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ then $\varrho_\varepsilon \star f \in C^\infty(\mathbb{R}^n)$ and $\varrho_\varepsilon \star f \to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$*
- $\mathrm{supp}(\varrho_\varepsilon \star f) \subset B_\varepsilon(0) \cdot \mathrm{supp}(f)$
- *$X_j(\varrho_\varepsilon \star \phi) = \varrho_\varepsilon \star X_j \phi$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$ and all $j = 1, \dots, m$*
- *$\int_{\mathbb{R}^n} (\varrho_\varepsilon \star f) g \, \mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^n} (\varrho_\varepsilon \star g) f \, \mathrm{d}\mathcal{L}^n$ for every $f \in L^\infty(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$*
- *if $f \in L^{\infty(\mathbb{R}^n)} \cap C^0(\Omega)$ for a suitable open set $\Omega \subset \mathbb{R}^n$, then $\varrho_\varepsilon \star f \to f$ uniformly on compact subsets of $\Omega$ as $\varepsilon \to 0$*

The result, obtained in [BASCV07], is the following

**Theorem 13.3.2.** *Let $E$ and $\Omega$ be respectively a measurable and an open set in $\mathbb{H}^n$, and define $\nu_E : \Omega \to H\mathbb{H}^n$ the horizontal normal to $E$ in $\Omega$. Let us assume*

- *$E$ has locally finite $\mathbb{H}$-perimeter in $\Omega$*
- *$\mathrm{div}_{\mathbb{H}}(\nu_E) = 0$ in $\Omega$ in distributional sense*
- *there exists an open set $\widetilde{\Omega} \subset \Omega$ such that $|\partial E|_{\mathbb{H}}(\Omega \setminus \widetilde{\Omega}) = 0$ and $\nu_E$ is continous in $\widetilde{\Omega}$*

*Then $E$ is a minimizer for the $\mathbb{H}$-perimeter in $\Omega$.*

*Proof. Step 1*: let $(\varrho_\varepsilon)_\varepsilon$ be a family of mollifiers such as in Lemma 13.3.1, and denote by $\overline{\nu} : \mathbb{H}^n \to H\mathbb{H}^n$ as $\overline{\nu} \equiv \nu_E$ in $\Omega$, $\overline{\nu} \equiv 0$ in $\mathbb{H}^n \setminus \Omega$. Let us denote

$$\nu_\varepsilon(x) := (\varrho_\varepsilon \star \overline{\nu})(x) = ((\varrho_\varepsilon \star \overline{\nu}_1)(x), \dots, (\varrho_\varepsilon \star \overline{\nu}_{2n})(x))$$

Fix an open set $\Omega' \Subset \Omega$; we want to prove that

$$\int_\Omega \varphi \mathrm{div}_{\mathbb{H}}(\nu_\varepsilon)\, \mathrm{d}\mathcal{L}^{2n+1} = 0 \tag{13.6}$$

for every $\varphi \in C_c^\infty(\Omega')$ and every $0 < \varepsilon < \frac{d(\Omega', \mathbb{R}^n \setminus \Omega)}{2}$.
Fix $\varphi \in C_c^\infty(\Omega')$; since from Lemma 13.3.1 we have that $\varphi_\varepsilon := \varrho_\varepsilon \star \varphi \in C_c^\infty(\Omega)$ and the operators $X_j$ are self-adjoint, where we write $X_j := Y_{j-n}$ for $j = n+1, \dots, 2n$, we have that

$$\begin{aligned}
\int_\Omega \mathrm{div}_{\mathbb{H}}(\nu_\varepsilon)\varphi\, \mathrm{d}\mathcal{L}^{2n+1} &= -\int_\Omega \sum_{j=1}^{2n} \langle \nu_\varepsilon, X_j\varphi\rangle\, \mathrm{d}\mathcal{L}^{2n+1} \\
&= -\int_\Omega \sum_{j=1}^{2n} \langle \nu, \varrho_\varepsilon \star (X_j\varphi)\rangle\, \mathrm{d}\mathcal{L}^{2n+1} \\
&= -\int_\Omega \sum_{j=1}^{2n} \langle \nu, X_j\varphi_\varepsilon\, \mathrm{d}\mathcal{L}^{2n+1} = 0
\end{aligned}$$

where in the last step we have take into account that $\mathrm{div}_{\mathbb{H}}(\nu) = 0$ in distributional sense. Hence from (13.6) we obtain that

$$\mathrm{div}_{\mathbb{H}}(\nu_\varepsilon) = 0 \quad \text{ in } \Omega \tag{13.7}$$

for every open set $\Omega' \Subset \Omega$ provided $0 < \varepsilon < \frac{d(\Omega', \mathbb{R}^n \setminus \Omega)}{2}$.

Now let $(\Omega_h)_h$ be a sequence such that $\Omega_h \Subset \Omega$, $\Omega_{h+1} \Subset \Omega_h$ and $\bigcup_{h=1}^\infty \Omega_h = \Omega$. From what we have just proved we can find for each $h$ a $\varepsilon_h$ such that (13.7) holds. Moreover from Lemma 13.3.1 we obtain that $\nu_{\varepsilon_h} \to \nu_E$ uniformly on compact subsets of $\widetilde{\Omega}$, and hence, since $|\partial E|_{\mathbb{H}}(\Omega \setminus \widetilde{\Omega}) = 0$, we obtain that $\nu_{\varepsilon_h}(x) \to \nu_E(x)$ for $|\partial E|_{\mathbb{H}}$-a.e. $x \in \Omega$.

*Step 2*: now we want to prove that $E$ is a minimizer for the $\mathbb{H}$-perimeter in $\Omega$. Fix an open set $\Omega' \Subset \Omega$ and a measurable set $F \subset \mathbb{H}^n$ such that $E \triangle F \Subset \Omega'$. Let $\Omega''$ any open set such that $E \triangle F \Subset \Omega'' \Subset \Omega'$. Let $\overline{h}$ and $\psi \in C_c^1(\Omega')$ be such that

$$\Omega' \subset \Omega_{\overline{h}}, \quad 0 \le \psi \le 1$$

$$\Omega'' \Subset \{\psi = 1\} \Subset \Omega' \Subset \Omega \tag{13.8}$$

Hence for every $h > \overline{h}$ it holds

$$\int_\Omega \langle \psi \nu_{\varepsilon_h}, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} = \int_\Omega \langle \psi \nu_{\varepsilon_h}, \nu_F \rangle \, \mathrm{d}|\partial F|_{\mathbb{H}} \tag{13.9}$$

In fact from (13.8) and $\mathrm{div}_{\mathbb{H}}(\nu_{\varepsilon_h}) = 0$ in $\Omega$ we have

$$\int_\Omega \langle \psi \nu_{\varepsilon_h}, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} - \int_\Omega \langle \psi \nu_{\varepsilon_h}, \nu_F \rangle \, \mathrm{d}|\partial F|_{\mathbb{H}}$$

$$= -\int_{\Omega'} (\chi_E - \chi_F) \mathrm{div}_{\mathbb{H}}(\psi \nu_{\varepsilon_h}) \, \mathrm{d}\mathcal{L}^{2n+1} = -\int_{\Omega''} (\chi_E - \chi_F) \mathrm{div}_{\mathbb{H}}(\nu_{\varepsilon_h}) \, \mathrm{d}\mathcal{L}^{2n+1} = 0$$

where we have also take into account that $E \equiv F$ in $\Omega' \setminus \Omega''$. Hence from (13.9) we obtain

$$|\partial F|_{\mathbb{H}}(\Omega') \geq \left| \int_\Omega \langle \psi \nu_{\varepsilon_h}, \nu_F \rangle \, \mathrm{d}|\partial F|_{\mathbb{H}} \right| = \left| \int_\Omega \langle \psi \nu_{\varepsilon_h}, \nu_E \rangle \, \mathrm{d}|\partial E|_{\mathbb{H}} \right|$$

Since $|\nu_{\varepsilon_h}| \equiv 1$ and $\nu_{\varepsilon_h}(x) \to \nu_E(x)$ for $|\partial E|_{\mathbb{H}}$-a-e $x \in \Omega$, letting $h \to \infty$ from the Lebesgue's convergence Theorem we obtain that

$$|\partial F|_{\mathbb{H}}(\Omega') \geq \int_{\Omega'} \psi \, \mathrm{d}|\partial E|_{\mathbb{H}} \geq |\partial E|_{\mathbb{H}}(\Omega'')$$

Now letting $\Omega'' \uparrow \Omega'$ we obtain the desired result. □

Thanks to this theorem we can prove that the vertical planes are minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^n$. In fact let $V$ be a vertical plane in $\mathbb{H}^n$ and let $\phi : \mathbb{R}^{2n} \to \mathbb{R}$

$$\phi(\eta, \nu\, \tau) := c + \langle (\eta, \nu), w \rangle_{\mathbb{R}^{2n-1}}$$

with $w \in \mathbb{R}^{2n-1}$, be a function that parametrize it (similar formula in the case $n = 1$). Since $\phi$ is of class $C^1$ from Theorem 12.7.11 and Theorem 12.7.8 we obtain that the inward normal to the $X_1$-subgraph $E_\phi$ of $\phi$ is constant, and hence, using Theorem 13.3.2, we obtain that $E_\phi$ is a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^n$.

## 13.4 Solutions to the Bernstein Problem in $\mathbb{H}^n$

In this section we want to state the nowaday results for the two formulations of the Bernstein Problems in $\mathbb{H}^n$ we have gave in Section 13.2.

### 13.4.1 The Bernstein Problem in $\mathbb{H}^1$

For the Bernstein Problem $(B2)$ in $\mathbb{H}^1$ we have the following result, obtained in [BASCV07]

**Theorem 13.4.1.** *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$ function, and let $E, S \subset \mathbb{H}^1$ be respectively the $X_1$-graph and the $X_1$-subgraph of $\phi$. Let us suppose that $E$ is a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^n$. Then $S$ is a vertical plane, i.e. $\phi(\eta\tau) = w\eta + c$ for all $(\eta, \tau) \in \mathbb{R}^2$ for some constants $w, c \in \mathbb{R}$.*

The assumption that $\phi$ is a $C^2$ function is crucial for the above result, because we can find a counterexample to the result is we drop that assumption. In fact it holds

**Theorem 13.4.2.** *Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$\theta(y, t) := -\mathrm{sgn}(t)\sqrt{|t|}$$

*Then the subgraph $E_\theta$ is a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^1$ and*

$$\partial E_\theta = \{(x, y, 2xy - x|x|) \in \mathbb{H}^1 \mid x, y \in \mathbb{R}\}$$

*is not a vertical plane.*

*Proof.* (*sketch*) Our aim is to apply Theorem 13.3.2 to obtain that $E_\theta$ is a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^1$. First of all we note that the intrinsic subgraph of $\theta$ is

$$\begin{aligned} E_\theta &= \{\iota((y,t)) \cdot se_1 \in \mathbb{H}^1 \mid (y,t) \in \mathbb{R}^2,\ s < \theta(y,t)\} \\ &= \{(s, y, t + 2sy) \in \mathbb{H}^1 \mid (y,t) \in \mathbb{R}^2,\ s < -\mathrm{sgn}(t)\sqrt{|t|}\} \\ &= \{(x,y,t) \in \mathbb{H}^1 \mid x < \theta(y, t - 2xy)\} \end{aligned}$$

Now, since the function $g(\tau) : \mathbb{R} \to \mathbb{R}$ defined by $g(\tau) := \mathrm{sgn}(\tau)\sqrt{|\tau|}$ is a strictly decreasing function and has as inverse the function $g^{-1}(x) := x|x|$, applying $g^{-1}$ to both member of $x < \theta(y, t - 2xy)$ and, for the decreasing, reverse the inequality sign, we obtain that

$$E_\theta = \{(x,y,t) \in \mathbb{H}^1 \mid f(x,y,t) < 0\}$$

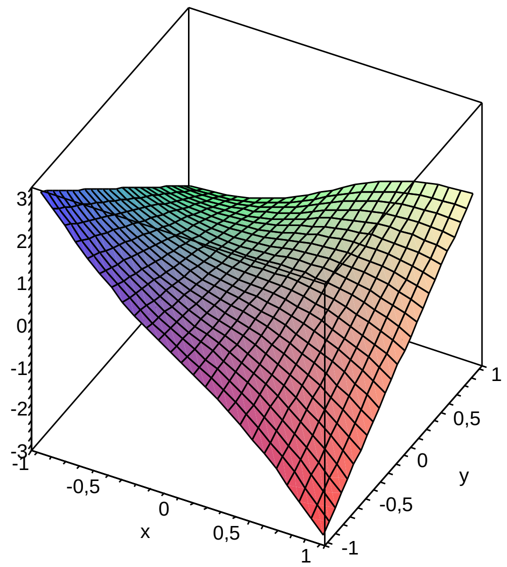



Figure 13.1: The $X_1$-graph of the function $\theta$

where $f(x, y, t) := t - 2xy + x|x|$. Hence $E_\theta$ can also be seen as the t-subgraph of the function $f$, and it is clearly not a vertical plane. Since $S := \partial E_\theta$ is (Euclidean) $C^{1,1}$-regular, for a result obtained in [FSSC01], we have that $E$ has locally finite Euclidean and $\mathbb{H}$-perimeters. Hence condition $(i)$ of Theorem 13.3.2 is satisfied.

Now let $S_0 := S \setminus \{(0, y, t) \in \mathbb{H}^1 \mid t \in \mathbb{R}\}$; since $f \in C^{1,1}(\mathbb{H}^1)$ and

$$X_1 f(x, y, t) = 2|x|, \quad Y_1 f(x, y, t) = -4x$$

from Theorem 12.7.8 we obtain that $S_0$ is an $\mathbb{H}$-regular hypersurface and

$$\nu_{E_\theta} = \nu_{S_0} = -\frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|}(x, y, t) = -\Big(\frac{1}{\sqrt{5}}, -\frac{x}{|x|}\frac{2}{\sqrt{5}}\Big)$$

If we set $\widetilde{\Omega} := \mathbb{H}^1 \setminus V_1 = \mathbb{H}^1 \setminus \{(x, y, t) \in \mathbb{H}^1 \mid x = 0\}$, and $K := \{(0, y, 0) \mid y \in \mathbb{R}\}$ we have

$$|\partial E_\theta|_{\mathbb{H}}(\Omega \setminus \widetilde{\Omega}) = |\partial E_\theta|_{\mathbb{H}}(K) \le \mathcal{S}^3_\infty(K) \le \mathcal{H}^2(K) = 0$$

where we have used the fact that

$$|\partial E_\theta|_{\mathbb{H}} \ll \mathcal{S}^3_\infty, \qquad \text{see [FSSC01]}$$

and

$$\mathcal{S}^3_\infty \ll \mathcal{H}^2, \qquad \text{see [FSSC03]}$$

Hence we have proved that $\nu_E \in C^0(\widetilde{\Omega})$ and $|\partial E|_{\mathbb{H}}(\Omega \setminus \widetilde{\Omega}) = 0$.

Finally we want to prove that $\mathrm{div}_{\mathbb{H}}(\nu_{E_\theta}) \equiv 0$ in $\mathbb{H}^1$ in distributional sense. In fact for each $\varphi \in C^1_c(\mathbb{H}^1)$ it holds

$$\int_{\mathbb{R}^3} (\nu_1 X_1\varphi + \nu_2 X_2\varphi)\,\mathrm{d}\mathcal{L}^3 = -\frac{1}{\sqrt{5}}\int_{\mathbb{R}^3} (\varphi_x + 2y\varphi_t)\,\mathrm{d}\mathcal{L}^3 + \frac{2}{\sqrt{5}}\int_{\mathbb{R}^3} \frac{x}{|x|}(\varphi_y - 2x\varphi_t)\,\mathrm{d}\mathcal{L}^3 = 0$$

because both integrals vanish.

Hence applying Theorem 13.3.2 we obtain that $E_\theta$ is a minimizer for the perimeter in $\mathbb{H}^1$. □

For the Bernstein Problem $(B1)$ we have already seen in Section 13.2 that there exists a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ that satisfied the minimal surface equation for $X_1$-subgraph in $He^1$ (13.4) and that does not parametrize a vertical plane.

### 13.4.2 The Bernstein Problem in $\mathbb{H}^n$ for $n \geq 2$

Let $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ be a function that does not depends on the variable $\tau$, that is

$$\phi(\eta, \nu, \tau) = \psi(\eta, \nu)$$

for some $\psi : \mathbb{R}^{2n-1} \to \mathbb{R}$. For such a function $\phi$ we have that

$$\widetilde{X}_j\phi = \frac{\partial\psi}{\partial\nu_j} \qquad \text{for } j = 2, \ldots, n$$

$$\widetilde{Y}_j\phi = \frac{\partial\psi}{\partial\nu_{n+j}} \qquad \text{for } j = 2, \ldots, n$$

and

$$W^\phi_{n+1}\phi = \frac{\partial\psi}{\partial\nu}$$

Hence the minimal surface equation (13.4) rewrites as the classical minimal surface equation for $\psi$

$$\mathrm{div}\left(\frac{\nabla\psi}{\sqrt{1 + |\nabla\psi|^2}}\right) \tag{13.10}$$

So, thanks to the result of the Euclidean case, we know that if $2n + 1 \geq 9$, that is $n \geq 5$, there exists functions $\psi : \mathbb{R}^{2n-1} \to \mathbb{R}$ that are solutions of the minimal surface equation (13.10) but that are not affine functions, i.e. the related function $\phi(\eta, \nu, \tau) = \psi(\eta, \nu)$ cannot be written as (13.5), and hence such that $\psi$ does not parametrize a vertical plane. Moreover, using Theorem 13.3.2, we can also prove that the $X_1$-subgraphs of such a functions are minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^1$. So fix a function $\psi : \mathbb{R}^{2n-1} \to \mathbb{R}$ that satisfied (13.10) that is not affine, and define the related function $\phi(\eta, \nu\tau) := \psi(\eta, \nu)$. Consider the smooth section $\nu : \mathbb{H}^n \to H\mathbb{H}^n$ given by

$$\begin{aligned}\nu(x,y,t) &:= \left( -\frac{1}{\sqrt{1+|W^\phi\phi|^2}}, \frac{W^\phi\phi}{\sqrt{1+|W^\phi\phi|^2}} \right)(\eta,\nu,0) \\ &= \left( -\frac{1}{\sqrt{1+|\nabla\psi|^2}}, \frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}} \right)(\eta,\nu)\end{aligned}$$

where we put $\eta := y_1$ and $\nu := (x_2, \dots, x_n, y_2, \dots, y_n)$.Hence $|\nu(P)|_P = 1$ for all $P \in \mathbb{H}^1$; moreover, thanks to Theorem 12.7.11, $\nu$ coincides with the horizzonatal normal to the $X_1$-graph of $\phi$. Finally it holds

$$\mathrm{div}_{\mathbb{H}}(\nu) = \sum_{j=1}^{2n} X_j \nu_j = 0$$

where we have used the fact the $\nu_1$ is indipendend from $x_1$ and that $\psi$ satisfied (13.10). Hence $\nu$ is a calibration for the $X_1$-graph of $\phi$ and hence from Theorem 13.3.2 we obtain that the $X_1$-graph of $\phi$ is a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^n$. This give an example of a function $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ solution of the minimal surface equation for $X_1$-graphs (13.4) that does not parametrize a vertical plane, and such that $\partial E_\phi$ is a minimizer for the $\mathbb{H}$-perimeter in $\mathbb{H}^n$.

The Bernstein Problem in the Heisenberg group $\mathbb{H}^n$ remains still open in the cases $n = 2, 3, 4$.

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