7. Convergence

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1 Classical results

Monotonicity. Every bounded monotone real sequence $a_1, a_2, \ldots$ converges to a limit.

Cauchy sequence (definition). A sequence $a_1, a_2, \ldots$ is called a *Cauchy sequence* if for every $\epsilon > 0$, there is a positive integer N such that for all $i, j > N$, we have $|a_i - a_j| < \epsilon$. The real and complex number systems have the property that every Cauchy sequence converges to a limit, which is a number in the system.

Absolute convergence. Let $z_1, z_2, \ldots$ be a sequence of complex numbers, for which $\sum_i |z_i|$ converges. Then $\sum_i z_i$ converges as well.

Abel summation. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sequences, and let B_k denote $\sum_{i=1}^k b_i$ for every k. Then

$$\sum_{i=1}^{n} a_i b_i = a_n B_n - \sum_{i=1}^{n-1} B_i (a_{i+1} - a_i)\,.$$

Classical. Prove that the sequence $\sqrt{7}$, $\sqrt{7+\sqrt{7}}$, $\sqrt{7+\sqrt{7+\sqrt{7}}}$, …converges, and determine its limit. This is often denoted as $\sqrt{7+\sqrt{7+\sqrt{7+\cdots}}}$.

2 Problems

1. Show that if $\sum_{i=1}^{\infty} a_i^2$ and $\sum_{i=1}^{\infty} b_i^2$ both converge, then so does $\sum_{i=1}^{\infty} (a_i - b_i)^2$.
2. Let $a_1, a_2, \ldots$ be positive integers such that $\sum \frac{1}{a_i}$ converges. For each n, let b_n denote the number of positive integers i for which $a_i \le n$. Prove that $\lim_{n\to\infty} \frac{b_n}{n} = 0$.
3. Let $a_1, a_2, \ldots$ be a sequence of real numbers for which the sum $\sum_{i=1}^{\infty} a_i$ converges. Show that the sum $\sum_{i=1}^{\infty} \frac{a_i}{i}$ also converges.
4. Let a_i be a monotonically decreasing sequence of positive real numbers, for which $\sum_{i=1}^{\infty} a_i$ converges. Show that $\sum_{i=1}^{\infty} i(a_i - a_{i+1})$ also converges.
5. Let α be an arbitrary real number. Define $a_1 = \alpha$, and for all $n \ge 1$, let $a_{n+1} = \cos a_n$. Prove that a_n converges to a limit, and that this limit does not depend on α.
6. Let $z_1, z_2, \ldots$ be nonzero complex numbers with the property that $|z_i - z_j| > 1$ for all i, j. Prove that $\sum \frac{1}{z_i^3}$ converges.
7. Let a_i be a sequence of positive real numbers. Show that $\limsup \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n \ge e$.
8. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers, for which $\sum_{i=1}^{\infty} \frac{1}{a_i}$ converges. For every n, let $b_n = \frac{a_1 + \cdots + a_n}{n}$. Show that $\sum_{i=1}^{\infty} \frac{1}{b_n}$ also converges.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.