EXACT SOLUTIONS FOR THE DENOISING PROBLEM OF PIECEWISE CONSTANT IMAGES IN DIMENSION ONE

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Abstract. In this paper we provide a method to compute explicitly the solution of the total variation denoising problem with a L^p fidelity term, where $p > 1$, for piecewise constant data in dimension one.

1. Introduction

The images and the signals we use in everyday life are not perfect. External conditions, other then defects or limitations of the instruments we use to obtain them, affect the quality of the acquired data. Thus, it is important to be able to recover the *clean* object in the best possible way, *i.e.*, with optimal fidelity. If we denote it by u and the acquired, corrupted signal by f, it is usually assumed that the two are related as[1]:

$$f = Au + n\,, \tag{1}$$

where A is a bounded linear operator representing the blurring effect and n is the random noise. One of the aims of image reconstruction is to deblurring and denoising f in order to recover u (see [8, 22]).

Here we are interested in the denoising problem, *i.e.*, when the operator A is the identity and we have to remove the noise. Problem (1) is, in general, ill-posed (in the sense of Hadamard) and thus we need to regularize it (see [1, 44]). A widely used variational technique for this purpose was introduced by Rudin, Osher and Fatemi in [42], where they proposed to recover u in an open set $\Omega \subset \mathbb{R}^N$ via the minimization problem

$$\min_{u\in BV(\Omega),\ \|u-f\|^2_{L^2}=\sigma^2} |Du|(\Omega)\,, \tag{2}$$

for some fixed $\sigma > 0$, where f is suppose to be in $L^2(\Omega)$ and $|Du|(\Omega)$ denotes the total variation of the function u in Ω. The choice of $BV(\Omega)$ as the functional space where to perform the minimization is motivated by the fact that it permits the presence of discontinuities in the solutions, *i.e.*, the sharp edges of the objects in the image (actually, it can be shown that, in general, real images are *not* of bounded variation (see [30])). Problem (2) has been shown to be equivalent to the following penalized minimum problem (known as the total variation denoising model with L^2 fidelity term)

$$\min_{u\in BV(\Omega)} |Du|(\Omega) + \lambda\|u-f\|^2_{L^2(\Omega)}\,, \tag{3}$$

for some Lagrange multiplier $\lambda > 0$ (see [17]). Today's literature on the study of problem (3) is extensive, and here we limit ourselves to recall that:

[1] We do not want to be precise here.

properties of the solutions have been studied, for instance, in [2, 3, 4, 9, 10, 14, 15, 20, 23, 26, 28, 31, 35, 41, 45, 46], the analysis of variants of (3) that use the generalized total variation have been performed in [11, 12, 38, 40, 39], anisotropic models are undertaken in [24, 27, 29, 34], while the effects of considering high-order models have been investigated in [19, 21, 25, 32, 39]. Finally, other variants of (2) have been addressed in [6, 7, 37], and algorithmic considerations may be found in [13, 16, 18, 36].

In this paper we study the one dimensional case where f is a piecewise constant function and we generalize the L^2 fidelity term as an L^p fidelity term, with $p \in [1, \infty)$, *i.e.*, we consider the minimum problem

$$\min_{u \in BV(\Omega)} \mathcal{G}(u)\,, \tag{4}$$

where $\Omega := (a, b) \subset \mathbb{R}$ and

$$\mathcal{G}(u) := |Du|(\Omega) + \lambda \|u - f\|^p_{L^p(\Omega)}\,, \tag{5}$$

for a given initial piecewise constant data f. In the case $p > 1$ we are able to provide an analytic method to find the exact solution of (4).

The principal obs0,–bvtruction in obtaining an analytic method to compute (one) solution of the problem (4) in the case $p = 1$ is that explicit computations are difficult to perform, analytically, in the case $p \neq 2$. Thus, albeit we know that the solution u^p of (4) for a fixed λ and corresponding to the L^p fidelity term, for $p > 1$, will converge to a solution u^1 of (4) for the same λ, but with L^1 fidelity term, we cannot obtain it as an *explicit* limit of such solutions. Nevertheless, a finer analysis of the behavior of the solution for $p > 1$ is currently under investigation.

Our result extends the one obtained by Strong and Chan in [43], where they are able to obtain the exact solution for large λ in the case of a noisy f.

2. Settings

In this section we review the basic definitions of one dimensional functions of bounded variation. For more details, see [5, 33]. Here $a, b \in \mathbb{R}$ and $a < b$.

Definition 2.1. Let $u : (a, b) \to \mathbb{R}$. The *pointwise variation* of u in (a, b) is defined as

$$pV(u; a, b) := \sup \left\{ \sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)| \,:\, a < x_1 < \cdots < x_n < b \right\}.$$

Definition 2.2. For $u \in L^1\big((a, b)\big)$ its *total variation* in (a, b) is given by

$$|Du|\big((a, b)\big) := \sup \left\{ \int_a^b \varphi' u \,\mathrm{d}x \,:\, \varphi \in C_0^\infty\big((a, b)\big)\,, |\varphi| \leq 1 \right\}.$$

If $|Du|\big((a, b)\big) < \infty$, we say that u belongs to the space $BV\big((a, b)\big)$ of functions of bounded variation in (a, b).

The relation among the above objects is given by the following result.

Theorem 2.3. *Let $u \in L^1\big((a,b)\big)$ and define the* essential variation *of u as*

$$eV(u;a,b) := \inf\{\, pV(v;a,b) \,:\, v = u \quad L^1 - a.e.\ in\ (a,b)\,\}\,.$$

The above infimum is achieved and it coincides with $|Du|\big((a,b)\big)$.

The above result allows us to single out some well behaving representative of a BV function.

Definition 2.4. Let $u \in BV\big((a,b)\big)$. Any v with $v = u$ L^1-a.e. in (a,b) such that $pV(v;a,b) = eV(u;a,b) = |Du|\big((a,b)\big)$ is called a *good representative* of u.

3. The general structure of the solutions

We start by proving that a solution to the minimum problem (4) needs to have the same structure as$f$, *i.e.*, it has to be a piecewise constant function with its jump set contained in the jump set of f. In higher dimension, the inclusion[2] $J_u \subset J_f$ is well known (see [14] and [45]) in the case $p > 1$, while it is not always true if $p = 1$ (see [20] and [28]). The following result has been proved, with a different argument, in [12].

Theorem 3.1. *Let $f \in L^1\big((a,b)\big)$ and let $u \in BV\big((a,b)\big)$ be a solution of* (4)*. If f is constant in $(c,d) \subset (a,b)$, then u is constant in (c,d).*

Proof. Let $u \in BV\big((a,b)\big)$ and suppose it is a good representative such that

$$u(c) = \lim_{y \to c^-} u(y)\,, \qquad u(d) = \lim_{y \to d^+} u(y)\,.$$

Define the function

$$\widetilde{u} := \begin{cases} u & \text{in } (a,b)\backslash(c,d)\,, \\ t & \text{in } (c,d)\,, \end{cases}$$

where $t := \fint_c^d u$. We claim that

$$\mathcal{F}(\widetilde{u}) \le \mathcal{F}(u)\,,$$

where equality holds if and only if $u \equiv t$ in (c,d). We show that the above inequality holds separately for each term of the energy. The fact that the fidelity term decreases is due to Jensen's inequality. Indeed, by recalling that f is constant on (c,d), say $f \equiv \bar{f}$ in (c,d), we have that

$$\Big|\fint_c^d u(y)\,\mathrm{d}y - \bar{f}\,\Big|^p = \Big|\fint_c^d \big(u(y) - f\big)\,\mathrm{d}y\Big|^p \le \fint_c^d |u(y) - \bar{f}|^p\,\mathrm{d}y\,,$$

and, by integrating both sides on (c,d), we obtain

$$\int_c^d |t - \bar{f}|^p\,\mathrm{d}x \le \int_c^d |u(x) - \bar{f}|^p\,\mathrm{d}x\,,$$

where the equality case holds if and only if $u \equiv t$ in (c,d).

We now consider the total variation term. We have that

$$|D\widetilde{u}|([c,d]) = |u(c) - t| + |u(d) - t|\,,$$

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[2]With J_u we denote the jump set of $u \in BV(\Omega)$

Suppose, without loss of generality, that $u(c) \leq u(d)$. We will consider three cases: $t \in [u(c), u(d)]$, $t \leq u(c)$ and $t \geq u(d)$. In the first one, we simply notice that

$$|D\widetilde{u}|([c,d]) = u(d) - u(c) \leq |Du|([c,d])\,.$$

If $t \leq u(c)$, then there exists $x \in [c,d)$ such that $u(x) \leq t$. Thus,

$$\begin{aligned}|Du|([c,d]) &\geq \big(u(c) - u(x)\big) + \big(u(d) - u(x)\big) \geq \big(u(c) - t\big) + \big(u(d) - t\big) \\ &= |D\widetilde{u}|([c,d])\,.\end{aligned}$$

The case $t \geq \max\{u(c), u(d)\}$ can be treated similarly. This concludes the proof. □

The above result allows us to get the structure of minimizers of problem (4) in the case in which f is a piecewise constant function.

Corollary 3.2. *Let f be a piecewise constant function in (a,b),* i.e.,

$$f(x) = \sum_{i=1}^{k} f_i\, \chi_{(x_{i-1},x_i)}(x)\,, \qquad f_i \in \mathbb{R}\,.$$

Then any solution u of the minimum problem (4) is of the form

$$u(x) = \sum_{i=1}^{k} u_i\, \chi_{(x_{i-1},x_i)}(x)\,, \tag{6}$$

for some $(u_i)_{i=1}^k \subset \mathbb{R}$, not necessarily distinct from each other.

In particular, a function u of the form (6) is a solution of (4) if and only $\bar{u} := (u_1, \dots, u_k) \in \mathbb{R}^k$ is a solution of the minimum problem

$$\min_{v \in \mathbb{R}^k} G(v)\,, \tag{7}$$

where $G : \mathbb{R}^k \to \mathbb{R}$ is the function defined as

$$G(v) := \sum_{i=2}^{k} |v_i - v_{i-1}| + \lambda \sum_{i=1}^{k} L_i |f_i - v_i|^p\,, \tag{8}$$

with $v = (v_1, \dots, v_k)$.

Thus, hereafter we will concentrate on the study of the minimum problem (7). The issue in finding minimizers of G is that the functional has some regions where it is not differentiable (due to the first summation), and so, albeit it is convex (strictly, if $p > 1$), minimizers cannot be found directly as critical points. The aim of this paper is to provide a method to overcome this difficulty.

The cases $p = 1$ and $p > 1$ turn out to be quite different. Heuristically, the difference lies in the fact that, in the first case, the two terms of the energy are *of the same order* while, for $p > 1$, the fidelity term is of higher order than the total variation one. This leads to very different behavior of the solutions. A first difference between the two cases is the lack of uniqueness in the case $p = 1$ (see Proposition 4.1). However, in this regime it is possible to obtain a more rigid structure result for a particular solution of the minimum problem for $p = 1$.

Corollary 3.3. *For $p = 1$, there exists a solution u of the problem (7) such that $u_i \in \{f_1, \dots, f_k\}$ for every $i = 1, \dots, k$.*

Proof. For any given a pair of functions $s_1 : \{2, \dots, k\} \to \{0, 1\}$ and $s_2 : \{1, \dots, k\} \to \{0, 1\}$ consider the set $\mathcal{A}_{s_1,s_2} \subset \mathbb{R}^k$ such that

$$G(u) = \sum_{i=2}^{k} (-1)^{s_1(i)} (u_i - u_{i-1}) + \lambda \sum_{i=1}^{k} L_i (-1)^{s_2(i)} (f_i - u_i) , \tag{9}$$

for all $u \in \mathcal{A}_{s_1,s_2}$. We note that $\mathcal{A}_{s_1,s_2}$ could be empty. Then

$$\min_{\mathbb{R}^k} G = \min_{s_1,s_2} \min_{\mathcal{A}_{s_1,s_2}} G_{|\mathcal{A}_{s_1,s_2}} .$$

If $u \in \mathcal{A}_{s_1,s_2}$, then (9) can be written as

$$G(u) = v_\lambda^{s_1,s_2} \cdot u + c_\lambda^{s_1,s_2} , \tag{10}$$

for some $c_\lambda^{s_1,s_2} \in \mathbb{R}$ and $v_\lambda^{s_1,s_2} \in \mathbb{R}^k$, hence for any s_1 and s_2, the function G restricted to $\mathcal{A}_{s_1,s_2}$ is always minimized by a vector of the form

$$u_i = f_{\sigma(i)} ,$$

for some function $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$. This concludes the proof. □

The above result allows us to select a particular solution for the minimum problem in the case $p = 1$.

Definition 3.4. We will denote by u^λ a solution of the minimum problem (7) corresponding to the value λ. This will be *the* solution, if $p > 1$, while, for $p = 1$, it will be understood as *a* solution whose structure is those given by the previous result.

Remark 3.5. It is easy to see that $u_i \in [\min f, \max f]$ for every solution u.

In the rest of this section we want to understand the behavior of the solution u^λ in the limiting cases for λ, *i.e.*, when $\lambda \ll 1$ and when $\lambda \gg 1$. In the first case the predominant term of the energy is given by the total variation, thus we expect u^λ to minimizes it.

Lemma 3.6. *Fix $p \geq 1$, positive numbers $(L_i)_{i=1}^k$ and two constants $m < M$. Then, there exists a constant $\bar{\lambda} > 0$, depending only on p, $(L_i)_{i=1}^k$, m and M with the following property: for any piecewise constant function f such that $f \in [m, M]$ and any $\lambda \in (0, \bar{\lambda}]$, we have that u^λ is constant.*

In particular, if $p > 1$ then there exists $c \in \mathbb{R}$ such that $u_i^\lambda \equiv c$ for all $\lambda \in (0, \bar{\lambda}]$ and all $i = 1, \dots, k$.

Proof. We first treat the case $p > 1$. Assume u^λ is not constant and let $i \in \{1, \dots, k\}$ be such that $u_i^\lambda = \min\{u_j^\lambda : j = 1, \dots, k\}$. Let $r := \inf\{j \leq i : u_s = u_i \text{ for all } j \leq s \leq i\}$ and let $t := \sup\{j \geq i : u_s = u_i \text{ for all } i \leq s \leq j\}$. By hypothesis, either $r > 1$ or $t < k$. Consider, for $\varepsilon > 0$, the vector $u^\varepsilon \in \mathbb{R}^k$ defined as $u_j^\varepsilon := u_j + \varepsilon$ for $j = r, \dots, t$ and $u_j^\varepsilon := u_j^\lambda$ for all the other j's. Then, recalling that $u_j \in [m, M]$ for all $j = 1, \dots, k$, we have that

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{G(u^\varepsilon) - G(u^\lambda)}{\varepsilon} &= a + p\lambda(-1)^{s_i} L_i |u_i - f_i|^{p-1} \\ &\leq a + p\lambda(M - m)^{p-1} \max_{i=1,\dots,k} L_i , \end{aligned} \tag{11}$$

where $a \in \{-1,-2\}$ (in particular, $a=-1$ if $r=1$ or $t=k$ and $a=-2$ otherwise), and $s_i \in \{0,1\}$. Let

$$\bar{\lambda} := \frac{1}{p(M-m)^{p-1}\max_i L_i}\,.$$

If $\lambda < \bar{\lambda}$, from (11) we get that $G(u^\varepsilon) < G(u^\lambda)$. This means that u^λ has to be constant for $\lambda < \bar{\lambda}$. Moreover, it is easy to see that the function G restricted to the set $\{(u_1,\dots,u_k)\in\mathbb{R}^k \,:\, u_1=\cdots=u_k\}$ admits a unique minimizer, that is independent of λ.

We now have to prove that $u^{\bar{\lambda}}$ is constant. Assume that $u^\lambda_i \equiv c$ for for all $\lambda \in (0,\bar{\lambda})$ and all $i=1,\dots,k$. Let $\bar{c}\in\mathbb{R}^k$ be the vector given by $\bar{c}_i := c$. Then $G_\lambda(c) < G_\lambda(v)$ for all $v\in\mathbb{R}^k$ with $v\neq\bar{c}$ and all $\lambda\in(0,\bar{\lambda})$, where the subscript λ is to underline the dependence of G on λ. By letting $\lambda \nearrow \bar{\lambda}$, we get $G_{\bar{\lambda}}(c) < G_{\bar{\lambda}}(v)$ for all $v\in\mathbb{R}^k$ and thus $u^{\bar{\lambda}} = \bar{c}$.

Let us now treat the case $p=1$. Suppose that u^λ is not constant. Recalling that $u^\lambda_i \in \{f_1,\dots,f_k\}$, we have that

$$|Du^\lambda|(\Omega) \geq \min_i |f_i - f_{i-1}|\,.$$

On the other hand, for any function v such that $v\equiv c\in[\min f,\max f]$ in (a,b), it holds that

$$G(v) \leq \lambda k(\max_i L_i)(M-m)\,.$$

Set

$$\bar{\lambda} := \frac{\min_i |f_i - f_{i-1}|}{k(\max_i L_i)(M-m)}\,.$$

For $\lambda < \bar{\lambda}$ the above estimates show that u^λ must be constant.

Finally, in order to prove that also $u^{\bar{\lambda}}$ is constant, we reason as follows: we know that $u^\lambda = \bar{c}^\lambda$ for $\lambda\in(0,\bar{\lambda})$, for some $\bar{c}_\lambda = (c_\lambda,\dots,c_\lambda)\in\mathbb{R}^k$. Take $\lambda_n \nearrow \bar{\lambda}$. Since $c_{\lambda_n}\in[\min f,\max f]$, up to a not relabelled subsequence we have that $c_{\lambda_n}\to c$. We conclude that $G_{\bar{\lambda}}\big((c,\dots,c)\big) \leq G_{\bar{\lambda}}(v)$ for all $v\in\mathbb{R}^k$. □

We now consider the case $\lambda \gg 1$. Since

$$\lambda L_i |u^\lambda_i - f_i|^p \leq G(u^\lambda) \leq G(f) < \infty\,,$$

we know that

$$u^\lambda \to f \qquad \text{as } \lambda\to\infty\,. \tag{12}$$

The following results underline a first important difference between the cases $p=1$ and $p>1$. Indeed, if $p=1$ the limit (12) is reached for $\lambda<\infty$, while if $p>1$ only asymptotically.

Lemma 3.7. *Let $p>1$ and assume f is not constant. Then $u^\lambda\in(\min f,\max f)$ for all $\lambda>0$. In particular, f can never be a solution of the minimum problem.*

Proof. We first prove that u^λ cannot achieve the value $\min f$. Assume that $u^\lambda_i = \min f$ for some $i\in\{1,\dots,k\}$. Let $r\leq i\leq s$ be such that $u_j = u_i$ for all $j = r, \dots, s$. Consider, for $\varepsilon > 0$, the vector $u^\varepsilon \in \mathbb{R}^k$ given by $u^\varepsilon_j := u^\lambda_j + \varepsilon$ for $j = r, \dots, s$ and $u^\varepsilon_j := u^\lambda_j$ for all other j's. Then

$$\lim_{\varepsilon \to 0^+} \frac{G(u^\varepsilon) - G(u)}{\varepsilon} = a - p\lambda \sum_{j=r}^{s} L_j (f_j - u^\lambda_i)^{p-1} < 0\,,$$

where $a \in \{-1, -2\}$. This is in contradiction with the minimality of u^λ.

With a similar argument it is possible to show that u does not achieve the value $\max f$. □

Lemma 3.8. *Let $p = 1$. Then there exists $\bar\lambda > 0$ such that for all $\lambda \geq \bar\lambda$ the solution of the minimum problem* (7) *is unique and is given by f itself.*

Proof. Suppose that there exists a sequence $\lambda_j \to \infty$ for which $u^{\lambda_j}_i \neq f_i$ for all j's (this is possible, since k is finite). By recalling that $u^{\lambda_j}_i \in \{f_1, \dots, f_k\}$, setting

$$\bar\lambda := \frac{G(f)}{\min_i L_i \, \min_i |f_i - f_{i-1}|}\,,$$

we have, for $\lambda_j > \bar\lambda$, that

$$G(u^{\lambda_j}) \geq \lambda_j L_i |u^{\lambda_j}_i - f_i| > G(f)\,,$$

contradicting the minimality of u^{λ_j}. □

4. Explicit solutions in a simple case

Here we study the case in which we have just two grey levels, *i.e.*, $k = 2$. This analysis will underline some important features of the cases $p = 1$ and $p > 1$.

Proposition 4.1. *Let $f_1 < f_2$. Then the solutions u^λ of the minimum problem* (7) *in the case $p = 1$ are the following:*

- *if $L_1 > L_2$, set $\lambda^1_T := \frac{1}{L_2}$. Then*

$$\begin{cases} u^\lambda_1 = u^\lambda_2 = f_1 & \text{for } \lambda < \lambda^1_T\,, \\ u^\lambda_1 = f_1, u^\lambda_2 \in [f_1, f_2] & \text{for } \lambda = \lambda^1_T\,, \\ u^\lambda_1 = f_1, u^\lambda_2 = f_2 & \text{for } \lambda > \lambda^1_T\,, \end{cases}$$

- *if $L_1 = L_2$, set $\lambda^1_T := \frac{1}{L_1}$. Then*

$$\begin{cases} u^\lambda_1 \in [f_1, f_2], u^\lambda_2 \geq u_1 & \text{for } \lambda \leq \lambda^1_T\,, \\ u^\lambda_1 = f_1, u^\lambda_2 = f_2 & \text{for } \lambda > \lambda^1_T\,, \end{cases}$$

- *if $L_1 < L_2$, set $\lambda^1_T := \frac{1}{L_1}$. Then*

$$\begin{cases} u^\lambda_1 = u^\lambda_2 = f_2 & \text{for } \lambda < \lambda^1_T\,, \\ u^\lambda_1 \in [f_1, f_2], u^\lambda_2 = f_2 & \text{for } \lambda = \lambda^1_T\,, \\ u^\lambda_1 = f_1, u^\lambda_2 = f_2 & \text{for } \lambda > \lambda^1_T\,, \end{cases}$$

Proof. It is easy to see that we must have $f_1 \le u_1 \le u_2 \le f_2$. Thus, we consider the region

$$\mathcal{T} := \{ (u_1, u_2) \in \mathbb{R}^2 \ : \ f_1 \le u_1 \le u_2 \le f_2 \}, \tag{13}$$

and we rewrite the function G in $\mathcal{T}$ as

$$G(\bar{u}) = [\lambda L_1 - 1]u_1 + [1 - \lambda L_2]u_2 + \lambda[f_2 L_2 - f_1 L_1] = v_\lambda \cdot u + c_\lambda .$$

When minimizing G in $\mathcal{T}$, we can drop the term c_λ. Then, the minimizers, according to the position of the vector $\frac{v_\lambda}{|v_\lambda|}$ (well defined for all λ's, except in the case $L_1 = L_2$ and $\lambda = \frac{1}{L_1}$), are the following:

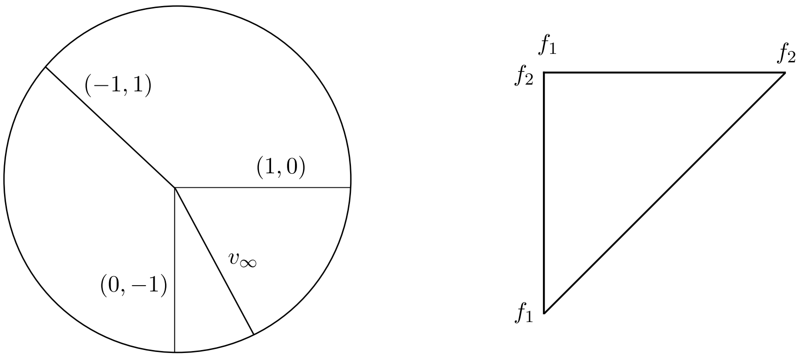


FIGURE 1. On the left it is displayed where the (renormalized) vector v_λ can vary: from v_1 for $\lambda = 0$ up tp (asymptotically) $v_\infty := \arctan \frac{L_2}{L_1}$. On the left the triangle where the vector u can vary.

Thus, by simply studying the sign of the components of v_λ, we obtain the desired result. Notice that the non uniqueness happens only when the vector v_λ is orthogonal to $\{x = y\} \subset \mathbb{R}^2$. □

In the case $p > 1$ the landscape of the solutions is quite different.

Proposition 4.2. *Let $f_1 < f_2$ and let $p > 1$. Define*

$$\lambda_T^p := \frac{1}{p} \frac{(L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}})^{p-1}}{L_1 L_2 (f_2 - f_1)^{p-1}} .$$

The solution u^λ of the minimum problem (7) is the following:

- *for $\lambda \le \lambda_T^p$*

$$u_1^\lambda = u_2^\lambda = \frac{L_1^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} f_1 + \frac{L_2^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} f_2 , \tag{14}$$

- *for $\lambda > \lambda_T^p$*

$$u_1^\lambda = f_1 + \frac{1}{(p\lambda L_1)^{\frac{1}{p-1}}} , \qquad u_2^\lambda = f_2 - \frac{1}{(p\lambda L_2)^{\frac{1}{p-1}}} . \tag{15}$$

Proof. Recalling that $f_1 \leq u_1 \leq u_2 \leq f_2$, we just have to consider the region $\mathcal{T}$ defined in (13) and to rewrite the function G in that region as

$$G(u_1, u_2) := u_2 - u_1 + \lambda L_1 (u_1 - f_1)^p + \lambda L_2 (f_2 - u_2)^p \,.$$

The critical point of G is given by

$$u_1 = f_1 + \frac{1}{(p\lambda L_1)^{\frac{1}{p-1}}} \,, \qquad u_2 = f_2 - \frac{1}{(p\lambda L_2)^{\frac{1}{p-1}}} \,,$$

and it belongs to the interior of $\mathcal{T}$, *i.e.*, $u_1^\lambda < u_2^\lambda$, only for $\lambda > \lambda_T^p$. Since G is strictly convex, this critical value turns out to be the global minimum of G for $\lambda > \lambda_T^p$. In the case $\lambda \leq \lambda_T^p$, the minimum point has to be on $\partial\mathcal{T}$. Instead of performing all the computations for finding the minimum point in all of the three edges of $\partial\mathcal{T}$ and to compare them, we will use the following argument based on the continuity of the minimum u^λ with respect to λ (see Lemma 5.1), *i.e.*, we invoke the fact that the function $\lambda \mapsto u^\lambda$ is continuous. Notice that for $\lambda \searrow \lambda_T^p$ we have

$$u_\lambda \to (\bar{u}, \bar{u}) \,,$$

where

$$\bar{u} := \frac{L_1^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} \, f_1 + \frac{L_2^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} \, f_2 \,,$$

is independent of λ. By using the continuity of the solution, we can conclude that, for $\lambda \leq \lambda_T^p$, the solution of the minimum problem is given by $(\bar{u}, \bar{u})$. $\square$

Remark 4.3. We remark a couple of facts:

(1) we have that $\lambda_T^p \to \lambda_T^1$ as $p \to 1^+$ (in each of the cases for the definition of the second one). Indeed, suppose that $L_1 < L_2$. Then,

$$\begin{aligned}
\lim_{p\to 1^+} \lambda_T^p &= \lim_{p\to 1^+} \frac{(L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}})^{p-1}}{L_1 L_2} \\
&= \frac{1}{L_1} \lim_{p\to 1^+} \left(1 + \left(\frac{L_1}{L_2} \right)^{\frac{1}{p-1}} \right)^{p-1} \\
&= \frac{1}{L_1} \lim_{t\to 0^+} exp \left[t \log \left[\left(\frac{L_1}{L_2} \right)^{\frac{1}{t}} + 1 \right] \right] = \frac{1}{L_2} = \lambda_T^1 \,.
\end{aligned}$$

Similar reasonings lead to the claimed result in the other two cases. In particular, notice that $\lambda_T^p > \lambda_T^1$.

(2) The solutions that converge to *a* solution for $p = 1$, as $p \searrow 1$. Indeed, suppose $\lambda > \lambda_T^1$, Then for p sufficiently close to 1, from the above bullet point, we have that $\lambda > \lambda_T^p$. Thus, the solution of the minimum problem for p is given by (15). In this case, it is easy to see that the solution converges to (f_1, f_2), as $p \searrow 1$. In the case $\lambda < \lambda_T^1$, we can assume as above that p is so close to 1 that the solution of the minimum problem for p is given by (14).

If $L_1 > L_2$, then

$$\frac{L_1^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} = \frac{1}{\left(\frac{L_2}{L_1}\right)^{\frac{1}{p-1}} + 1} \to 1\,, \qquad \text{as } p \to 1\,,$$

$$\frac{L_2^{\frac{1}{p-1}}}{L_1^{\frac{1}{p-1}} + L_2^{\frac{1}{p-1}}} = \frac{1}{\left(\frac{L_1}{L_2}\right)^{\frac{1}{p-1}} + 1} \to 0\,, \qquad \text{as } p \to 1\,.$$

In the case $L_1 = L_2$, both coefficients are equal to $\frac{1}{2}$.

Finally, in the case $\lambda = \lambda^1_T$, since $\lambda^p_T > \lambda^1_T$ we have that the solution of the minimum problem is given by (14). The result follows by arguing as before.

5. The behavior of the solution for $p > 1$

In this section we will describe a method to obtain *explicitly* the solution u^λ in the case $p > 1$. This analytic method will be derived by using qualitative properties of the solution.

We start by proving a continuity property of the solution u^λ with respect to λ.

Lemma 5.1. *Let $p > 1$. Then $\lambda \mapsto u^\lambda$ is continuous.*

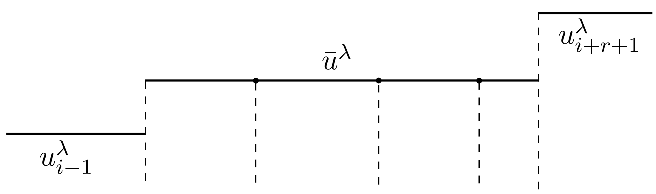
Proof. Fix $\bar{\lambda} > 0$ and let $\lambda_n \to \lambda$. Then $G(u^{\lambda_n}) \leq G(v)$ for all $v \in \mathbb{R}^k$, where equality holds if and only if $v = u^{\lambda_n}$. Since $|u^{\lambda_n}| \leq \sqrt{k}|\max_i f_i|$, up to a (not relabeled) subsequence, we have that $u^{\lambda_n} \to \bar{v}$. Then, by using the continuity of G in both v and λ, we have that $G(\bar{v}) \leq G(v)$ for all $v \in \mathbb{R}^k$. By the uniqueness of the solution, we deduce that $\bar{v} = u^\lambda$, and that $u^{\lambda_n} \to u^\lambda$ for *all* sequences $\lambda_n \to \lambda$. □

We now prove several properties that will be used to deduce the behavior of the solution u^λ when λ varies. Albeit some of the following results can be stated in a more inclusive way, we prefer to consider each single case separately since they are useful to describe the qualitative behavior of the solution when no analytic computations can be done (*i.e.*, when $p \neq 2$).

Lemma 5.2. *Let $p > 1$. Then, the following properties hold true:*

(i) *Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar{u}^\lambda$ such that, for some $r \geq 0$,*

$$\begin{cases} u^\lambda_i = u^\lambda_{i+1} = \cdots = u^\lambda_{i+r} = \bar{u}^\lambda\,, \\ u^\lambda_{i-1} < \bar{u} < u^\lambda_{i+r+1} \quad \text{or} \quad u^\lambda_{i-1} > \bar{u} > u^\lambda_{i+r+1}\,. \end{cases}$$



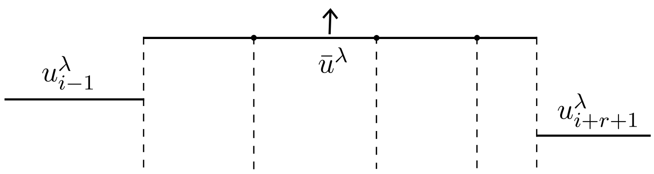
Then $\bar{u}^\lambda$ is the solution of

$$\min_{c\in(u^\lambda_{i-1},u^\lambda_{i+r+1})} \sum_{j=i}^{i+r} L_j|c-f_j|^p\,.$$

In particular, $\bar{u}^\lambda$ is constant in (λ_1,λ_2).

(ii) *Assume that, for $\lambda\in(\lambda_1,\lambda_2)$, there exists a function $\lambda\mapsto\bar{u}^\lambda$ such that, for some $r\geq 0$,*

$$\begin{cases} u^\lambda_i = u^\lambda_{i+1} = \cdots = u^\lambda_{i+r} = \bar{u}^\lambda\,, \\ \qquad u^\lambda_{i-1}\,,\, u^\lambda_{i+r+1} < \bar{u}^\lambda\,. \end{cases}$$

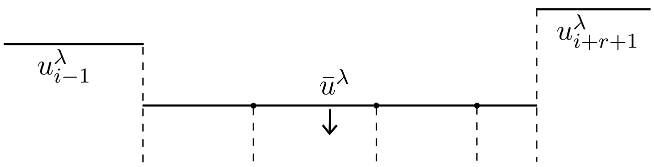


Then $\lambda\mapsto\bar{u}^\lambda$ is increasing.
In particular, in the case $r=0$, we have

$$u^\lambda_i = f_i - \Big(\frac{2}{p\lambda L_i}\Big)^{\frac{1}{p-1}}\,.$$

(iii) *Assume that, for $\lambda\in(\lambda_1,\lambda_2)$, there exists a function $\lambda\mapsto\bar{u}^\lambda$ such that, for some $r\geq 0$,*

$$\begin{cases} u^\lambda_i = u^\lambda_{i+1} = \cdots = u^\lambda_{i+r} = \bar{u}^\lambda\,, \\ \qquad u^\lambda_{i-1}\,,\, u^\lambda_{i+r+1} > \bar{u}^\lambda\,. \end{cases}$$

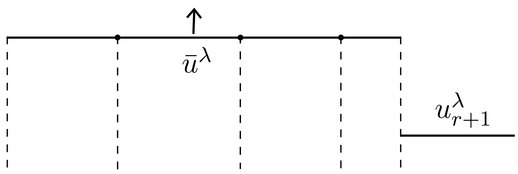


Then $\lambda\mapsto\bar{u}^\lambda$ is decreasing.
In particular, in the case $r=0$, we have

$$u^\lambda_i = f_i + \Big(\frac{2}{p\lambda L_i}\Big)^{\frac{1}{p-1}}\,.$$

(iv) *Assume that, for* $\lambda \in (\lambda_1, \lambda_2)$*, there exists a function* $\lambda \mapsto \bar{u}^\lambda$ *such that, for some* $r \geq 0$*,*

$$\begin{cases} u_1^\lambda = u_2^\lambda = \cdots = u_r^\lambda = \bar{u}^\lambda \,, \\ u_{r+1}^\lambda < \bar{u}^\lambda \,. \end{cases}$$

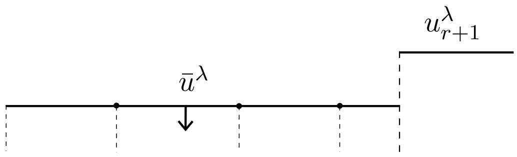


Then $\lambda \mapsto \bar{u}^\lambda$ *is increasing.*
In particular, in the case $r = 0$*, we have*

$$u_i^\lambda = f_1 - \left(\frac{1}{p\lambda L_1} \right)^{\frac{1}{p-1}} .$$

(v) *Assume that, for* $\lambda \in (\lambda_1, \lambda_2)$*, there exists a function* $\lambda \mapsto \bar{u}^\lambda$ *such that, for some* $r \geq 0$*,*

$$\begin{cases} u_1^\lambda = u_{i+1}^\lambda = \cdots = u_r^\lambda = \bar{u}^\lambda \,, \\ u_{r+1}^\lambda > \bar{u}^\lambda \,. \end{cases}$$

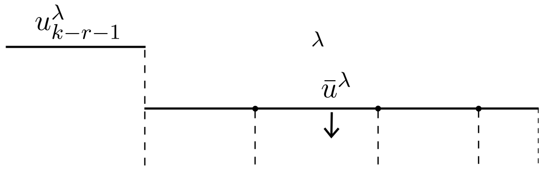


Then $\lambda \mapsto \bar{u}^\lambda$ *is decreasing.*
In particular, in the case $r = 0$*, we have*

$$u_i^\lambda = f_1 + \left(\frac{1}{p\lambda L_1} \right)^{\frac{1}{p-1}} .$$

(vi) *Assume that, for* $\lambda \in (\lambda_1, \lambda_2)$*, there exists a function* $\lambda \mapsto \bar{u}^\lambda$ *such that, for some* $r \geq 0$*,*

$$\begin{cases} u_{k-r}^\lambda = \cdots = u_k^\lambda = \bar{u}^\lambda \,, \\ u_{k-r-1}^\lambda > \bar{u}^\lambda \,. \end{cases}$$

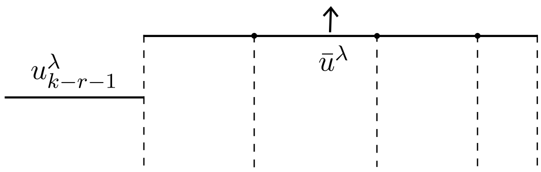


Then $\lambda \mapsto \bar u^\lambda$ is decreasing.
In particular, in the case $r = 0$, we have

$$u_k^\lambda = f_k + \Big(\frac{1}{p\lambda L_k}\Big)^{\frac{1}{p-1}} .$$

(vii) *Assume that, for $\lambda \in (\lambda_1, \lambda_2)$, there exists a function $\lambda \mapsto \bar u^\lambda$ such that, for some $r \geq 0$,*

$$\begin{cases} u^\lambda_{k-r} = \cdots = u^\lambda_k = \bar u^\lambda , \\ \qquad u^\lambda_{k-r-1} < \bar u^\lambda . \end{cases}$$



Then $\lambda \mapsto \bar u^\lambda$ is increasing.
In particular, in the case $r = 0$, we have

$$u_k^\lambda = f_k - \Big(\frac{1}{p\lambda L_k}\Big)^{\frac{1}{p-1}} .$$

Proof. We start by proving property (i). Suppose that $u^\lambda_{i-1} < \bar u^\lambda < u^\lambda_{i+r+1}$. In the other case we argue in a similar way. By hypothesis, the vector u^λ minimizes the function G in the set

$$\{ (u_1, \dots, u_k) \in \mathbb{R}^k : u_{i-1} < u_i = \cdots = u_{i+r} < u_{i+r+1} \} ,$$

and in this set, the function G can be written as

$$G(u) = \widetilde G(u_1 \dots, u_{i-1}, u_{i+r+1}, \dots, u_k) + \lambda \sum_{j=i}^{i+r} L_j |\bar u - f_j|^p .$$

By keeping $u_1, \dots, u_{i-1}$ and $u_{i+r+1}, \dots, u_k$ fixed, the claim follows by minimizing the above quantity with respect to $\bar u$.

Since all the other properties can be proved with an argument whose general lines are the same, we just prove property (ii), leaving the details of the others proofs to the reader.

In the hypothesis of (ii), it holds that u^λ is a minimum of G in the set

$$\{\, (u_1, \dots, u_k) \in \mathbb{R}^k : u_{i-1},\, u_{i+r+1} < u_i = \cdots = u_{i+r} \,\}\,.$$

Restricted to this set, the function G can be written as

$$G(u) = \widetilde{G}(u_1 \dots, u_{i-1}, u_{i+r+1}, \dots, u_k) + 2\bar{u} + \lambda \sum_{j=i}^{i+r} L_j |\bar{u} - f_j|^p\,.$$

So, for $\lambda \in (\lambda_1, \lambda_2)$ and $u_1 \dots, u_{i-1}, u_{i+r+1}, \dots, u_k$ fixed, $\bar{u}^\lambda$ is the minimum of the strictly convex function

$$H(c) := 2c + \lambda \sum_{j=i}^{i+r} L_j |c - f_j|^p$$

in the set $(\max\{u^\lambda_i, u^\lambda_{i+r}\}, \max f)$.

To study the minimum of H, we can assume without loss of generality that $f_i < f_{i+1} < \cdots < f_{i+r}$. Indeed, we notice that the order of the f_j's doesn't matter. Moreover, in the case in which $f_p = f_q$ for some $p \neq q$, we can simply collect the two terms in a single one and use $L_p + L_q$ as a corresponding factor in the above summation. We now want to prove that $\lambda \mapsto \bar{u}$ is decreasing. Note that the function H can be written as

$$H(c) = 2c + \lambda \sum_{j=i}^{m} L_j (c - f_j)^p + \lambda \sum_{j=m+1}^{i+r} L_j (f_j - c)^p =: H_m(c)\,,$$

if $c \in (f_m, f_{m+1}]$, for some $m \in \{i, \dots, i+r-1\}$, and

$$H(c) = 2c + \lambda \sum_{j=i}^{i+r} L_j (c - f_j)^p\,,$$

if $c \in [f_{i+r}, \max f)$. Consider the function H_m in the interval (f_m, f_{m+1}). We have that

$$H'_m(c) = 2 + p\lambda \left[\sum_{j=i}^{m} L_j (c - f_j)^{p-1} - \sum_{j=m+1}^{i+r} L_j (f_j - c)^{p-1} \right]\,.$$

Here $H'_m(c) = 0$ has a solution only if the term in the parenthesis is negative and if so, the let $\lambda \mapsto c^\lambda$ be such a solution. It is easy to see that this function is regular in (f_m, f_{m+1}). By differentiating the expression $H'_m(c^\lambda)$ with respect to λ, we obtain

$$\begin{aligned} p &\left[\sum_{j=i}^{m} L_j (c - f_j)^{p-1} - \sum_{j=m+1}^{i+r} L_j (f_j - c)^{p-1} \right] \\ &+ \lambda \frac{\mathrm{d}c^\lambda}{\mathrm{d}\lambda} p(p-1) \left[\sum_{j=i}^{m} L_j (c - f_j)^{p-2} + \sum_{j=m+1}^{i+r} L_j (f_j - c)^{p-2} \right] = 0\,. \end{aligned}$$

Thus, by recalling that the term in the first parenthesis is negative, we get $\frac{\mathrm{d}c^\lambda}{\mathrm{d}\lambda} < 0$, as desired.

In the case in which the minimum of the function H is reached at a point $c = f_{m+1}$, we simply consider the function H_m and we apply the argument above.

Finally, the same reasoning applies when $c \in [f_{i+r}, \max f)$. $\square$

We are now in position to prove the fundamental result we will use to develop our strategy for finding the solution.

Theorem 5.3. *For each* $i = 1, \dots, k-1$ *there exists* $\lambda_i \in (0, \infty)$ *such that* $u_i^\lambda = u_{i+1}^\lambda$ *for* $\lambda \le \lambda_i$*, while* $u_i^\lambda \neq u_{i+1}^\lambda$ *for* $\lambda > \lambda_i$.

Proof. Step 1. We claim that if $u_i^{\widetilde{\lambda}} = u_{i+1}^{\widetilde{\lambda}}$ for some $\widetilde{\lambda} > 0$, then $u_i^\lambda = u_{i+1}^\lambda$ for all $\lambda \in (0, \widetilde{\lambda}]$. Indeed, let

$$\bar{\lambda} := \min\{\, \lambda \,:\, u_i^\mu = u_{i+1}^\mu \text{ fo all } \mu \in [\lambda, \widetilde{\lambda}] \,\},$$

and assume that $\bar{\lambda} > 0$. By continuity of $\lambda \mapsto u^\lambda$ there exists $\varepsilon > 0$ such that $u_i^\lambda \neq u_{i+1}^\lambda$ for $\lambda \in (\bar{\lambda} - \varepsilon, \bar{\lambda})$. Consider the case in which $u_i^\lambda < u_{i+1}^\lambda$ in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$ (the other case can be treated similarly).

If $i = 1$, then property (v) of Lemma 5.2 tells us that $\lambda \mapsto u_i^\lambda$ is decreasing in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$, and thus it is not possible to have $u_i^{\bar{\lambda}} = u_{i+1}^{\bar{\lambda}}$.

If $i > 1$, we can focus, without loss of generality, only on the following two cases: $u_{i-1}^\lambda > u_i^\lambda$ and $u_{i-1}^\lambda < u_i^\lambda$ in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$.

In the first case, we get a contradiction since by property (iii) of Lemma 5.2, the map $\lambda \mapsto u_i^\lambda$ is decreasing in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$ and thus, as above, we cannot have $u_i^{\bar{\lambda}} = u_{i+1}^{\bar{\lambda}}$.

In the other case, we have $u_{i-1}^\lambda < u_i^\lambda < u_{i+1}^\lambda$ in $(\bar{\lambda} - \varepsilon, \bar{\lambda})$. By using property (i) of Lemma 5.2, we see that this is possible only if $u_i^\lambda = f_i$ for all $\lambda \in (\bar{\lambda} - \varepsilon, \bar{\lambda})$. This yields the desired contradiction.

Step 2. Let us define

$$\lambda_i := \max\{\, \lambda \,:\, u_i^\mu = u_{i+1}^\mu\,, \text{ for all } \mu \le \lambda \,\}.$$

Step 1 and the continuity of $\lambda \mapsto u^\lambda$ ensure that λ_i is well defined. Moreover, by Lemma 3.6, we also get that $\lambda_i > 0$ for all $i = i, \dots, k-1$. Finally, the fact that $u^\lambda \to f$ as $\lambda \to \infty$, tells us that $\lambda_i < \infty$ for all $i = 1, \dots, k-1$. This concludes the proof. $\square$

Remark 5.4. By direct inspection of the proof of property (ii) of Proposition 5.2, we see that the function $\lambda \mapsto u^\lambda$ is continuous. Moreover, we can also say that it is smooth for all $\lambda \in (0, \infty) \setminus S$, where $S := \{\lambda_1, \dots, \lambda_{k-1}\} \cup T$, where the λ_i's are given by Theorem 5.3, and $T := \{\mu_1, \dots, \mu_k\}$ where $\mu_i := \inf\{\lambda \,:\, u_i^\lambda = f_i\}$.

Finally, we derive another consequence of Lemma 5.2, that will ensure that the solution is monotone where f is, and with the same monotonicity.

Proposition 5.5. *Suppose that* $f_i < f_{i+1} < \cdots < f_{i+r}$. *Then the solution* u *of the minimum problem* (7) *is such that* $u_i \le u_{i+1} \le \cdots \le u_{i+r}$.

In particular, it has the following structure:

- *if* $u_i \ge f_{i+r}$, *then* $u_j = u_i$ *for all* $j = i, \dots, i+1$,
- *if* $u_{i+r} \le f_i$, *then* $u_j = u_{i+r}$ *for all* $j = i, \dots, i+1$,

- *otherwise, u is of the form*

$$u_j = \begin{cases} u_i & \text{for } j = i, \dots, j_1 \,, \\ f_j & \text{for } j = j_1 + 1, \dots, j_2 - 1 \,, \\ u_{i+r} & \text{for } j = j_2, \dots, k \,, \end{cases}$$

for some $f_{j_1} \leq u_i < f_{j_1+1}$ *and* $f_{j_2} \leq u_{i+r} < f_{j_2+1}$*, where* $i_1 < i_2$*.*

Proof. Step 1. We claim that $u_i \leq u_{i+1} \leq \dots \leq u_{i+r}$.

Suppose that $u_{j-1} > u_j$ for some $j \in \{i+1, \dots, i+r\}$. We have to treat three cases: $u_j < f_j$, $u_j = f_j$ and $u_j > f_j$.

In the first case, we get a contradiction with the minimality of u^λ since it is easy to see that

$$G(u_1^\lambda, \dots, u_{j-1}^\lambda, u_j^\lambda + \varepsilon, u_{j+1}^\lambda, \dots, u_k) < G(u^\lambda) \,,$$

for $\varepsilon > 0$ small.

Now, suppose $u_j > f_j$ and that $u_j > u_{j+1}$. Then, for $\varepsilon > 0$ small,

$$G(u_1^\lambda, \dots, u_{j-1}^\lambda, u_j^\lambda - \varepsilon, u_{j+1}^\lambda, \dots, u_k) < G(u^\lambda) \,,$$

yielding the desired contradiction.

Finally, we can treat all the remaining cases (namely $u_j = f_j$ or $u_j > f_j$ and $u_{j+1} > u_j$) simultaneously as follows: let $j_m \in \{i, \dots, j\}$ be the minimum index r such that $u_r > u_{r+1}$. In both cases we have $u_{j_m} > f_{j_m}$, and thus,

$$G(u_1^\lambda, \dots, u_{j_m-1}^\lambda, u_{j_m}^\lambda - \varepsilon, u_{j_m+1}^\lambda, \dots, u_k) < G(u^\lambda) \,,$$

for $\varepsilon > 0$ small.

Step 2. Using Step 1, we have that

$$\sum_{j=i+1}^{i+r} |u_j^\lambda - u_{j-1}^\lambda| = u_{i+r}^\lambda - u_i^\lambda \,.$$

Since this value is invariant under modification of u_j^λ for $j = i+1, \dots, i+r-1$, if we keep u_i and u_{i+r} fixed, the minimality of u^λ implies that

$$\sum_{j=i}^{i+r} |u_j - f_j|^p = \min_{\mathcal{A}} \sum_{j=i}^{i+r} |v_i - f_i|^p \,,$$

where

$$\mathcal{A} := \{(v_{i+1}, \dots, v_{i+r-1}) \in \mathbb{R}^{i+r-2} \,:\, u_i \leq v_{i+1} \leq \dots \leq v_{i+r-1} \leq u_{i+r}\} \,.$$

This proves the second part of the statement of the proposition. □

Finally, thanks to the above properties, we can now provide a method to compute *explicitly* the solution u^λ to the minimum problem (4) in the case $p > 1$.

A method for finding the solution. The idea is the following: we know that $u^\lambda \to f$ as $\lambda \to \infty$. Thus, for $\lambda \gg 1$, $u_i^\lambda \neq u_{i+1}^\lambda$ and

$$\frac{u_{i+1} - u_i}{|u_{i+1} - u_i|} = \frac{f_{i+1} - f_i}{|f_{i+1} - f_i|} \,,$$

for all $i = 1, \dots, k-1$.

By Theorem 5.3, once $u_i^\lambda \neq u_{i+1}^\lambda$, for some λ, the same holds true for all bigger values of λ. Moreover, thanks to Lemma 5.2, if u_i^λ is close to f_i, then we can also assert $u_i^\lambda > f_i$, $u_i^\lambda < f_i$, or $u_i^\lambda = f_i$. Thus, we are able to determine $\bar{s}_i, \bar{t}_i \in \{0,1\}$ for which

$$G(u^\lambda) = \sum_{i=2}^{k} (-1)^{\bar{s}_i}(u_i^\lambda - u_{i-1}^\lambda) + \lambda \sum_{i=1}^{k} L_i\big((-1)^{\bar{t}_i}(f_i - u_i^\lambda)\big)^p \,, \qquad (16)$$

for all $\lambda \gg 1$. In particular, (16) holds for all $\lambda > \max\{\, \lambda_1, \dots, \lambda_{k-1} \,\}$, where the λ_i's are the ones given by Theorem 5.3, and the problem is solved for $\lambda \gg 1$, by simply computing the critical points of (16).

We now let λ decrease. Eventually, a *critical* event will happen, that is either $\lambda = \lambda_i$ for some i, or σ_i^λ will change, where

$$\sigma_i^\lambda := \begin{cases} 0 & \text{if } u_i^\lambda = f_i \,, \\ \frac{u_i^\lambda - f_i}{|u_i^\lambda - f_i|} & \text{otherwise} \;. \end{cases}$$

Such a critical event will determine a change in the values of s and t for which

$$G(u^\lambda) = G_{s,t}(u) := \sum_{i=2}^{k} (-1)^{s_i}(u_i - u_{i-1}) + \lambda \sum_{i=1}^{k} L_i\big((-1)^{t_i}(f_i - u_i)\big)^p \,,$$

where $s_i, t_i \in \{0,1\}$. Notice that each $G_{s,t}$ is differentiable. Let us recall that, if $\lambda = \lambda_i$ for some i, by Theorem 5.3 we know that, for $\lambda \leq \lambda_i$, we will have $u_i^\lambda = u_{i+1}^\lambda$. So, for $\lambda \leq \lambda_i$, we will have to consider the functional G restricted to the subspace $\{u_i = u_{i+1}\}$. Again, solutions are found as critical points of the function under consideration.

Example. To understand better the above strategy, we provide an example. We will treat the case $p = 2$, where explicit analytic computations can be made.

Suppose that $k = 6$, take $L_1 = L_3 = L_5 = 1$ and $L_2 = L_4 = L_6 = 2$. Consider the initial data f given by $f_1 = 2$, $f_2 = 1$, $f_3 = 3$, $f_4 = 5$, $f_5 = 6$, $f_6 = 4$.

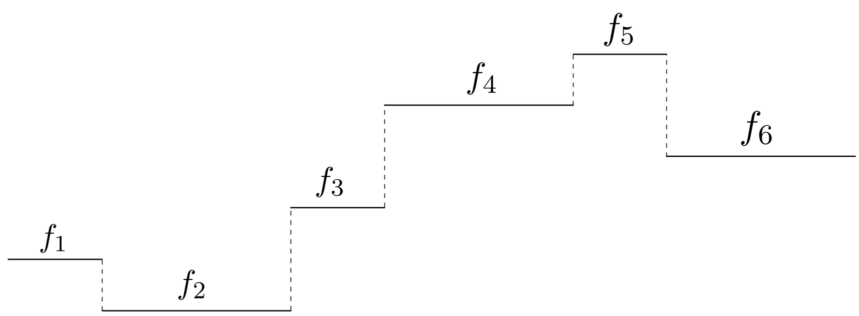


FIGURE 2. The initial data f.

For $\lambda \gg 1$, we know that we have to consider the following functional

$$G(u_1,u_2,u_3,u_4,u_5,u_6) := u_1 - 2u_2 + 2u_5 - u_6 + \lambda[\,(2-u_1)^2 + 2(1-u_2)^2 + |u_3-3|^2 + 2|u_4-5|^2 + (6-u_5)^2 + 2(u_6-4)^2\,]\,.$$

In particular, we obtain that the solution u^λ is given by

$$u_1^\lambda := 2 - \tfrac{1}{2\lambda}\,, \qquad u_2^\lambda := 1 + \tfrac{1}{2\lambda}\,, \qquad u_3^\lambda := 3\,,$$
$$u_4^\lambda := 5\,, \qquad u_5^\lambda := 6 - \tfrac{1}{\lambda}\,, \qquad u_6^\lambda := 4 + \tfrac{1}{4\lambda}\,.$$

for $\lambda > 1$.

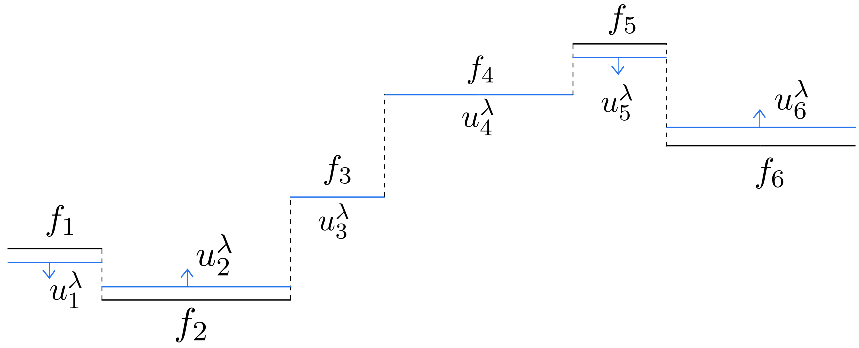


FIGURE 3. The behavior of the solution for $\lambda > 1$ for λ decreasing.

The first *critical event* happens for $\lambda = 1$, when $u_1^\lambda = u_2^\lambda$ and $u_4^\lambda = u_5^\lambda$. For smaller values of λ, we have to consider the functional

$$G(v_1,v_2,v_3,v_4) := 2v_3 - v_1 - v_4 + \lambda[\,(2-v_1)^2 + 2(v_1-1)^2 + |v_3-3|^2 + 2(5-v_3)^2 + (6-v_3)^2 + 2(v_4-4)^2\,]\,.$$

Here, the solution is given by

$$u_1^\lambda := \tfrac{4}{3} + \tfrac{1}{6\lambda}\,, \qquad u_2^\lambda := \tfrac{4}{3} + \tfrac{1}{6\lambda}\,, \qquad u_3^\lambda := 3\,,$$
$$u_4^\lambda := \tfrac{16}{3} - \tfrac{1}{3\lambda}\,, \qquad u_5^\lambda := \tfrac{16}{3} - \tfrac{1}{3\lambda}\,, \qquad u_6^\lambda := 4 + \tfrac{1}{4\lambda}\,.$$

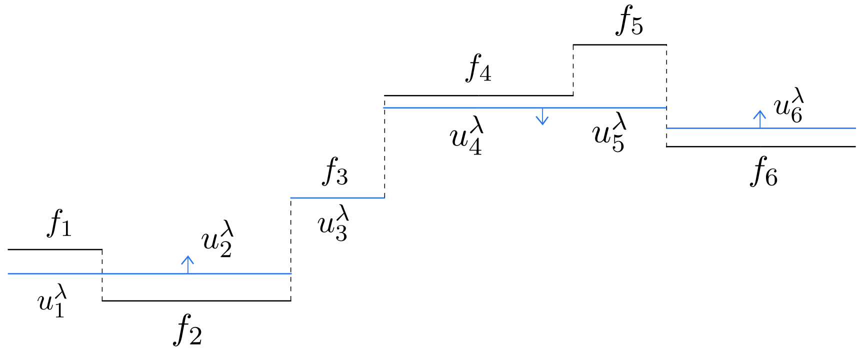


FIGURE 4. The behavior of the solution for $\lambda \in (\frac{9}{14}, 1]$ for λ decreasing.

Then, for $\lambda = \frac{9}{14}$ we have that $u_6^\lambda = u_5^\lambda$. Hence, the new functional we have to consider is

$$G(v_1, v_2, v_3) := v_3 - v_1 + \lambda[\,(2 - v_1)^2 + 2(v_1 - 1)^2 + |v_2 - 3|^2 \\ + 2(5 - v_3)^2 + (6 - v_3)^2 + 2(v_3 - 4)^2\,]\,.$$

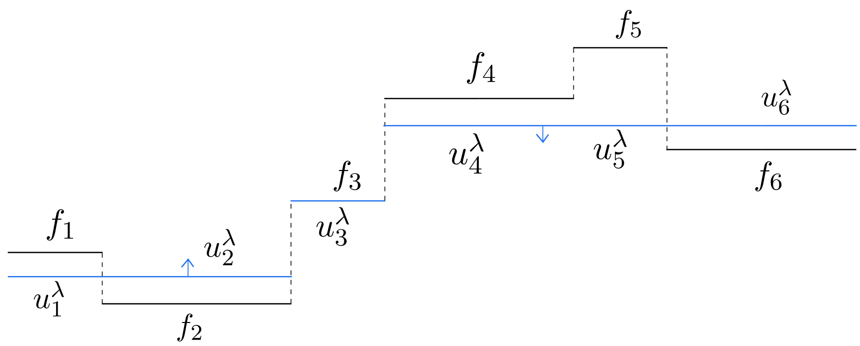


FIGURE 5. The behavior of the solution for $\lambda \in (, \frac{1}{10}, \frac{9}{14}]$ for λ decreasing.

The solution is now

$$u_1^\lambda := \tfrac{4}{3} + \tfrac{1}{6\lambda}\,, \qquad u_2^\lambda := \tfrac{4}{3} + \tfrac{1}{6\lambda}\,, \qquad u_3^\lambda := 3\,,$$

$$u_4^\lambda := \tfrac{16}{3} - \tfrac{1}{6\lambda}\,, \qquad u_5^\lambda := \tfrac{16}{3} - \tfrac{1}{6\lambda}\,, \qquad u_6^\lambda := \tfrac{16}{3} - \tfrac{1}{6\lambda}\,.$$

Notice that for $\lambda = \frac{1}{4}$ we have $u_1^\lambda = f_1$. Thus, for $\lambda < \frac{1}{4}$, we have to consider the functional

$$G(v_1, v_2, v_3) := v_3 - v_1 + \lambda[\,(v_1 - 2)^2 + 2(v_1 - 1)^2 + |v_2 - 3|^2 \\ + 2(5 - v_3)^2 + (6 - v_3)^2 + 2(v_3 - 4)^2\,]\,.$$

Hence, the solution remains equal to the previous ones. For $\lambda = \frac{1}{10}$ we get $u_2^\lambda = u_3^\lambda$. Then we consider the functional

$$G(v_1, v_2) := v_2 - v_1 + \lambda[\,(2 - v_1)^2 + 2(v_1 - 1)^2 + |v_2 - 3|^2 \\ + 2(5 - v_2)^2 + (6 - v_2)^2 + 2(v_2 - 4)^2\,]\,.$$

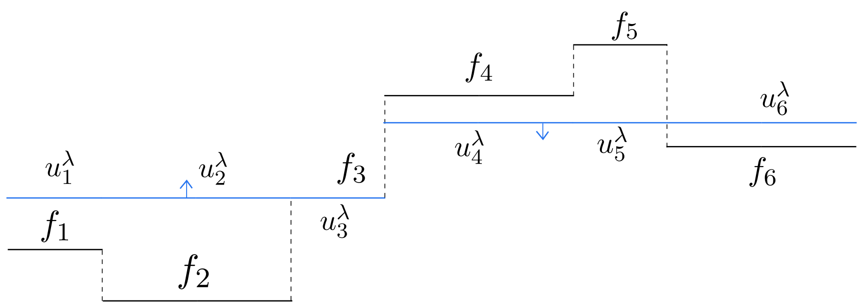


FIGURE 6. The behavior of the solution for $\lambda \in (\frac{9}{122}, \frac{1}{10}]$ for λ decreasing.

Such a functional is minimized by

$$u_1^\lambda := \tfrac{7}{4} + \tfrac{1}{8\lambda}\,, \qquad u_2^\lambda := \tfrac{7}{4} + \tfrac{1}{8\lambda}\,, \qquad u_3^\lambda := \tfrac{7}{4} + \tfrac{1}{8\lambda}\,,$$

$$u_4^\lambda := \tfrac{24}{5} - \tfrac{1}{10\lambda}\,, \qquad u_5^\lambda := \tfrac{24}{5} - \tfrac{1}{10\lambda}\,, \qquad u_6^\lambda := \tfrac{24}{5} - \tfrac{1}{10\lambda}\,.$$

Finally, for $\lambda \leq \frac{9}{122}$ we have that the solution is given by

$$u_1^\lambda = u_2^\lambda = u_3^\lambda = u_4^\lambda = u_5^\lambda = u_6^\lambda := \frac{31}{9}\,.$$

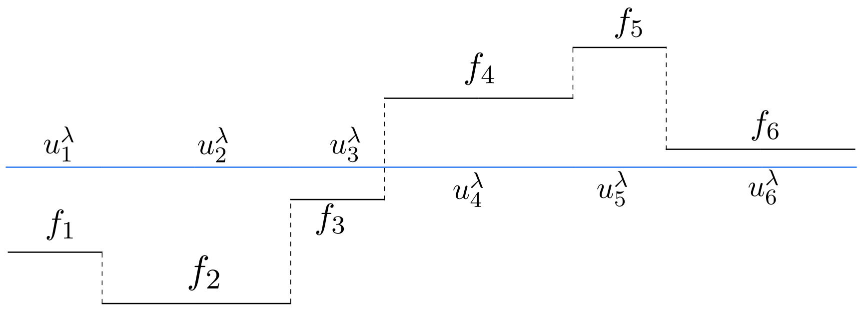


FIGURE 7. The behavior of the solution for $\lambda < \frac{9}{122}$.

Remark 5.6. The previous example allows us to identify some properties of the solution u^λ:

(1) it is *not* true that if $u_i^{\bar\lambda} = f_i$, then $u_i^\lambda = f_i$ for all $\lambda \geq \bar\lambda$,
(2) the function $\lambda \mapsto u_i^\lambda$ is *not* monotone in general. Nevertheless, a change in the monotonicity can happen only if $\lambda = \lambda_i$ or $\lambda = \lambda_{i-1}$,

Remark 5.7. As we saw in the example, the value of λ for which the solution u^λ is such that $u_i^\lambda \neq u_{i+1}^\lambda$ for all $i = 1, \dots, k-1$, can be determined *a posteriori*.

Remark 5.8. Let us denote by $u^{\lambda,p}$ the solution of problem (7) corresponding to p and λ. Albeit we know that, for every λ fixed, $u^{\lambda,p} \to v$ as $p \searrow 1$, where v is a solution of the problem (7) corresponding to λ and $p = 1$, we cannot apply directly our method to find v, since analytic computations are difficult to perform in the case $p \in (1,2)$. Nevertheless, a finer analysis of the behavior of the solution $u^{\lambda,p}$ for $p \in (1,2)$ is currently under investigation.

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