A homogenization result in the gradient theory of phase transitions

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Abstract

A homogenization problem arising in the gradient theory of fluid-fluid phase transitions is addressed in the vector-valued setting by means of Γ-convergence.

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1 Introduction

The asymptotic behavior of functionals of the type

$$E_\varepsilon(u) := \int_\Omega \left(\frac{1}{\varepsilon}W(u(x)) + \varepsilon|\nabla u(x)|^2\right) dx \tag{1.1}$$

has received much attention in the last two decades in the context of fluid-fluid phase transitions. If Ω is an open, bounded domain in $\mathbb{R}^N$, with Lipschitz boundary, and if W is a nonnegative bulk energy density with $\{W = 0\} = \{a, b\}$, then Gibbs' criterion for equilibria leads to the study of the problem

$$(P) \quad \text{minimize} \int_\Omega W(u(x))dx \quad \text{subject to the constraint} \int_\Omega u(x)dx = m.$$

If $m = \theta a + (1-\theta)b,\ 0 < \theta < \mathcal{L}^N(\Omega)$, then the minimum problem (P) admits infinitely many solutions. In order to select physically preferred solutions to this problem, and following the ideas of the gradient theory of phase transitions proposed in 1893 by van der Waals, Cahn and Hilliard [8] introduced a model where to each configuration u of the two-fluid system an energy E_ε which penalizes the original energy of the system $u \mapsto \int_\Omega W(u(x))dx$ through a term containing the gradient of u and a small parameter $\varepsilon > 0$, i.e. $u \mapsto \int_\Omega \left(W(u(x)) + \varepsilon^2|\nabla u|^2\right) dx$. The competing effects of the resulting two integrals favor separation of phases (i.e. those configurations where u takes values close to a and b), while penalizing spatial inhomogeneities of u and, consequently, the introduction of too many transition regions.

The connection between the classical theory of phase transition based on Gibbs' criterion and the gradient theory is due to Gurtin [18], [19], who conjectured in 1983 that solutions of

$$(P_\varepsilon) \quad \text{minimize } F_\varepsilon(u) \quad \text{subject to the constraint } \int_\Omega u(x)dx = m$$

converge to minimizers of (P) having minimal interfacial energy. Gurtin's conjecture was proved by Carr, Gurtin, and Slemrod [9] in the scalar case (N = 1), and independently by Modica [20] and Sternberg [22], in the higher dimensional case $N \geq 2$. The approach in [20] and [22] uses the notion of Γ-convergence, due to De Giorgi [12] (see also [1], [6], [10]), and follows the ideas of Modica and Mortola [21] who studied a similar functional proposed by De Giorgi in a completely different physical context.

The vector-valued case, where $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^d\,(d, N \geq 2)$ was considered by Fonseca and Tartar [16], Sternberg [23], and Barroso and Fonseca [5]. The case where W has more than two wells was addressed by Baldo [4] (see also Sternberg [23]), and later generalized by Ambrosio [2].

Let $Q \subset \mathbb{R}^N$ be the open unit cube centered at the origin, and given $\nu \in S^{N-1} := \{x \in \mathbb{R}^{\mathbb{N}} : \|x\| = 1\}$, we denote by Q_ν the cube centered at the origin with two of its faces normal to ν. Precisely, if $\{\nu_1, ..., \nu_{N-1}, \nu\}$ is an orthonormal basis of $\mathbb{R}^N$, then

$$Q_\nu := \left\{x \in \mathbb{R}^{\mathbb{N}} : |x \cdot \nu_i| < \frac{1}{2}, \quad |x \cdot \nu| < \frac{1}{2}, \quad i = 1, ..., N-1\right\}.$$

In this paper we study a homogenization problem within the context of the gradient theory of phase transitions, in the vector-valued setting. Let $W : \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty)$ be a continuous function satisfying the following hypotheses

(H1) $W(\cdot, u)$ is Q-periodic for every $u \in \mathbb{R}^d$;

(H2) $W(x, u) = 0$ if and only if $u \in \{a, b\}$;

(H3) there exist $C > 0$ and $q \geq 2$ such that

$$\frac{1}{C}|u|^q - C \leq W(x, u) \leq C(1 + |u|^q)$$

for all $(x, u) \in \Omega \times \mathbb{R}^d$,

and let $I_\varepsilon : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ be defined by

$$I_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}, u(x)\right) + \varepsilon|\nabla u(x)|^2\right) dx & \textit{if } u \in H^1(\Omega; \mathbb{R}^d) \\ +\infty & \textit{otherwise.} \end{cases}$$

The main result of the paper is the following theorem

Theorem 1.1 *Assume that (H1)-(H3) hold, let $\nu \in S^{N-1}$, $\rho : \mathbb{R} \to [0, +\infty)$ be a mollifier, and let $\rho_{T,\nu}(x) := T^N \rho(Tx \cdot \nu)$. Define*

$$K_1(\nu) = \lim_{T \to \infty} \frac{1}{T^{N-1}} \inf \left\{ \int_{TQ_\nu} \left(W(y, u(y)) + |\nabla u(y)|^2\right) dy \ : \ u \in H^1(TQ_\nu; \mathbb{R}^d), \right.$$
$$\left. u = \rho_{T,\nu} * u_0 \text{ on } \partial(TQ_\nu) \right\}$$

with

$$u_0(x) = \begin{cases} b & \text{if } x\cdot\nu > 0, \\ a & \text{if } x\cdot\nu < 0. \end{cases}$$

Consider the functional $I_0 : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ *defined by*

$$I_0(u) := \begin{cases} \displaystyle\int_{\partial^* A_0\cap\Omega} K_1(\nu(x))d\mathcal{H}^{N-1}(x) & \text{if } u \in BV(\Omega;\{a,b\}), \\ +\infty & \text{otherwise}, \end{cases}$$

where $A_0 := \{x \in \Omega : u(x) = a\}$. *Then*

(i) $\Gamma(L^1(\Omega;\mathbb{R}^d)) - \liminf_{\varepsilon\to 0} I_\varepsilon = I_0$;

(ii) *Assume that the set* A_0 *is polyhedral, and that the outward unit normal* $\nu(x)$ *to the reduced boundary* $\partial^* A_0$ *is such that* $\nu(x) \in \{\pm e_1, \cdots, \pm e_N\}$, *for* $\mathcal{H}^{N-1}$*-a.e.* $x \in (\partial^* A_0)\cap\Omega$. *Then*

$$\Gamma(L^1(\Omega;\mathbb{R}^d)) - \lim_{\varepsilon\to 0} I_\varepsilon = I_0.$$

The paper is organized as follows: In Section 2 we recall some facts about functions of bounded variation, sets of finite perimeter, and Γ-convergence, in Section 3 we prove a compactness result (Theorem 3.2), and the Γ-liminf inequality, and in Section 4 of the paper we perform the construction of a recovering sequence for the Γ-limit.

Remark 1.2 Without the additional assumption in part (ii) of Theorem 1.1, some of the techniques used in Section 4.3 to construct a recovering sequence for the Γ-limit would only go through under the (far too strong) requirement that $W(R\cdot, u)$ be Q-periodic for all rotations $R \in \mathrm{SO}(N)$, and $u \in \mathbb{R}^d$. Future work will address the general case. The geometry of A_0 is important here, as it can be seen in (4.19), where the periodicity of $W(\cdot, u)$ with respect to the directions orthogonal to $\nu(x_0)$ is strongly used.

2 Preliminaries

We begin this section by recalling some facts about functions of bounded variations (we refer the reader to [3] for details). A function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^d)$, if for all $i = 1,\cdots,d$, and $j = 1,\cdots,N$, there exists a Radon measure μ_{ij} such that

$$\int_\Omega u_i(x)\frac{\partial v}{\partial x_j}(x)\,dx = -\int_\Omega v(x)\,d\mu_{ij}$$

for every $v \in C^1_c(\Omega;\mathbb{R})$. The distributional derivative Du is the matrix-valued measure with components μ_{ij}. Given $u \in BV(\Omega;\mathbb{R}^d)$ the *approximate upper* and *lower limit* of each component u_i, $i = 1,\cdots,d$, are given by

$$u_i^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\mathcal{L}^N(\{y \in \Omega\cap Q(x,\varepsilon) : u_i(y) > t\}) = 0\right\}$$

and

$$u_i^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\mathcal{L}^N(\{y \in \Omega\cap Q(x,\varepsilon) : u_i(y) < t\}) = 0\right\},$$

while the *jump set* of u, or *singular set*, is defined by

$$S(u) := \bigcup_{i=1}^{d} \{x \in \Omega : u_i^-(x) < u_i^+(x)\}.$$

It is well known that $S(u)$ is $N-1$ rectifiable, i.e.

$$S(u) = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and K_n is a compact subset of a C^1 hypersurface. If $x \in \Omega \setminus S(u)$ then $u(x)$ is taken to be the common value of $(u_1^+(x), \cdots, u_d^+(x))$ and $(u_1^-(x), \cdots, u_d^-(x))$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus S(u)$. Furthermore, for $\mathcal{H}^{N-1}$-a.e. $x \in S(u)$ there exists a unit vector $\nu_u(x) \in S^{N-1}$, *normal to* $S(u)$ *at* x, and two vectors $u^-(x), u^+(x) \in \mathbb{R}^d$ (*the traces of* u *on* $S(u)$ *at the point* x) such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \int_{\{y \in Q(x_0,\varepsilon) : (y-x)\cdot \nu_u(x) > 0\}} |u(y) - u^+(x)|^{N/(N-1)} \, dy = 0$$

and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \int_{\{y \in Q(x_0,\varepsilon) : (y-x)\cdot \nu_u(x) < 0\}} |u(y) - u^-(x)|^{N/(N-1)} \, dy = 0.$$

Note that, in general, $(u_i)^+ \neq (u^+)_i$ and $(u_i)^- \neq (u^-)_i$. We denote the *jump of* u *across* $S(u)$ by $[u] := u^+ - u^-$. The distributional derivative Du may be decomposed as

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \lfloor \, S(u) + C(u),$$

where ∇u is the density of the absolutely continuous part of Du with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$ and $C(u)$ is the Cantor part of Du. These three measures are mutually singular, and the total variation of u,

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u \text{ div } \phi(x) \, dx : \phi \in C_c^1(\Omega; \mathbb{R}^N), \; \|\phi\|_\infty \leq 1 \right\},$$

is now

$$|Du|(\Omega) = \int_\Omega |\nabla u| dx + \int_{S(u)} |u^+ - u^-| d\mathcal{H}^{N-1} + |C(u)|(\Omega).$$

We recall that if $\{u_n\} \subset BV(\Omega; \mathbb{R}^d)$ and $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$, then

$$|Du|(\Omega) \leq \liminf_{n \to \infty} |Du_n|(\Omega).$$

We say that a set $E \subset \Omega$ is of *finite perimeter* if $\chi_E \in BV(\Omega; \mathbb{R})$, and we denote by $\text{Per}_\Omega(E)$ the perimeter of E in Ω, i.e. $\text{Per}_\Omega(E) := |D\chi_E|(\Omega)$ given by

$$\text{Per}_\Omega(E) := \sup \left\{ \int_E \text{div } \phi(x) \, dx : \phi \in C_c^1(\Omega; \mathbb{R}^N), \; \|\phi\|_\infty \leq 1 \right\}.$$

Definition 2.1 *Let $A \subset \mathbb{R}^N$ be a set of locally finite perimeter and let $x_0 \in \mathbb{R}^N$. We say that x_0 belongs to the* reduced boundary *of A (and we write $x_0 \in \partial^* A$) if, with $D\chi_A = -\nu|D\chi_A|$, $\nu \in L^1_{\rm loc}(\mathbb{R}^N; S^{N-1})$ with respect to the Radon measure $|D\chi_A|$, we have*

(i) $|D\chi_A|(B(x_0,\varepsilon)) > 0$ *for all* $\varepsilon > 0$*;*

(ii) $\displaystyle\lim_{\varepsilon\to 0^+} \frac{1}{\mathcal{L}^N(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} \nu(x)\ d|D\chi_A|(x) = \nu(x_0)$*;*

(iii) $\|\nu(x_0)\| = 1$.

ν *is said to be* the outward unit normal to the boundary of A at x.

Theorem 2.2 (see [13], [17]) *If $x \in \partial^* A$ then*

$$\lim_{\delta\to 0^+} \frac{1}{\delta^N} \mathcal{L}^N(\{y \in B(x,\delta) \setminus A : (y-x)\cdot\nu(x) < 0\}) = 0,$$

$$\lim_{\delta\to 0^+} \frac{1}{\delta^N} \mathcal{L}^N(\{y \in B(x,\delta) \cap A : (y-x)\cdot\nu(x) > 0\}) = 0.$$

It can be shown (see [15]) that if $\mathrm{Per}_\Omega(A) < +\infty$ then for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap \partial^* A$

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{N-1}} \mathcal{H}^{N-1}((\Omega \cap \partial^* A) \cap (x + \delta Q_{\nu(x)})) = 1. \tag{2.1}$$

Theorem 2.3 (see [4, Lemma 3.1]) *Let A be a subset of Ω such that $\mathrm{Per}_\Omega(A) < +\infty$. There exists a sequence of polyhedral sets $\{A_k\}$ (i.e. A_k are bounded, Lipschitz domains with $\partial A_k = H_1 \cup H_2 \cup ... \cup H_p$, where each H_i is a closed subset of a hyperplane $\{x \in \mathbb{R}^N : x \cdot \nu_i = \alpha_i\}$) satisfying the following properties:*

(i) $\displaystyle\lim_{k\to\infty} \mathcal{L}^N[((A_k \cap \Omega) \setminus A) \cup (A \setminus (A_k \cap \Omega))] = 0$*;*

(ii) $\displaystyle\lim_{k\to\infty} \mathrm{Per}_\Omega(A_k) = \mathrm{Per}_\Omega(A)$*;*

(iii) $\mathcal{H}^{N-1}(\partial A_k \cap \partial\Omega) = 0$*;*

(iv) $\mathcal{L}^N(A_k) = \mathcal{L}^N(A)$.

Let $\varepsilon_n \to 0^+$. A functional

$$I : L^1(\Omega;\mathbb{R}^d) \to [0,\infty]$$

is called the Γ-*liminf* (resp. Γ-*limsup*) of a sequence of functionals $\{I_{\varepsilon_n}\}$ with respect to the strong convergence in $L^1(\Omega;\mathbb{R}^d)$ if for every $u \in L^1(\Omega;\mathbb{R}^d)$

$$I(u) = \inf\left\{\liminf_{n\to\infty} \text{ (resp. } \limsup_{n\to\infty}) I_{\varepsilon_n}(u_n) : u_n \in L^1(\Omega;\mathbb{R}^d),\ u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d)\right\},$$

and we write

$$I = \Gamma - \liminf_{n\to\infty} I_{\varepsilon_n} \left(\text{resp. } I = \Gamma - \limsup_{n\to\infty} I_{\varepsilon_n}\right).$$

We say that the sequence $\{I_{\varepsilon_n}\}$ Γ-*converges* to I if the Γ-liminf and the Γ-limsup coincide, and we write

$$I = \Gamma - \lim_{n\to\infty} I_{\varepsilon_n}.$$

The functional I is said to be the Γ-liminf (resp. Γ-limsup) of the family of functionals $\{I_\varepsilon\}$ with respect to the strong convergence in $L^1(\Omega;\mathbb{R}^d)$ if for every sequence $\varepsilon_n \to 0^+$ we have that

$$I = \Gamma - \liminf_{n\to\infty} I_{\varepsilon_n} \quad \left(\text{resp. } I = \Gamma - \limsup_{n\to\infty} I_{\varepsilon_n}\right),$$

and we write

$$I = \Gamma - \liminf_{\varepsilon\to 0} I_\varepsilon \quad \left(\text{resp. } I = \Gamma - \limsup_{\varepsilon\to 0} I_\varepsilon\right).$$

Finally, if Γ-liminf and Γ-limsup coincide, we say that I is the Γ-*limit* of the family of functionals $\{I_\varepsilon\}$, and we write

$$I = \Gamma - \lim_{\varepsilon\to 0} I_\varepsilon.$$

The following lemma is very useful in many diagonalization arguments.

Lemma 2.4 (Lemma 7.1 in [7]) *Let $\{a_{k,j}\}$ be a doubly indexed sequence of real numbers. If*

$$\lim_{k\to\infty}\lim_{j\to\infty} a_{k,j} = L,$$

then there exists an increasing subsequence $\{k(j)\} \nearrow +\infty$ such that $\lim_{j\to\infty} a_{k(j),j} = L$.

In order to prove Theorem 1.1, it is enough to show that every sequence $\{\varepsilon_n\}$ of positive numbers converging to zero has a subsequence $\{\varepsilon_{n_k}\}$ such that $I_{\varepsilon_{n_k}}\Gamma(L^1(\Omega;\mathbb{R}^d))$-converges to I_0 (see [10], [11]). We divide the proof of Theorem 1.1 into two parts, which are dealt with in Sections 3 and 4 of the paper. In the sequel, C will denote a generic positive constant that may vary from line to line, and expression to expression.

3 Compactness and the $\Gamma - \liminf$ inequality

We first show that the limit in the definition of $K_1(\nu)$ is well defined.

Lemma 3.1 *For all $\nu \in S^{N-1}$ the limit*

$$\lim_{T\to\infty}\frac{1}{T^{N-1}}\inf\left\{\int_{TQ_\nu}\left(W(y,u(y)) + |\nabla u(y)|^2\right)dy \;:\; u\in H^1(TQ_\nu;\mathbb{R}^d), u = \rho_{T,\nu} * u_0 \text{ on } \partial(TQ_\nu)\right\}$$

exists.

Proof. Assume, without loss of generality that $\nu = e_N$, and write ρ_T for ρ_{T,e_N}. For any $T > 0$, define

$$g(T) := \frac{1}{T^{N-1}}\inf\left\{\int_{TQ}\left(W(y,u(y)) + |\nabla u(y)|^2\right)dy \;:\; u\in H^1(TQ;\mathbb{R}^d), u = \rho_T * u_0 \text{ on } \partial(TQ)\right\},$$

and let $u_T \in H^1(TQ;\mathbb{R}^d)$ be such that $u_T = \rho_T * u_0$ on $\partial(TQ)$, and

$$\frac{1}{T^{N-1}}\int_{TQ}\left(W(y,u_T(y)) + |\nabla u_T(y)|^2\right)dy \le g(T) + \frac{1}{T}. \tag{3.1}$$

Let $S > T+3$, and let $E_{T,S}, E^*_{T,S} \subset \left(S - \frac{1}{T}\right) Q \cap \{x \in \mathbb{R}^N \ : \ x_N = 0\}$, $M_{S,T} := \left[\left(\frac{S-\frac{1}{T}}{[T]+2}\right)^{N-1}\right] \in \mathbb{N}$, and $z_i \in \mathbb{Z}^{N-1} \times \{0\}$ $(i = 1, \cdots, M_{S,T})$ be such that

$$\left(S - \frac{1}{T}\right) Q \cap \left\{x \in \mathbb{R}^N \ : \ x_N = 0\right\} = \left(\bigcup_{i=1}^{M_{S,T}} (z_i + ([T]+2)Q) \cap \left\{x \in \mathbb{R}^N \ : \ x_N = 0\right\}\right) \cup E_{T,S}$$
$$= \left(\bigcup_{i=1}^{M_{S,T}} (z_i + TQ) \cap \left\{x \in \mathbb{R}^N \ : \ x_N = 0\right\}\right) \cup E^*_{T,S}.$$

We have

$$\mathcal{L}^{N-1}\left(E_{T,S}\right) = \left(S - \frac{1}{T}\right)^{N-1} - M_{S,T}([T]+2)^{N-1}$$

and so, since

$$E^*_{T,S} = E_{T,S} \cup \left(\bigcup_{i=1}^{M_{S,T}} ((z_i + ([T]+2)Q) \setminus (z_i + TQ)) \cap \left\{x \in \mathbb{R}^N \ : \ x_N = 0\right\}\right),$$

we obtain

$$\mathcal{L}^{N-1}\left(E^*_{T,S}\right) = \left(S - \frac{1}{T}\right)^{N-1} - M_{S,T}\, T^{N-1}. \tag{3.2}$$

Consider cut-off functions $\varphi_{S,T} \in C_c(SQ;[0,1])$ and, for $2 \le m < T$, $i \in \{1, \cdots, M_{S,T}\}$, $\varphi_{m,i} \in C_c\left(z_i + \left(T + \frac{1}{m}\right) Q; [0,1]\right)$ such that

$$\varphi_{S,T}(x) = 0 \text{ if } x \in \partial(SQ),\ \varphi_{S,T}(x) = 1 \text{ if } x \in \left(S - \frac{1}{T}\right) Q,\ \|\nabla\varphi_{S,T}\|_\infty \le CT,$$

and

$$\varphi_{m,i}(x) = 0 \text{ if } x \in \partial\left(z_i + \left(T + \frac{1}{m}\right) Q\right),\ \varphi_{m,i}(x) = 1 \text{ if } x \in z_i + TQ,\ \|\nabla\varphi_{m,i}\|_\infty \le Cm.$$

Define $v_S \in H^1(SQ;\mathbb{R}^d)$ by

$$v_S(x) := \begin{cases} u_T\left(x - z_i\right) & \text{if } x \in z_i + TQ, \\ \varphi_{m,i}(x)(\rho_T * u_0)(x) + (1 - \varphi_{m,i}(x))(\rho_m * u_0)(x) & \text{if } x \in \left(z_i + \left(T + \frac{1}{m}\right) Q\right) \setminus (z_i + TQ), \\ \varphi_{S,T}(x)(\rho_m * u_0)(x) + (1 - \varphi_{S,T}(x))(\rho_S * u_0)(x) & \text{if } x \in SQ \setminus \left(S - \frac{1}{T}\right) Q. \end{cases}$$

Note that since $z_i \cdot e_N = 0$, we have

$$u_T(\cdot - z_i)_{|\partial(z_i+TQ)} = (\rho_T * u_0)(\cdot - z_i)_{|\partial(z_i+TQ)} = (\rho_T * u_0)(\cdot)_{|\partial(z_i+TQ)},$$

and thus v_S is a Sobolev function, admissible for the infimum in the definition of $g(S)$. We obtain that

$$g(S) \le \frac{1}{S^{N-1}} \int_{SQ} \left(W(x, v_S(x)) + |\nabla v_S(x)|^2\right) dx$$
$$\le I_1(S,T) + I_2(S,T,m) + I_3(S,T,m) + I_4(S,T,m), \tag{3.3}$$

where

$$I_1(S,T) := \frac{1}{S^{N-1}} \sum_{i=1}^{M_{S,T}} \int_{z_i+TQ} \left(W(x, u_T(x-z_i)) + |\nabla u_T(x-z_i)|^2 \right) dx,$$

$$I_2(S,T,m) := \frac{1}{S^{N-1}} \sum_{i=1}^{M_{S,T}} \int_{\left(z_i+\left(T+\frac{1}{m}\right)Q\right)\setminus(z_i+TQ)} \Bigg(W\left(x, \left(\varphi_{m,i}(\rho_T * u_0) + (1-\varphi_{m,i})(\rho_m * u_0)\right)(x)\right) \\ + \left|\nabla\left(\varphi_{m,i}(\rho_T * u_0) + (1-\varphi_{m,i})(\rho_m * u_0)\right)(x)\right|^2 \Bigg) dx,$$

$$I_3(S,T,m) = \frac{1}{S^{N-1}} \int_{E^*_{T,S}\times\left(-\frac{1}{m},\frac{1}{m}\right)} \left(W\left(x, (\rho_m * u_0)(x)\right) + |\nabla(\rho_m * u_0)(x)|^2 \right) dx,$$

and

$$I_4(S,T,m) := \frac{1}{S^{N-1}} \int_{SQ\setminus\left(S-\frac{1}{T}\right)Q} \Bigg(W\left(x, \left(\varphi_{S,T}(\rho_m * u_0) + (1-\varphi_{S,T})(\rho_S * u_0)\right)(x)\right) \\ + \left|\nabla\left(\varphi_{S,T}(\rho_m * u_0) + (1-\varphi_{S,T})(\rho_S * u_0)\right)(x)\right|^2 \Bigg) dx.$$

In view of (H1) and (3.1), and because $z_i \in \mathbb{Z}^N$, we get $W(\cdot + z_i, \cdot) = W(\cdot\,, \cdot)$, and

$$I_1(S,T) = \frac{1}{S^{N-1}} M_{S,T} \int_{TQ} \left(W(x, u_T(x)) + |\nabla u_T(x)|^2 \right) dx \le \frac{1}{S^{N-1}} M_{S,T}\, T^{N-1} \left(g(T) + \frac{1}{T} \right) \\ \le g(T) + \frac{1}{T}. \tag{3.4}$$

Using (H2), (H3), and the facts that $(\rho_T * u_0)(x) \in \{a,b\}$ if $|x_N| \ge \frac{1}{T}$ and $(\rho_m * u_0)(x) \in \{a,b\}$ if $|x_N| \ge \frac{1}{m}$, we obtain that

$$I_2(S,T,m) \le \frac{C}{S^{N-1}} \sum_{i=1}^{M_{S,T}} \int_{\left(\left(z_i+\left(T+\frac{1}{m}\right)Q\right)\setminus(z_i+TQ)\right)\cap\left\{x\in\mathbb{R}^N\,:\,|x_N|<\frac{1}{T}\right\}} \Big(1 + |\rho_T * u_0|^q \\ + \|\nabla(\rho_T * u_0)\|_\infty^2 \Big) dx \\ + \frac{C}{S^{N-1}} \sum_{i=1}^{M_{S,T}} \int_{\left(\left(z_i+\left(T+\frac{1}{m}\right)Q\right)\setminus(z_i+TQ)\right)\cap\left\{x\in\mathbb{R}^N\,:\,|x_N|<\frac{1}{m}\right\}} \Big(1 + |\rho_T * u_0|^q + |\rho_m * u_0|^q \\ + \|\nabla\varphi_{m,i}\|_\infty^2 + \|\nabla(\rho_m * u_0)\|_\infty^2 \Big) dx$$

$$
\begin{aligned}
&\le \frac{C}{S^{N-1}} M_{S,T} \left(\left(T + \frac{1}{m} \right)^{N-1} - T^{N-1} \right) \frac{1+T^2}{T} \\
&\qquad\qquad + \frac{C}{S^{N-1}} M_{S,T} \left(\left(T + \frac{1}{m} \right)^{N-1} - T^{N-1} \right) \frac{1+m^2}{m} \\
&\le C \left(\frac{1}{m} \cdot \frac{T^{N-2} + T^N}{T^N} + \frac{1+m^2}{m^2} \cdot \frac{1}{T} \right).
\end{aligned} \tag{3.5}
$$

Using again (H3), and in view of (3.2), we have

$$
\begin{aligned}
I_3(S,T,m) &\le \frac{C}{S^{N-1}} \int_{E^*_{T,S} \times \left(-\frac{1}{m}, \frac{1}{m}\right)} \left(1 + |(\rho_m * u_0)(x)|^q + |\nabla(\rho_m * u_0)(x)|^2\right) dx \\
&\le \frac{C}{S^{N-1}} \cdot \frac{1+m^2}{m} \left(\left(S - \frac{1}{T} \right)^{N-1} - T^{N-1} \left(\left(\frac{S - \frac{1}{T}}{[T]+2} \right)^{N-1} - 1 \right) \right) \\
&= \frac{C(1+m^2)}{m} \left(\left(1 - \frac{1}{ST} \right)^{N-1} - \left(\frac{T}{[T]+2} \right)^{N-1} \left(1 - \frac{1}{ST} \right)^{N-1} + \left(\frac{T}{S} \right)^{N-1} \right).
\end{aligned} \tag{3.6}
$$

Finally,

$$
\begin{aligned}
I_4(S,T,m) &\le \frac{C}{S^{N-1}} \int_{\left(SQ \setminus \left(S - \frac{1}{T}\right)Q\right) \cap \left\{x \in \mathbb{R}^N \,:\, |x_N| < \frac{1}{S}\right\}} \left(1 + |\rho_S * u_0|^q + \|\nabla(\rho_S * u_0)\|^2_\infty \right) dx \\
&\quad + \frac{C}{S^{N-1}} \int_{\left(SQ \setminus \left(S - \frac{1}{T}\right)Q\right) \cap \left\{x \in \mathbb{R}^N \,:\, |x_N| < \frac{1}{m}\right\}} \Big(1 + |\rho_S * u_0|^q + |\rho_m * u_0|^q \\
&\qquad\qquad + \|\nabla \varphi_{S,T}\|^2_\infty + \|\nabla(\rho_m * u_0)\|^2_\infty \Big) dx \\
&\le \frac{C(1+S^2)}{S^{N-1}} \left(S^{N-1} - \left(S - \frac{1}{T} \right)^{N-1} \right) \frac{1}{S} \\
&\quad + \frac{C(1+T^2+m^2)}{S^{N-1}} \left(S^{N-1} - \left(S - \frac{1}{T} \right)^{N-1} \right) \frac{1}{m} \\
&\le C \frac{S^{N-2} + S^N}{S^N} \cdot \frac{1}{T} + C \frac{(1+T^2+m^2)}{Tm} \cdot \frac{1}{S}.
\end{aligned} \tag{3.7}
$$

Taking into account (3.4), (3.5), (3.6), and (3.7), we obtain

$$
\limsup_{m\to\infty} \liminf_{T\to\infty} \limsup_{S\to\infty} \left(I_1(S,T) + I_2(S,T,m) + I_3(S,T,m) + I_4(S,T,m) \right) \le \liminf_{T\to\infty} g(T).
$$

Thus, in view of (3.3), we deduce that

$$
\limsup_{S\to\infty} g(S) \le \liminf_{T\to\infty} g(T).
$$

□

We continue this section with the following compactness result

Theorem 3.2 *Let $\varepsilon_n \to 0^+$, and $\{u_n\} \subset H^1(\Omega;\mathbb{R}^d)$ be such that*

$$\sup_{n\in\mathbb{N}} \int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx < +\infty.$$

Then there exists $u \in L^1(\Omega;\mathbb{R}^d)$, with $u(x) \in \{a,b\}$ $\mathcal{L}^N$-a.e $x \in \Omega$ such that, up to a subsequence, $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^d)$.

Proof. First, note that

$$\lim_{n\to\infty} \int_\Omega W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) dx = 0. \tag{3.8}$$

By the coercivity condition in (H3), there exists a constant $R > 0$ such that

$$W(y,u) \geq C|u| \text{ for } \mathcal{L}^N - \text{a.e. } y \in \mathbb{R}^N, \ |u| > R.$$

Define $w_n(x) := u_n(x)\chi_{\{x\in\Omega \,:\, |u_n(x)|>R\}}(x)$, and set $v_n(x) := u_n(x) - w_n(x)$, $x \in \Omega$. Thus,

$$\int_\Omega |w_n(x)|dx = \int_{\{x\in\Omega \,:\, |u_n(x)|>R\}} |u_n(x)|dx \leq \frac{1}{C} \int_\Omega W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) dx,$$

which gives, in view of (3.8),

$$w_n \to 0 \text{ strongly in } L^1(\Omega;\mathbb{R}^d). \tag{3.9}$$

Taking into account (3.8) one more time, we also have that

$$\begin{aligned}\int_\Omega W\left(\frac{x}{\varepsilon_n}, v_n(x)\right) dx &= \int_{\{x\in\Omega \,:\, |u_n(x)|\leq R\}} W\left(\frac{x}{\varepsilon_n}, v_n(x)\right) dx + \int_{\{x\in\Omega \,:\, |u_n(x)|>R\}} W\left(\frac{x}{\varepsilon_n}, 0\right) dx \\ &\leq \left(1 + \frac{C}{R}\right) \int_\Omega W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) dx \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.10}$$

Set $\overline{W}(u) := \min_{y\in\overline{Q}} W(y,u)$, and note that $\overline{W} : \mathbb{R}^d \to [0,\infty)$ is continuous on $\mathbb{R}^d$, $\overline{W}(u) = 0$ if and only if $u \in \{a,b\}$, and that by the coercivity condition in (H3) there exists $C > 0$ such that $\overline{W}(u) \geq C|u|$, for $|u|$ sufficiently large. We have

$$0 \leq \int_\Omega \overline{W}(v_n(x))dx \leq \int_\Omega W\left(\frac{x}{\varepsilon_n}, v_n(x)\right) dx.$$

Thus, by (3.10),

$$\lim_{n\to\infty} \int_\Omega \overline{W}(v_n(x))dx = 0.$$

We may now proceed as in Fonseca and Tartar [16] (see the proof of their Theorem 4.1) to conclude that there exists $u \in L^1(\Omega;\mathbb{R}^d)$, and a subsequence (not relabelled) of $\{v_n\}$, which converges to u strongly in $L^1(\Omega;\mathbb{R}^d)$. Taking into account (3.9), and since $u_n = v_n + w_n$, we deduce that $u_n \to u$ strongly in $L^1(\Omega;\mathbb{R}^d)$. □

In the remainder of the section we prove part (i) of Theorem 1.1. Precisely,

Proposition 3.3 *Let (H1)-(H3) hold, and let $u \in L^1(\Omega;\mathbb{R}^d)$ be given. If $\varepsilon_n \to 0^+$ and if $\{u_n\} \subset H^1(\Omega;\mathbb{R}^d)$ is such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, then*

$$\liminf_{n\to\infty} I_{\varepsilon_n}(u_n) \geq I_0(u).$$

Proof. *Step 1.* If $u \in L^1(\Omega;\mathbb{R}^d)$ and $\mathcal{L}^N(\{x \in \Omega \ : \ u(x) \notin \{a,b\}\}) > 0$ then

for any sequence $\varepsilon_n \to 0^+$ and for any $\{u_n\} \subset H^1(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, we have

$$\int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2\right) dx \to +\infty. \tag{3.11}$$

Indeed, if for some sequences $\varepsilon_n \to 0^+$, and $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$

$$\sup_{n\in\mathbb{N}} \int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2\right) dx < +\infty,$$

then

$$\lim_{n\to\infty} \int_\Omega W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) dx = 0. \tag{3.12}$$

For $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, consider the Q-periodic function $v(x) = \{x\} := (\{x_1\}, \cdots, \{x_N\})$, where, for each $i \in \{1, 2, \cdots, N\}$, $\{x_i\} := x_i - [x_i]$ ($[y]$ stands for the integer part of the real number y), and define $v_n(x) := v\left(\frac{x}{\varepsilon_n}\right)$. Up to a subsequence (not relabelled), $\{v_n\}$ and $\{u_n\}$ generate the Young measures $\{\nu_x\}_{x\in\Omega}$ and $\{\mu_x\}_{x\in\Omega}$ respectively, where $\{\nu_x\}_{x\in\Omega}$ is homogeneous

$$\langle \nu_x, \varphi\rangle = \langle \nu, \varphi\rangle := \int_Q \varphi(v(y))dy \text{ for } \mathcal{L}^N - \text{a.e. } x \in \Omega,$$

and, in view of the strong convergence of u_n to u in $L^1(\Omega;\mathbb{R}^d)$,

$$\mu_x = \delta_{u(x)} \text{ for } \mathcal{L}^N - \text{a.e. } x \in \Omega.$$

Thus, the sequence $(v_n, u_n) : \Omega \to \mathbb{R}^N \times \mathbb{R}^d$ generates the Young measure $\{\nu \otimes \delta_{u(x)}\}_{x\in\Omega}$. By the Fundamental Theorem on Young measures, and using the periodicity of W in its first variable, we have that

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) dx &= \liminf_{n\to\infty} \int_\Omega W\left(v_n(x), u_n(x)\right) dx \\ &\geq \int_\Omega \int_{\mathbb{R}^N\times\mathbb{R}^d} W(A,B) d(\nu \otimes \delta_{u(x)})(A,B) dx = \int_\Omega \int_Q W(y, u(x)) dy dx.\end{aligned}$$

Thus, in view of (3.12),

$$\int_\Omega \int_Q W(y, u(x)) dy dx \leq 0.$$

The fact that W is nonnegative, together with (H2), implies that $u(x) \in \{a,b\}$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, and we have reached a contradiction.

Step 2. Let $u(x) = \chi_{A_0}(x) \cdot a + (1 - \chi_{A_0}(x)) \cdot b$ and assume that $u \notin BV(\Omega;\mathbb{R}^d)$, that is, $\mathrm{Per}_\Omega(A_0) = +\infty$. We will show once again that (3.11) is satisfied. We argue by contradiction. Suppose that there exists a subsequence (not relabelled) such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, and

$$\sup_{n\in\mathbb{N}} \int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx < +\infty.$$

Then, by the Cauchy-Schwarz inequality, we obtain that,

$$\sup_{n\in\mathbb{N}} \int_\Omega \sqrt{W\left(\frac{x}{\varepsilon_n}, u_n(x)\right)}\, |\nabla u_n(x)|\, dx \leq C.$$

Set $\overline{W}(u) := \min_{y\in\overline{Q}} W(y,u)$. As we have already observed (see the proof of Theorem 3.2), $\overline{W} : \mathbb{R}^d \to [0,\infty)$ is continuous on $\mathbb{R}^d$, $\overline{W}(u) = 0$ if and only if $u \in \{a,b\}$, and there exists $C > 0$ such that $\overline{W}(u) \geq C|u|$, for $|u|$ sufficiently large. In view of Lemma 3.7 in [16], for suitable $M > 0$ the function

$$\Phi(u) := \inf\left\{ \int_{-1}^{1} \sqrt{\min\{\overline{W}(\gamma(s)), M\}}\, |\gamma'(s)|\, ds : \gamma \text{ is piecewise } C^1, \gamma(-1) = a,\ \gamma(1) = u \right\}$$

is Lipschitz continuous and $|\nabla(\Phi \circ v)(x)| \leq \sqrt{\overline{W}(v(x))}|\nabla v(x)|$ for any $v \in H^1(\Omega;\mathbb{R}^d)$, and $\mathcal{L}^N$-a.e. $x \in \Omega$. Thus

$$\sup_{n\in\mathbb{N}} \|\nabla(\Phi \circ u_n)\|_{L^1(\Omega;\mathbb{R}^d)} < +\infty.$$

Therefore $|D(\Phi \circ u)|(\Omega) < +\infty$, and since $\Phi \circ u = \chi_{A_0}\Phi(a) + (1 - \chi_{A_0})\Phi(b)$, we obtain that $\mathrm{Per}_\Omega(A_0) < +\infty$, which contradicts our initial assumption on u.

Step 3. It remains to prove the proposition in the case where $u(x) = \chi_{A_0}(x) \cdot a + (1 - \chi_{A_0}(x)) \cdot b$ with $\mathrm{Per}_\Omega(A_0) < +\infty$. Here, $I_0(u) = \int_{\Omega\cap\partial^* A_0} K_1(\nu(x)) d\mathcal{H}^{N-1}(x)$, and it suffices to show that for any sequence $\varepsilon_n \to 0^+$ and for any $\{u_n\} \subset H^1(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, we have

$$\int_{\Omega\cap\partial^* A_0} K_1(\nu(x)) d\mathcal{H}^{N-1}(x) \leq \liminf_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx. \tag{3.13}$$

Upon extracting a subsequence (not relabelled) we may assume, without loss of generality, that

$$\begin{aligned} &\liminf_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx \\ &= \lim_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx < +\infty. \end{aligned}$$

and that there exists a finite Radon measure $\mu \geq 0$, such that

$$\frac{1}{\varepsilon_n} W\left(\frac{\cdot}{\varepsilon_n}, u_n(\cdot)\right) + \varepsilon_n |\nabla u_n(\cdot)|^2 \rightharpoonup \mu, \tag{3.14}$$

weakly* in the sense of measures. We claim that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\,(\Omega\cap\partial^*A_0)}(x)\geq K_1(\nu(x)),\ \text{for } \mathcal{H}^{N-1}-\text{a.e. } x\in\Omega\cap\partial^*A_0. \tag{3.15}$$

Let $\delta_k\to 0^+$ be such that for $\mathcal{H}^{N-1}-$ a.e. $x_0\in\Omega\cap\partial^*A_0$ we have $\mu(\partial Q_{\nu(x_0)}(x_0,\delta_k))=0$ for all $k\in\mathbb{N}$, and

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\,(\Omega\cap\partial^*A_0)}(x_0)=\lim_{\delta\to 0}\frac{\mu(Q_{\nu(x_0)}(x_0,\delta))}{\delta^{N-1}},$$

where we have taken into account (2.1). Thus, in view of (3.14), we have that

$$\begin{aligned}\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\,(\Omega\cap\partial^*A_0)}(x_0)&=\lim_{k\to\infty}\frac{\mu(Q_{\nu(x_0)}(x_0,\delta_k))}{\delta_k^{N-1}}\\&=\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\delta_k^{N-1}}\int\limits_{Q_{\nu(x_0)}(x_0,\delta_k)}\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},u_n(x)\right)+\varepsilon_n|\nabla u_n(x)|^2\right)dx.\end{aligned}$$

Let $u_{k,n}(x):=u_n(x_0+\delta_k x),\ x\in Q_{\nu(x_0)}$. Changing variables, we deduce that

$$\begin{aligned}&\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\,(\Omega\cap\partial^*A_0)}(x_0)\\&=\lim_{k\to\infty}\lim_{n\to\infty}\int\limits_{Q_{\nu(x_0)}}\left(\frac{\delta_k}{\varepsilon_n}W\left(\frac{x_0+\delta_k y}{\varepsilon_n},u_n(x_0+\delta_k y)\right)+\varepsilon_n\delta_k|\nabla u_n(x_0+\delta_k y)|^2\right)dy\\&=\lim_{k\to\infty}\lim_{n\to\infty}\int\limits_{Q_{\nu(x_0)}}\left(\frac{\delta_k}{\varepsilon_n}W\left(\frac{x_0+\delta_k y}{\varepsilon_n},u_{k,n}(y)\right)+\frac{\varepsilon_n}{\delta_k}|\nabla u_{k,n}(y)|^2\right)dy.\end{aligned} \tag{3.16}$$

Let $m_n\in\mathbb{Z}^N$ and $s_n\in[0,1)^N$ be such that $\frac{x_0}{\varepsilon_n}=m_n+s_n$. Put $x_{k,n}:=-\frac{\varepsilon_n}{\delta_k}s_n$, and note that we have $\lim\limits_{k\to\infty}\lim\limits_{n\to\infty}x_{k,n}=0$. Changing variables, and using the periodicity of $W(\cdot,u)$, we obtain that

$$\begin{aligned}&\lim_{k\to\infty}\lim_{n\to\infty}\int\limits_{Q_{\nu(x_0)}}\left(\frac{\delta_k}{\varepsilon_n}W\left(\frac{x_0+\delta_k y}{\varepsilon_n},u_{k,n}(y)\right)+\frac{\varepsilon_n}{\delta_k}|\nabla u_{k,n}(y)|^2\right)dy\\&=\lim_{k\to\infty}\lim_{n\to\infty}\int\limits_{-x_{k,n}+Q_{\nu(x_0)}}\left(\frac{\delta_k}{\varepsilon_n}W\left(\frac{x_0+\delta_k(z+x_{k,n})}{\varepsilon_n},u_{k,n}(z+x_{k,n})\right)+\frac{\varepsilon_n}{\delta_k}|\nabla u_{k,n}(z+x_{k,n})|^2\right)dz\\&=\lim_{k\to\infty}\lim_{n\to\infty}\int\limits_{-x_{k,n}+Q_{\nu(x_0)}}\left(\frac{\delta_k}{\varepsilon_n}W\left(\frac{\delta_k}{\varepsilon_n}z,u_{k,n}(z+x_{k,n})\right)+\frac{\varepsilon_n}{\delta_k}|\nabla u_{k,n}(z+x_{k,n})|^2\right)dz.\end{aligned} \tag{3.17}$$

Recall that

$$u_0(x)=\begin{cases}b & \text{if } x\cdot\nu(x_0)>0,\\ a & \text{if } x\cdot\nu(x_0)<0.\end{cases}$$

We claim that

$$\lim_{k\to\infty}\lim_{n\to\infty}\|u_{k,n}-u_0\|_{L^1(Q_{\nu(x_0)};\mathbb{R}^d)}=0. \tag{3.18}$$

Indeed, after changing variables, and making use of Lebesgue's Dominated Convergence Theorem,

$$
\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_{\nu(x_0)}} |u_{k,n}(x)-u_0(x)|dx \\
&= \lim_{k\to\infty}\lim_{n\to\infty}\left\{\int_{Q_{\nu(x_0)}\cap\{x\in\mathbb{R}^N : x\cdot\nu(x_0)>0\}} |u_n(x_0+\delta_k x)-b|dx \right.\\
&\qquad\qquad\left. + \int_{Q_{\nu(x_0)}\cap\{x\in\mathbb{R}^N : x\cdot\nu(x_0)<0\}} |u_n(x_0+\delta_k x)-a|dx\right\} \\
&= \lim_{k\to\infty}\left\{\int_{Q_{\nu(x_0)}\cap\{x\in\mathbb{R}^N : x\cdot\nu(x_0)>0\}} |u(x_0+\delta_k x)-b|dx + \int_{Q_{\nu(x_0)}\cap\{x\in\mathbb{R}^N : x\cdot\nu(x_0)<0\}} |u(x_0+\delta_k x)-a|dx\right\} \\
&= \lim_{k\to\infty}\frac{1}{\delta_k^N}\left\{\int_{Q_{\nu(x_0)}(x_0,\delta_k)\cap\{x\in\mathbb{R}^N : x\cdot\nu(x_0)>x_0\cdot\nu(x_0)\}} |u(x)-b|dx \right.\\
&\qquad\qquad\left. + \int_{Q_{\nu(x_0)}(x_0,\delta_k)\cap\{x\in\mathbb{R}^N : x\cdot\nu(x_0)<x_0\cdot\nu(x_0)\}} |u(x)-a|dx\right\} \\
&= |b-a|\lim_{k\to\infty}\left\{\frac{\mathcal{L}^N(\{x\in Q_{\nu(x_0)}(x_0,\delta_k)\cap A_0 : x\cdot\nu(x_0)>x_0\cdot\nu(x_0)\})}{\delta_k^N}\right.\\
&\qquad\qquad\left. + \frac{\mathcal{L}^N(\{x\in Q_{\nu(x_0)}(x_0,\delta_k)\setminus A_0 : x\cdot\nu(x_0)<x_0\cdot\nu(x_0)\})}{\delta_k^N}\right\} = 0,
\end{aligned}
$$

where the last equality follows by Theorem 2.2.

A diagonalization process allows us to find an increasing sequence $\{n_k\} \nearrow \infty$ such that, denoting $\eta_k := \frac{\varepsilon_{n_k}}{\delta_k}$, $x_k := x_{k,n_k}$, $w_k(z) := u_{k,n_k}(z+x_k)$, we have

$$
\lim_{k\to\infty}\eta_k = \lim_{k\to\infty}\lim_{n\to\infty}\frac{\varepsilon_n}{\delta_k} = 0,
$$

$$
\lim_{k\to\infty}x_k = \lim_{k\to\infty}\lim_{n\to\infty}x_{k,n} = 0,
$$

in view of (3.18),

$$
\lim_{k\to\infty}\|w_k-u_0\|_{L^1(Q_{\nu(x_0)};\mathbb{R}^d)} = \lim_{k\to\infty}\lim_{n\to\infty}\|u_{k,n}-u_0\|_{L^1(Q_{\nu(x_0)};\mathbb{R}^d)} = 0,
$$

and, in addition,

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{-x_k+Q_{\nu(x_0)}}\left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k},w_k(z)\right)+\eta_k|\nabla w_k(z)|^2\right)dz \\
&= \lim_{k\to\infty}\lim_{n\to\infty}\int_{-x_{k,n}+Q_{\nu(x_0)}}\left(\frac{\delta_k}{\varepsilon_n}W\left(\frac{\delta_k}{\varepsilon_n}z,u_{k,n}(z+x_{k,n})\right)+\frac{\varepsilon_n}{\delta_k}|\nabla u_{k,n}(z+x_{k,n})|^2\right)dz.
\end{aligned}
$$

By (3.16) and (3.17), we obtain

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\,(\Omega\cap\partial^* A_0)}(x_0) = \lim_{k\to\infty}\int_{-x_k+Q_{\nu(x_0)}} \left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k}, w_k(z)\right) + \eta_k|\nabla w_k(z)|^2\right)dz. \tag{3.19}$$

Since $x_k \to 0$ as $k \to \infty$, for k sufficiently large there exists a cube $Q_k \subset\subset Q_{\nu(x_0)}$, such that $Q_k \subset (-x_k + Q_{\nu(x_0)})$, and $\lim_{k\to\infty} \mathcal{L}^N(Q_{\nu(x_0)} \setminus Q_k) = 0$. In view of (3.19), we deduce that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\,(\Omega\cap\partial^* A_0)}(x_0) \geq \limsup_{k\to\infty}\int_{Q_k} \left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k}, w_k(z)\right) + \eta_k|\nabla w_k(z)|^2\right)dz. \tag{3.20}$$

Let $L_{k,j} := \{x \in Q_k : \operatorname{dist}(x, \partial Q_k) < 1/j\}$. Divide $L_{k,j}$ into $M_{k,j}$ equidistant layers $L^{(i)}_{k,j}$ ($i = 1, \cdots, M_{k,j}$) of width $\eta_k\|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|^{1/2}_{L^2(Q_{\nu(x_0)};\mathbb{R}^d)}$, so that

$$M_{k,j}\eta_k\|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|^{1/2}_{L^2(Q_{\nu(x_0)};\mathbb{R}^d)} = O(1/j) \tag{3.21}$$

Select now one of these layers $L^{(i_0)}_{k,j}$ such that

$$\begin{aligned}
&\int_{L^{(i_0)}_{k,j}} \Bigg(1 + |w_k|^q + |\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^q + \eta_k^2|\nabla w_k|^2 + \eta_k^2|\nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)|^2 \\
&\qquad\qquad + \frac{|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^2}{\|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|_{L^2(Q_{\nu(x_0)};\mathbb{R}^d)}}\Bigg)dx \\
&\leq \frac{1}{M_{k,j}}\int_{L_{k,j}} \Bigg(1 + |w_k|^q + |\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^q + \eta_k^2|\nabla w_k|^2 + \eta_k^2|\nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)|^2 \\
&\qquad\qquad + \frac{|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^2}{\|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|_{L^2(Q_{\nu(x_0)};\mathbb{R}^d)}}\Bigg)dx.
\end{aligned} \tag{3.22}$$

Consider cut-off functions $\varphi_{k,j} \in C_c^\infty(Q_{\nu(x_0)}; [0, 1])$ such that

$$\varphi_{k,j}(x) = 0 \text{ if } x \in \left(\bigcup_{i=i_0+1}^{M_{k,j}} L^{(i)}_{k,j}\right) \cup (Q_{\nu(x_0)} \setminus Q_k),$$

$$\varphi_{k,j}(x) = 1 \text{ if } x \in \left(\bigcup_{i=1}^{i_0-1} L^{(i)}_{k,j}\right) \cup (Q_k \setminus L_{k,j}),$$

and

$$\|\nabla\varphi_{k,j}\|_\infty = O\left(\frac{1}{\eta_k\|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|^{1/2}_{L^2(Q_{\nu(x_0)};\mathbb{R}^d)}}\right).$$

Define

$$w_{k,j} := \varphi_{k,j}w_k + (1 - \varphi_{k,j})(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0).$$

We have

$$\int_{Q_k} \left(\frac{1}{\eta_k} W\left(\frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) dz$$

$$= \int_{\left(\bigcup_{i=1}^{i_0-1} L_{k,j}^{(i)}\right) \cup (Q_k \setminus L_{k,j})} \left(\frac{1}{\eta_k} W\left(\frac{z}{\eta_k}, w_k(z) \right) + \eta_k |\nabla w_k(z)|^2 \right) dz$$

$$+ \int_{L_{k,j}^{(i_0)}} \left(\frac{1}{\eta_k} W\left(\frac{z}{\eta_k}, w_{k,j}(z) \right) + \eta_k |\nabla w_{k,j}(z)|^2 \right) dz$$

$$+ \int_{\bigcup_{i=i_0+1}^{M_{k,j}} L_{k,j}^{(i)}} \left(\frac{1}{\eta_k} W\left(\frac{z}{\eta_k}, (\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)(z) \right) + \eta_k |\nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)(z)|^2 \right) dz$$

$$=: A_{k,j} + B_{k,j} + C_{k,j}. \tag{3.23}$$

Taking into account the growth condition in (H3), we have

$$B_{k,j} \le C \int_{L_{k,j}^{(i_0)}} \Bigg(\frac{1}{\eta_k} \left(1 + |w_k|^q + |\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^q \right) + \eta_k \Big(|\nabla w_k - \nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)|^2$$

$$+ |\nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)|^2 + \|\nabla \varphi_{k,j}\|_\infty^2 |w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^2 \Big) \Bigg) dx$$

$$\le \frac{C}{\eta_k} \int_{L_{k,j}^{(i_0)}} \Bigg(1 + |w_k|^q + |\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^q + \eta_k^2 |\nabla w_k|^2 + \eta_k^2 |\nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)|^2$$

$$+ \frac{|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^2}{\|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|_{L^2(Q_{\nu(x_0)};\mathbb{R}^d)}} \Bigg) dx$$

In view of (3.21) and (3.22) we obtain the estimate

$$B_{k,j} \le \frac{C}{\eta_k M_{k,j}} \int_{L_{k,j}} \Bigg(1 + |w_k|^q + |\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^q + \eta_k^2 |\nabla w_k|^2 + \eta_k^2 |\nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)|^2$$

$$+ \frac{|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^2}{\|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|_{L^2(Q;\mathbb{R}^d)}} \Bigg) dx$$

$$= O(j) \|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|_{L^2(Q_{\nu(x_0)};\mathbb{R}^d)}^{1/2} \int_{L_{k,j}} \Bigg(1 + |w_k|^q + |\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^q + \eta_k^2 |\nabla w_k|^2 +$$

$$+ \eta_k^2 |\nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)|^2 + \frac{|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0|^2}{\|w_k - \rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0\|_{L^2(Q_{\nu(x_0)};\mathbb{R}^d)}} \Bigg) dx,$$

which gives

$$\limsup_{j \to \infty} \limsup_{k \to \infty} B_{k,j} = 0. \tag{3.24}$$

Using again the growth condition in (H3), we have that

$$\begin{aligned}
&\limsup_{j\to\infty}\limsup_{k\to\infty} C_{k,j}\\
&\le \lim_{j\to\infty}\limsup_{k\to\infty}\int\limits_{\bigcup_{i=i_0+1}^{M_{k,j}} L_{k,j}^{(i)}} \frac{1}{\eta_k}\left(1+|(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)(z)|^q+\eta_k^2|\nabla(\rho_{\frac{1}{\eta_k},\nu(x_0)} * u_0)(z)|^2\right)dz\\
&\le \limsup_{j\to\infty}\limsup_{k\to\infty}\frac{C}{\eta_k}\mathcal{L}^N\left(\left(\bigcup_{i=i_0+1}^{M_{k,j}} L_{k,j}^{(i)}\right)\cap\{x\in Q_{\nu(x_0)} : |x\cdot\nu(x_0)|<\eta_k\}\right)=0. \qquad (3.25)
\end{aligned}$$

Similarly, and in view of our choice of the cubes Q_k, we obtain that

$$\begin{aligned}
&\limsup_{k\to\infty}\int\limits_{Q_{\nu(x_0)}\setminus Q_k}\left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k},w_{k,j}(z)\right)+\eta_k|\nabla w_{k,j}(z)|^2\right)dz\\
&\le \limsup_{k\to\infty}\frac{C}{\eta_k}\mathcal{L}^N\left((Q_{\nu(x_0)}\setminus Q_k)\cap\{x\in Q_{\nu(x_0)} : |x\cdot\nu(x_0)|<\eta_k\}\right)=0,
\end{aligned}$$

and thus, taking into account (3.23), (3.24), and (3.25),

$$\begin{aligned}
&\limsup_{j\to\infty}\limsup_{k\to\infty}\int\limits_{Q_{\nu(x_0)}}\left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k},w_{k,j}(z)\right)+\eta_k|\nabla w_{k,j}(z)|^2\right)dz\\
&=\limsup_{j\to\infty}\limsup_{k\to\infty}\left(A_{k,j}+B_{k,j}+C_{k,j}+\int\limits_{Q_{\nu(x_0)}\setminus Q_k}\left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k},w_{k,j}(z)\right)+\eta_k|\nabla w_{k,j}(z)|^2\right)dz\right)\\
&=\limsup_{j\to\infty}\limsup_{k\to\infty}A_{k,j}\le\limsup_{k\to\infty}\int\limits_{Q_k}\left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k},w_k(z)\right)+\eta_k|\nabla w_k(z)|^2\right)dz.
\end{aligned}$$

In view of (3.20), we obtain that

$$\begin{aligned}
&\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor(\Omega\cap\partial^*A_0)}(x_0)\\
&\ge\limsup_{j\to\infty}\limsup_{k\to\infty}\int\limits_{Q_{\nu(x_0)}}\left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k},w_{k,j}(z)\right)+\eta_k|\nabla w_{k,j}(z)|^2\right)dz. \qquad (3.26)
\end{aligned}$$

A diagonalization procedure (see Lemma 2.4) allows us to construct an increasing subsequence $\{k(j)\}\nearrow\infty$ such that

$$\begin{aligned}
&\limsup_{j\to\infty}\limsup_{k\to\infty}\int\limits_{Q_{\nu(x_0)}}\left(\frac{1}{\eta_k}W\left(\frac{z}{\eta_k},w_{k,j}(z)\right)+\eta_k|\nabla w_{k,j}(z)|^2\right)dz\\
&=\lim_{j\to\infty}\int\limits_{Q_{\nu(x_0)}}\left(\frac{1}{\eta_{k(j)}}W\left(\frac{y}{\eta_{k(j)}},w_{k(j),j}(y)\right)+\eta_{k(j)}|\nabla w_{k(j),j}(y)|^2\right)dy\\
&=\lim_{j\to\infty}\eta_{k(j)}^N\int\limits_{\frac{1}{\eta_{k(j)}}Q_{\nu(x_0)}}\left(\frac{1}{\eta_{k(j)}}W\left(z,w_{k(j),j}(\eta_{k(j)}z)\right)+\eta_{k(j)}|\nabla w_{k(j),j}(\eta_{k(j)}z)|^2\right)dz, \qquad (3.27)
\end{aligned}$$

after making the change of variables $y = \eta_{k(j)} z$. Define $v_j \in H^1\left(\frac{1}{\eta_{k(j)}} Q_{\nu(x_0)}; \mathbb{R}^d\right)$ by $v_j(z) := w_{k(j),j}(\eta_{k(j)} z)$. Since $w_{k(j),j} = \rho_{\frac{1}{\eta_{k(j)}},\nu(x_0)} * u_0$ on $\partial Q_{\nu(x_0)}$, we have that

$$v_j = \rho_{\frac{1}{\eta_{k(j)}},\nu(x_0)} * u_0 \text{ on } \partial\left(\frac{1}{\eta_{k(j)}} Q_{\nu(x_0)}\right),$$

and, in addition,

$$\begin{aligned}
&\lim_{j\to\infty} \eta_{k(j)}^N \int_{\frac{1}{\eta_{k(j)}} Q_{\nu(x_0)}} \left(\frac{1}{\eta_{k(j)}} W\left(z, w_{k(j),j}(\eta_{k(j)} z)\right) + \eta_{k(j)} |\nabla w_{k(j),j}(\eta_{k(j)} z)|^2\right) dz \\
&= \lim_{j\to\infty} \eta_{k(j)}^{N-1} \int_{\frac{1}{\eta_{k(j)}} Q_{\nu(x_0)}} \left(W\left(z, v_j(z)\right) + |\nabla v_j(z)|^2\right) dz \qquad (3.28) \\
&\geq \lim_{j\to\infty} \left(\eta_{k(j)}^{N-1} \inf\left\{ \int_{\frac{1}{\eta_{k(j)}} Q_{\nu(x_0)}} \left(W\left(z, u(z)\right) + |\nabla u(z)|^2\right) dz \ : \ u \in H^1\left(\frac{1}{\eta_{k(j)}} Q_{\nu(x_0)}; \mathbb{R}^d\right), \right.\right. \\
&\qquad\qquad \left.\left. u = \rho_{\frac{1}{\eta_{k(j)}},\nu(x_0)} * u_0 \text{ on } \partial\left(\frac{1}{\eta_{k(j)}} Q_{\nu(x_0)}\right) \right\}\right),
\end{aligned}$$

where we have used the fact that v_j is admissible for the infimum in the definition of $K_1(\nu(x_0))$. Combining (3.26), (3.27), and (3.28), we deduce that

$$\begin{aligned}
&\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\, (\Omega \cap \partial^* A_0)}(x_0) \\
&\geq \liminf_{T\to\infty} \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_{\nu(x_0)}} \left(W(y, u(y)) + |\nabla u(y)|^2\right) dy \ : \ u \in H^1(TQ_{\nu(x_0)}; \mathbb{R}^d), \right. \\
&\qquad\qquad \left. u = \rho_{T,\nu(x_0)} * u_0 \text{ on } \partial(TQ_{\nu(x_0)}) \right\} \\
&= K_1(\nu(x_0)),
\end{aligned}$$

asserting (3.15).

In view of (3.14), we have that

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2\right) dx &\geq \mu(\Omega) \\
&\geq \int_\Omega \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor\, (\Omega \cap \partial^* A_0)}(x) d\mathcal{H}^{N-1}\lfloor\, (\Omega \cap \partial^* A_0)(x) \\
&\geq \int_{\Omega\cap\partial^* A_0} K_1(\nu(x)) d\mathcal{H}^{N-1}(x),
\end{aligned}$$

and we deduce that (3.13) holds, which concludes the proof. □

4 The construction of a recovering sequence for the Γ-limit

In this section we prove part (ii) of Theorem 1.1. In view of Steps 1 and 2 in the proof of Proposition 3.3, it suffices to prove

Proposition 4.1 *Given any $u \in BV(\Omega; \{a, b\})$ and any sequence $\varepsilon_n \to 0^+$, there exists a sequence $\{u_n\} \subset H^1(\Omega; \mathbb{R}^d)$ such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and*

$$\lim_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n|^2 \right) dx = \int_{\Omega\cap\partial^* A_0} K_1(\nu(x)) d\mathcal{H}^{N-1}(x). \tag{4.1}$$

For the proof of Proposition 4.1, it will be enough to show that given any sequence $\varepsilon_n \to 0^+$, (4.1) holds for a subsequence $\{\varepsilon_n^{\mathcal{R}}\}$ of $\{\varepsilon_n\}$. Indeed, recalling the main result of the previous section (Proposition 3.3) we then obtain that the $\Gamma(L^1)$-limit of $I_{\varepsilon_n^{\mathcal{R}}}$ is I_0, which is independent on the specific subsequence $\{\varepsilon_n^{\mathcal{R}}\}$. In light of Proposition 7.11 in [6], we deduce that, in fact, I_ε $\Gamma(L^1)$-converges to I_0. The proof of Proposition 4.1 relies on the following result which will allow us to modify competing sequences near the boundary without increasing the total energy.

Lemma 4.2 *Assume that (H1)-(H3) hold, let ν be a unit vector and let*

$$u_0(x) := \begin{cases} b & if\ x\cdot\nu > 0, \\ a & if\ x\cdot\nu < 0. \end{cases}$$

Let $\rho : \mathbb{R} \to [0, +\infty)$ be a mollifier and set $v_n := \rho_{\frac{1}{\varepsilon_n},\nu} * u_0$, where $\rho_{\frac{1}{\varepsilon_n},\nu}(x) := \left(\frac{1}{\varepsilon_n}\right)^N \rho\left(\frac{x\cdot\nu}{\varepsilon_n}\right)$, and $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \to 0^+$. If $\{u_n\}$ is a sequence in $H^1(Q_\nu; \mathbb{R}^d)$ converging in $L^1(Q_\nu; \mathbb{R}^d)$ to u_0, then there exists a sequence $\{w_n\}$ in $H^1(Q_\nu; \mathbb{R}^d)$ such that $w_n \to u_0$ in $L^1(Q_\nu; \mathbb{R}^d), w_n = v_n$ on ∂Q_ν, and

$$\limsup_{n\to\infty} \int_{Q_\nu} \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, w_n(x)\right) + \varepsilon_n |\nabla w_n(x)|^2 \right) dx$$
$$\leq \liminf_{n\to\infty} \int_{Q_\nu} \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx.$$

Proof. Assume, without loss of generality, that

$$\liminf_{n\to\infty} \int_{Q_\nu} \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx$$
$$= \lim_{n\to\infty} \int_{Q_\nu} \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx < +\infty,$$

and that $u_n(x) \to u_0(x)$ $\mathcal{L}^N$- a.e. $x \in Q_\nu$. Thus,

$$\lim_{n\to\infty} \int_{Q_\nu} \left(W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n^2 |\nabla u_n(x)|^2 \right) dx = 0. \tag{4.2}$$

By (H3) we have

$$|u_n(x) - u_0(x)|^q \leq C \left(W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + 1 \right),$$

and we deduce that

$$C\mathcal{L}^N(Q_\nu) - \limsup_{n\to\infty} \|u_n - u_0\|^2_{L^q(Q_\nu;\mathbb{R}^d)} = \liminf_{n\to\infty} \int\limits_{Q_\nu} \left(CW\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + C - |u_n(x) - u_0(x)|^q \right) dx$$

$$\geq \int\limits_{Q_\nu} \liminf_{n\to\infty} \left(CW\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + C - |u_n(x) - u_0(x)|^q \right) dx$$

$$\geq C\mathcal{L}^N(Q_\nu),$$

where we have used (4.2), and Fatou's lemma. Therefore,

$$\limsup_{n\to\infty} \int\limits_{Q_\nu} |u_n - u_0|^q dx = 0, \tag{4.3}$$

and, in particular, since $q \geq 2$ we conclude that $u_n \to u_0$ in $L^2(Q_\nu;\mathbb{R}^d)$ as $n \to \infty$.

For simplicity, assume in what follows that $\nu = e_N$ and denote Q_ν by Q. Note that

$$v_n(x) = \begin{cases} b & \text{if } x_N > \varepsilon_n, \\ a & \text{if } x_N < -\varepsilon_n, \end{cases}$$

and

$$\|\nabla v_n\|_\infty = O(1/\varepsilon_n), \ \operatorname{supp}\nabla v_n \subset \{x \in Q : \ |x_N| < \varepsilon_n\}, \text{ and } v_n \to u_0 \text{ in } L^q(Q;\mathbb{R}^d). \tag{4.4}$$

For each $k \in \mathbb{N}$ define

$$L_k := \left\{ x \in Q : \operatorname{dist}(x, \partial Q) \leq \frac{1}{k} \right\}.$$

Consider n sufficiently large, and divide L_k into $M_{k,n}$ layers $L^{(i)}_{k,n}$ ($i = 1, ..., M_{k,n}$) of width $\varepsilon_n \|u_n - v_n\|^{1/2}_{L^2(Q;\mathbb{R}^d)}$, so that $M_{k,n}\varepsilon_n \|u_n - v_n\|^{1/2}_{L^2(Q;\mathbb{R}^d)} = O(1/k)$. Since

$$\sum_{i=1}^{M_{k,n}} \int\limits_{L^{(i)}_{k,n}} \left(1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q;\mathbb{R}^d)}} \right) dx$$
$$= \int\limits_{L_k} \left(1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q;\mathbb{R}^d)}} \right) dx,$$

there exists $i = i(k,n) \in \{1, ..., M_{k,n}\}$ such that

$$\int\limits_{L^{(i)}_{k,n}} \left(1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q;\mathbb{R}^d)}} \right) dx$$
$$\leq \frac{1}{M_{k,n}} \int\limits_{L_k} \left(1 + |u_n|^q + |v_n|^q + \varepsilon_n^2 |\nabla u_n|^2 + \frac{|u_n - v_n|^2}{\|u_n - v_n\|_{L^2(Q;\mathbb{R}^d)}} \right) dx. \tag{4.5}$$

Choose cut-off functions $\varphi_{k,n} \in C_c^\infty(Q;[0,1])$ such that $\varphi_{k,n} = 0$ on $\bigcup_{j=i+1}^{M_{k,n}} L^{(j)}_{k,n} =: A_{k,n}$, $\varphi_{k,n} = 1$ on $(Q \setminus L_k) \cup (\bigcup_{j=1}^{i-1} L^{(j)}_{k,n}) =: B_{k,n}$, and define

$$w_{k,n} := \varphi_{k,n} u_n + (1 - \varphi_{k,n}) v_n.$$

We have

$$\lim_{k\to\infty}\lim_{n\to\infty}\|w_{k,n}-u_0\|_{L^1(Q;\mathbb{R}^d)}=0.$$

Also

$$\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},w_{k,n}(x)\right)+\varepsilon_n|\nabla w_{k,n}(x)|^2\right)dx\\
&\le\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{A_{k,n}}\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},v_n(x)\right)+\varepsilon_n|\nabla v_n(x)|^2\right)dx\\
&+\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{L^{(i)}_{k,n}}\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},w_{k,n}(x)\right)+\varepsilon_n|\nabla w_{k,n}(x)|^2\right)dx\\
&+\lim_{n\to\infty}\int_Q\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},u_n(x)\right)+\varepsilon_n|\nabla u_n(x)|^2\right)dx. \qquad (4.6)
\end{aligned}$$

By (H3) and (4.4) we have

$$\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{A_{k,n}}\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},v_n(x)\right)+\varepsilon_n|\nabla v_n(x)|^2\right)dx\\
&\le\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{L_k\cap\{x\in Q:|x_N|<\varepsilon_n\}}\frac{C}{\varepsilon_n}\left(1+|v_n|^q+\varepsilon_n^2|\nabla v_n|^2\right)\,dx=0,
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{L^{(i)}_{k,n}}\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},w_{k,n}(x)\right)+\varepsilon_n|\nabla w_{k,n}(x)|^2\right)dx\\
&\le\limsup_{k\to\infty}\limsup_{n\to\infty}\frac{C}{\varepsilon_n M_{k,n}}\int_{L_k}\left(1+|u_n|^q+|v_n|^q+\varepsilon_n^2|\nabla u_n|^2+\frac{|u_n-v_n|^2}{\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}}\right)\,dx\\
&\le\limsup_{k\to\infty}\limsup_{n\to\infty}Ck\|u_n-v_n\|^{1/2}_{L^2(Q;\mathbb{R}^d)}\left(\int_Q\left(1+|u_n|^q+|v_n|^q+\varepsilon_n^2|\nabla u_n|^2\right)\,dx\right.\\
&\qquad\left.+\|u_n-v_n\|_{L^2(Q;\mathbb{R}^d)}\right)=0,
\end{aligned}$$

where we have used (4.3) and (4.5). Thus, (4.6) becomes

$$\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},w_{k,n}(x)\right)+\varepsilon_n|\nabla w_{k,n}(x)|^2\right)dx\\
&\le\lim_{n\to\infty}\int_Q\left(\frac{1}{\varepsilon_n}W\left(\frac{x}{\varepsilon_n},u_n(x)\right)+\varepsilon_n|\nabla u_n(x)|^2\right)dx.
\end{aligned}$$

Using a diagonalization process (see Lemma 2.4) we extract a subsequence $\{k(n)\}$ of $\{k\}$ such that, upon letting $w_n := w_{k(n),n}$, we have $w_n = v_n$ on ∂Q,

$$\lim_{n\to\infty} \|w_n - u_0\|_{L^1(\Omega;\mathbb{R}^d)} = 0,$$

and

$$\limsup_{n\to\infty} \int_Q \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, w_n(x)\right) + \varepsilon_n |\nabla w_n(x)|^2 \right) dx$$
$$\leq \liminf_{n\to\infty} \int_Q \left(\frac{1}{\varepsilon_n} W\left(\frac{x}{\varepsilon_n}, u_n(x)\right) + \varepsilon_n |\nabla u_n(x)|^2 \right) dx.$$

□

Proof of Proposition 4.1. Let $\mathcal{A}(\Omega)$ be the family of all open subsets of Ω, and let $\mathcal{C}$ be the family of all open cubes in Ω with faces parallel to the axes, centered at points $x \in \Omega \cap \mathbb{Q}^N$ and with rational edgelength. Denote by $\mathcal{R}$ the countable subfamily of $\mathcal{A}(\Omega)$ obtained by taking all finite unions of elements of $\mathcal{C}$, i.e.,

$$\mathcal{R} := \left\{ \bigcup_{i=1}^k C_i : k \in \mathbb{N},\ C_i \in \mathcal{C} \right\}.$$

Let $\varepsilon_n \to 0^+$. Since $L^1(\Omega;\mathbb{R}^d)$ is a separable metric space, using Kuratowski's Compactness Theorem (see, e.g. [10]), a diagonalization argument, and in the spirit of Γ-convergence (see Proposition 7.9 in [6]), we can assert the existence of a subsequence $\{\varepsilon_n^{\mathcal{R}}\}$ of $\{\varepsilon_n\}$ such that, if

$$\mathcal{W}_{\{\delta_n\}}(u;A) := \inf \left\{ \liminf_{n\to\infty} \int_\Omega \left(\frac{1}{\delta_n} W\left(\frac{x}{\delta_n}, v_n(x)\right) + \delta_n |\nabla v_n|^2 \right) dx \ : \right.$$
$$\left. v_n \to u \text{ in } L^1(A;\mathbb{R}^d),\ v_n \in H^1(A;\mathbb{R}^d) \right\},$$

for $A \in \mathcal{A}(\Omega)$ and $\delta_n \to 0^+$, then for every $u \in L^1(\Omega;\mathbb{R}^d)$ and $C \in \mathcal{R}$, there exists a sequence $\{u^C_{\varepsilon_n^{\mathcal{R}}}\} \subset H^1(C;\mathbb{R}^d)$ such that

$$u^C_{\varepsilon_n^{\mathcal{R}}} \to u \text{ in } L^1(C;\mathbb{R}^d)$$

and

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;C) = \lim_{n\to\infty} \int_C \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u^C_{\varepsilon_n^{\mathcal{R}}}(x)\right) + \varepsilon_n^{\mathcal{R}} |\nabla u^C_{\varepsilon_n^{\mathcal{R}}}(x)|^2 \right) dx$$

We will first prove that

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot) \text{ is a finite nonnegative Radon measure, absolutely continuous with respect to } \mathcal{H}^{N-1}\lfloor\, \partial^* A_0. \tag{4.7}$$

For each $k \in \mathbb{N}$, let $\{v_n^k\} \subset H^1(\Omega;\mathbb{R}^d)$ be such that $\lim_{n\to\infty} \|v_n^k - u\|_{L^1(\Omega;\mathbb{R}^d)} = 0$, and

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) \leq \liminf_{n\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, v_n^k(x)\right) + \varepsilon_n^{\mathcal{R}} |\nabla v_n^k(x)|^2 \right) dx \leq \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) + \frac{1}{k}.$$

Extract an increasing subsequence $\{n(j,k)\}_j$ of $\{n\}$ such that

$$\liminf_{n\to\infty}\int_\Omega \left(\frac{1}{\varepsilon_n^{\mathcal{R}}}W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}},v_n^k(x)\right)+\varepsilon_n^{\mathcal{R}}|\nabla v_n^k(x)|^2\right)\,dx$$
$$=\lim_{j\to\infty}\int_\Omega \left(\frac{1}{\varepsilon_{n(j,k)}^{\mathcal{R}}}W\left(\frac{x}{\varepsilon_{n(j,k)}^{\mathcal{R}}},v_{n(j,k)}^k(x)\right)+\varepsilon_{n(j,k)}^{\mathcal{R}}|\nabla v_{n(j,k)}^k(x)|^2\right)\,dx.$$

We have

$$\lim_{k\to\infty}\lim_{j\to\infty}\int_\Omega \left(\frac{1}{\varepsilon_{n(j,k)}^{\mathcal{R}}}W\left(\frac{x}{\varepsilon_{n(j,k)}^{\mathcal{R}}},v_{n(j,k)}^k(x)\right)+\varepsilon_{n(j,k)}^{\mathcal{R}}|\nabla v_{n(j,k)}^k(x)|^2\right)\,dx=\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega).$$

A diagonalization process allows us to extract a subsequence $\{j(k)\}$ of $\{j\}$, such that, upon denoting $n_k:=n(j(k),k)$ and $v_k:=v^k_{n(j(k),k)}$, we have

$$\lim_{k\to\infty}\|v_k-u\|_{L^1(\Omega;\mathbb{R}^d)}=0,$$

and

$$\lim_{k\to\infty}\int_\Omega \left(\frac{1}{\varepsilon_{n_k}^{\mathcal{R}}}W\left(\frac{x}{\varepsilon_{n_k}^{\mathcal{R}}},v_k(x)\right)+\varepsilon_{n_k}^{\mathcal{R}}|\nabla v_k(x)|^2\right)\,dx=\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega).$$

The sequence of measures $\{\mu_k\}$, where $\mu_k:=\left(\frac{1}{\varepsilon_{n_k}^{\mathcal{R}}}W\left(\frac{x}{\varepsilon_{n_k}^{\mathcal{R}}},v_k(x)\right)+\varepsilon_{n_k}^{\mathcal{R}}|\nabla v_k(x)|^2\right)\mathcal{L}^N\lfloor\ \Omega$, is bounded in $\mathcal{M}(\Omega)$. Thus, there exists a nonnegative Radon measure μ such that, up to a subsequence (not relabelled), $\mu_k\rightharpoonup\mu$ weakly* in $\mathcal{M}(\Omega)$. We want to show that $\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A)=\mu(A)$ for all $A\in\mathcal{A}(\Omega)$. To this end, and in view of Lemma 7.3 in [7] (see also [14]), it suffices to show that for any $A,\ B,\ C\in\mathcal{A}(\Omega)$, $\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot):\mathcal{A}(\Omega)\to[0,\infty)$ satisfies

(i) if $\overline{C}\subset B\subset A$, then $\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A)\le\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A\backslash\overline{C})+\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;B)$,

(ii) for any $\varepsilon>0$, there exists $C_\varepsilon\in\mathcal{A}(\Omega)$ with $\overline{C_\varepsilon}\subset A$ and $\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A\backslash\overline{C_\varepsilon})\le\varepsilon$,

(iii) $\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega)\ge\mu(\mathbb{R}^N)$,

(iv) $\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A)\le\mu(\overline{A})$.

We will first prove (i). To this aim, let $A,\ B,\ C\in\mathcal{A}(\Omega)$ be such that $\overline{C}\subset B\subset A$. For $\delta>0$, let B^δ and D^δ be two elements of $\mathcal{R}$ such that $B^\delta\subset B,\ D^\delta\subset A\setminus\overline{C}$, and

$$\mathcal{H}^{N-1}\left(\left(A\setminus(B^\delta\cup D^\delta)\right)\cap\partial^*A_0\right)<\delta. \tag{4.8}$$

Let $\left\{u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}\right\}$ and $\left\{u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\right\}$ be sequences in $H^1(B^\delta;\mathbb{R}^d)$ and $H^1(D^\delta;\mathbb{R}^d)$, respectively, such that $u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}\to u$ in $L^1(B^\delta;\mathbb{R}^d)$, $u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\to u$ in $L^1(D^\delta;\mathbb{R}^d)$,

$$\lim_{n\to\infty}\int_{B^\delta}\left(\frac{1}{\varepsilon_n^{\mathcal{R}}}W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}},u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}(x)\right)+\varepsilon_n^{\mathcal{R}}|\nabla u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}(x)|^2\right)dx=\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;B^\delta)<+\infty, \tag{4.9}$$

and

$$\lim_{n\to\infty}\int_{D^\delta}\left(\frac{1}{\varepsilon_n^{\mathcal{R}}}W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}},u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}(x)\right)+\varepsilon_n^{\mathcal{R}}|\nabla u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}(x)|^2\right)dx=\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;D^\delta)<+\infty. \tag{4.10}$$

Let $\rho : \mathbb{R}^N \to [0,+\infty)$ be a symmetric mollifier, and define $\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}}(x) := \frac{1}{(\varepsilon_n^{\mathcal{R}})^N}\rho\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}\right)$. We may assume, without loss of generality, that

$$u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} = \rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \text{ on } \partial B^\delta,\ u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} \to u \text{ in } L^2(B^\delta;\mathbb{R}^d) \text{ and } \mathcal{L}^N-\text{a.e. } x \in B^\delta.$$

The idea of the proof is along the lines of the proof of Lemma 4.2, where we replace Q by B^δ, and v_n by $\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u$ (with $\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}}$ as defined above). Note that in this case supp $\nabla(\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u) \subset \{x : \text{dist}(x, \partial^* A_0) < \varepsilon_n^{\mathcal{R}}\}$, and for each $k \in \mathbb{N}$, the layer L_k in the proof of Lemma 4.2 should be taken to be $L_k := \{x \in B^\delta\ :\ \text{dist}(x, \partial B^\delta) \le \frac{1}{k}\}$.
Similarly, we may assume that

$$u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} = \rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \text{ on } \partial D^\delta,\ u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} \to u \text{ in } L^2(D^\delta;\mathbb{R}^d) \text{ and } \mathcal{L}^N-\text{a.e. } x \in D^\delta.$$

Extend $u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}$ and $u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}$ as $\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u$ outside B^δ and D^δ, respectively. Note that, in view of (4.3),

$$\lim_{n\to\infty} \|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u\|_{L^2(A;\mathbb{R}^d)} = \lim_{n\to\infty} \|u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} - u\|_{L^2(A;\mathbb{R}^d)} = 0. \tag{4.11}$$

Write $B \setminus \overline{C}$ as a union of M_n layers $L_n^{(i)}$ $(i = 1, ..., M_n)$ of width $\varepsilon_n^{\mathcal{R}} \|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\|_{L^2(A;\mathbb{R}^d)}^{1/2}$ so that

$$M_n \varepsilon_n^{\mathcal{R}} \|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\|_{L^2(A;\mathbb{R}^d)}^{1/2} = O(1). \tag{4.12}$$

We have

$$\begin{aligned}
&\sum_{i=1}^{M_n} \int_{L_n^{(i)}} \left(1 + |u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^q + |u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^q + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2 + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^2 + \frac{|u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2}{\|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\|_{L^2(A;\mathbb{R}^d)}}\right) dx \\
&= \int_{B\setminus\overline{C}} \left(1 + |u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^q + |u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^q + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2 + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^2 + \frac{|u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2}{\|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\|_{L^2(A;\mathbb{R}^d)}}\right) dx,
\end{aligned}$$

and thus there exists $i_0 \in \{1, \cdots, M_n\}$ such that

$$\begin{aligned}
&\int_{L_n^{(i_0)}} \left(1 + |u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^q + |u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^q + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2 + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^2 + \frac{|u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2}{\|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\|_{L^2(A;\mathbb{R}^d)}}\right) dx \qquad (4.13) \\
&\le \frac{1}{M_n} \int_{B\setminus\overline{C}} \left(1 + |u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^q + |u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^q + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2 + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^2 + \frac{|u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2}{\|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\|_{L^2(A;\mathbb{R}^d)}}\right) dx.
\end{aligned}$$

We remark that by (4.9), (4.10), (4.11), and (H3),

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \int_{B\setminus\overline{C}} \left(1 + |u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^q + |u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^q + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2 + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^2 + \frac{|u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2}{\|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\|_{L^2(A;\mathbb{R}^d)}}\right) dx \\
&=: c_0 < +\infty. \qquad (4.14)
\end{aligned}$$

Consider cut-off functions $\varphi_n \in C_c^\infty(\Omega; [0,1])$ such that

$$\varphi_n(x) = 0 \text{ if } x \in \left(\bigcup_{j=i_0+1}^{M_n} L_n^{(j)} \right) \cup (A \setminus \overline{B}),$$

$$\varphi_n(x) = 1 \text{ if } x \in \left(\bigcup_{j=1}^{i_0-1} L_n^{(j)} \right) \cup C,$$

and

$$\|\nabla \varphi_n\|_\infty = O\left(\frac{1}{\varepsilon_n^{\mathcal{R}} \|u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} - u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}\|_{L^2(A;\mathbb{R}^d)}^{1/2}} \right).$$

Define

$$u_n := \varphi_n u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} + (1-\varphi_n) u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} + \chi_{(A \setminus (B^\delta \cup D^\delta))} \left(\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \right).$$

We have that $u_n \to u$ in $L^1(A;\mathbb{R}^d)$ as $n \to \infty$, and in view of (4.9), (4.10),

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A) \leq \liminf_{n\to\infty} \int_A \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_n(x) \right) + \varepsilon_n^{\mathcal{R}} |\nabla u_n(x)|^2 \right) dx$$

$$\begin{aligned}
&\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;A) \leq \liminf_{n\to\infty} \int_A \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_n(x) \right) + \varepsilon_n^{\mathcal{R}} |\nabla u_n(x)|^2 \right) dx \\
&\leq \liminf_{n\to\infty} \Bigg\{ \int_{A\setminus(B^\delta \cup D^\delta)} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, \rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \right) + \varepsilon_n^{\mathcal{R}} \left| \nabla \left(\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \right) \right|^2 \right) dx \\
&\qquad + \int_{D^\delta} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta} \right) \varepsilon_n^{\mathcal{R}} + |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{D^\delta}|^2 \right) dx + \int_{B^\delta} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta} \right) + \varepsilon_n^{\mathcal{R}} |\nabla u_{\varepsilon_n^{\mathcal{R}}}^{B^\delta}|^2 \right) dx \\
&\qquad\qquad + \int_{L_n^{(i_0)}} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_n \right) + \varepsilon_n^{\mathcal{R}} |\nabla u_n|^2 \right) dx \Bigg\} \\
&\leq \limsup_{n\to\infty} \int_{A\setminus(B^\delta \cup D^\delta)} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, \rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \right) + \varepsilon_n^{\mathcal{R}} \left| \nabla \left(\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \right) \right|^2 \right) dx + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;D^\delta) \\
&\qquad + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;B^\delta) + \limsup_{n\to\infty} \int_{L_n^{(i_0)}} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_n \right) + \varepsilon_n^{\mathcal{R}} |\nabla u_n|^2 \right) dx \\
&\leq \mathcal{H}^{N-1}\left((A \setminus (B^\delta \cup D^\delta)) \cap \partial^* A_0 \right) + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;D^\delta) + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;B^\delta) \\
&\qquad + \limsup_{n\to\infty} \int_{L_n^{(i_0)}} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_n \right) + \varepsilon_n^{\mathcal{R}} |\nabla u_n|^2 \right) dx. \qquad (4.15)
\end{aligned}$$

By (4.12), (4.13), and the growth conditions in (H3), we obtain that

$$\begin{aligned}
&\int_{L_n^{(i_0)}} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_n \right) + \varepsilon_n^{\mathcal{R}} |\nabla u_n|^2 \right) \, dx \\
&\le \frac{C}{\varepsilon_n^{\mathcal{R}}} \int_{L_n^{(i_0)}} \left(1 + |u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}|^q + |u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}|^q + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}|^2 + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}|^2 + (\varepsilon_n^{\mathcal{R}})^2 \|\nabla \varphi_n\|_\infty^2 |u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}} - u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}|^2 \right) dx \\
&\le \frac{C}{\varepsilon_n^{\mathcal{R}} M_n} \int_{B \setminus \overline{C}} \left(1 + |u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}|^q + |u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}|^q + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}|^2 + (\varepsilon_n^{\mathcal{R}})^2 |\nabla u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}|^2 + \frac{|u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}} - u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}}|^2}{\|u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}} - u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\|_{L^2(A;\mathbb{R}^d)}} \right) dx \\
&\le c_0 \, C \|u^{B^\delta}_{\varepsilon_n^{\mathcal{R}}} - u^{D^\delta}_{\varepsilon_n^{\mathcal{R}}}\|^{1/2}_{L^2(A;\mathbb{R}^d)},
\end{aligned}$$

where we have used (4.14). Thus,

$$\limsup_{n\to\infty} \int_{L_n^{(i_0)}} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_n \right) + \varepsilon_n^{\mathcal{R}} |\nabla u_n|^2 \right) \, dx = 0,$$

and we deduce from (4.8) and (4.15) that

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; A) \le \delta + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; B^\delta) + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; D^\delta) \le \delta + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; B) + \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; A \setminus \overline{C}).$$

Letting $\delta \to 0^+$, we obtain that (i) holds.

Next, we note that (ii) follows by the inner regularity of the Radon measure $C\mathcal{H}^{N-1} \lfloor \, \partial^* A_0$. Indeed, it suffices to remark that for all $A \in \mathcal{A}(\Omega)$, by the growth condition in (H3), and since A_0 is polyhedral,

$$\begin{aligned}
\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; A) &\le \liminf_{n\to\infty} \int_A \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, \left(\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \right)(x) \right) + \varepsilon_n^{\mathcal{R}} \left| \nabla \left(\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \right)(x) \right|^2 \right) dx \\
&\le \liminf_{n\to\infty} \frac{C}{\varepsilon_n^{\mathcal{R}}} \int_{\{x \in A:\ \operatorname{dist}(x, \partial^* A_0) \le \varepsilon_n^{\mathcal{R}}\}} \left(1 + |\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u|^q + (\varepsilon_n^{\mathcal{R}})^2 \left| \nabla \left(\rho_{\frac{1}{\varepsilon_n^{\mathcal{R}}}} * u \right) \right|^2 \right) dx \\
&\le C \liminf_{n\to\infty} \frac{\mathcal{L}^N(\{x \in A :\ \operatorname{dist}(x, \partial^* A_0) \le \varepsilon_n^{\mathcal{R}}\})}{\varepsilon_n^{\mathcal{R}}} = C\mathcal{H}^{N-1}(A \cap \partial^* A_0).
\end{aligned}$$

Property (iii) follows immediately from

$$\mu(\mathbb{R}^N) \le \liminf_{k\to\infty} \mu_k(\mathbb{R}^N) = \lim_{k\to\infty} \int_\Omega \left(\frac{1}{\varepsilon_{n_k}^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_{n_k}^{\mathcal{R}}}, v_k(x) \right) + \varepsilon_{n_k}^{\mathcal{R}} |\nabla v_k(x)|^2 \right) \, dx = \mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; \Omega).$$

Finally, since the sequence $\{v_k\} \subset H^1(A;\mathbb{R}^d)$ is admissible for the definition of $\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; A)$, we obtain that

$$\begin{aligned}
\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; A) &\le \liminf_{k\to\infty} \int_A \left(\frac{1}{\varepsilon_{n_k}^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_{n_k}^{\mathcal{R}}}, v_k(x) \right) + \varepsilon_{n_k}^{\mathcal{R}} |\nabla v_k(x)|^2 \right) \, dx \\
&\le \liminf_{k\to\infty} \mu_k(A) \le \mu(\overline{A}),
\end{aligned}$$

thus asserting (iv). Since $\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)$ satisfies (i)-(iv), we conclude that (4.7) holds.

We claim that

$$\frac{d\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\, \partial^* A_0}(x_0) \le K_1(\nu(x_0)) \text{ for } \mathcal{H}^{N-1} - \text{a.e. } x_0 \in \Omega \cap \partial^* A_0. \tag{4.16}$$

Assume, without loss of generality, that $x_0 \in \Omega \cap \partial^* A_0$ is such that $\nu(x_0) = e_N$ and

$$\frac{d\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\, \partial^* A_0}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u; Q(x_0,\varepsilon))}{\varepsilon^{N-1}}, \tag{4.17}$$

and denote $Q_{\nu(x_0)}$ by Q, $\rho_{T,\nu(x_0)}$ by ρ_T, and $K_1(\nu(x_0))$ by K_1. In view of Lemma 3.1, let $\{T_k\} \subseteq \mathbb{N}$, with $T_k \to \infty$, and $\{u_k\} \subset H^1(T_k Q;\mathbb{R}^d)$ be such that $u_k = \rho_{T_k} * u_0$ on $\partial(T_k Q)$, and

$$\lim_{k\to\infty} \frac{1}{T_k^{N-1}} \int_{T_k Q} \left(W(y,u_k(y)) + |\nabla u_k(y)|^2\right) dy = K_1.$$

Changing variables, we obtain that

$$K_1 = \lim_{k\to\infty} \int_Q \left(T_k W(T_k x, v_k(x)) + \frac{1}{T_k}|\nabla v_k(x)|^2\right) dx, \tag{4.18}$$

where $v_k(x) := u_k(T_k x),\ x \in Q$. For $x_N \in \left(-\frac{1}{2},\frac{1}{2}\right)$, extend $v_k(\cdot,x_N)$ by periodicity outside Q', and define

$$v_{n,k}^{(\varepsilon)}(x) := \begin{cases} u_0(x) & \text{if } |x_N| > \frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon} \\ v_k\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}} T_k}\right) & \text{if } |x_N| \le \frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}. \end{cases}$$

For $\varepsilon > 0$, we have

$$\begin{aligned}
&\int_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x)\right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}|\nabla v_{n,k}^{(\varepsilon)}(x)|^2\right) dx \\
&= \int_{x\in Q\,:\,|x_N| \le \frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_k\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}} T_k}\right)\right) + \frac{\varepsilon}{\varepsilon_n^{\mathcal{R}} T_k^2}\left|\nabla v_k\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}} T_k}\right)\right|^2\right) dx \\
&= \int_{-\frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}}^{\frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}} \int_{Q'} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}}}, \frac{\varepsilon x_N}{\varepsilon_n^{\mathcal{R}}}\right), v_k\left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k}, \frac{\varepsilon x_N}{\varepsilon_n^{\mathcal{R}} T_k}\right)\right) + \frac{\varepsilon}{\varepsilon_n^{\mathcal{R}} T_k^2}\left|\nabla v_k\left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k}, \frac{\varepsilon x_N}{\varepsilon_n^{\mathcal{R}} T_k}\right)\right|^2\right) dx' dx_N \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Q'} \left(T_k W\left(\left(\frac{\varepsilon T_k x'}{\varepsilon_n^{\mathcal{R}} T_k}, T_k y_N\right), v_k\left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k}, y_N\right)\right) + \frac{1}{T_k}\left|\nabla v_k\left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k}, y_N\right)\right|^2\right) dx' dy_N.
\end{aligned}$$

Thus, by the Riemann-Lebesgue Lemma (recall that $T_k \in \mathbb{N}$, and thus $W(T_k\cdot, z)$ is Q'-periodic) and the Dominated Convergence Theorem, together with (4.18), we obtain

$$\begin{aligned}
&\lim_{k\to\infty} \lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \int_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x)\right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon}|\nabla v_{n,k}^{(\varepsilon)}(x)|^2\right) dx \\
&= \lim_{k\to\infty} \int_Q \left(T_k W(T_k x, v_k(x)) + \frac{1}{T_k}|\nabla v_k(x)|^2\right) dx = K_1.
\end{aligned} \tag{4.19}$$

Let $m_n \in \mathbb{Z}^N$ and $s_n \in [0,1)^N$ be such that $\frac{x_0}{\varepsilon_n^{\mathcal{R}}} = m_n + s_n$, and let $x_{\varepsilon,n} := -\frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} s_n$. Note that for any $\varepsilon > 0$, $\lim_{n\to\infty} x_{\varepsilon,n} = 0$. Define $u_{n,\varepsilon,k} \in H^1(Q(x_0,\varepsilon);\mathbb{R}^d)$ by $u_{n,\varepsilon,k}(x) := v_{n,k}^{(\varepsilon)}\left(\frac{x-x_0}{\varepsilon} - x_{\varepsilon,n}\right)$.

We claim that for any $k \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \|u_{n,\varepsilon,k} - u\|_{L^1(Q(x_0,\varepsilon);\mathbb{R}^d)} = 0. \tag{4.20}$$

Indeed, changing variables,

$$\begin{aligned}
&\int_{Q(x_0,\varepsilon)} |u_{n,\varepsilon,k}(x) - u(x)|dx = \int_{Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n}} \left|v_{n,k}^{(\varepsilon)}\left(\frac{z}{\varepsilon}\right) - u(x_0 + z + \varepsilon x_{\varepsilon,n})\right| dz \\
&= \int_{(Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n})\cap \ z:|z_N|\le \frac{\varepsilon_n^{\mathcal{R}} T_k}{2}} \left|v_k\left(\frac{z}{\varepsilon_n^{\mathcal{R}} T_k}\right) - u(x_0 + z + \varepsilon x_{\varepsilon,n})\right| dz \\
&\qquad + \int_{(Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n})\cap \ z:|z_N|> \frac{\varepsilon_n^{\mathcal{R}} T_k}{2}} \left|u_0\left(\frac{z}{\varepsilon}\right) - u(x_0 + z + \varepsilon x_{\varepsilon,n})\right| dz \qquad (4.21) \\
&\le \int_{(Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n})\cap \ z:|z_N|\le \frac{\varepsilon_n^{\mathcal{R}} T_k}{2}} C\left(1 + \left|v_k\left(\frac{z}{\varepsilon_n^{\mathcal{R}} T_k}\right)\right|\right) dz + \int_{(Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n})\cap\{z:0<z_N<-\varepsilon(x_{\varepsilon,n})_N\}} |b-a|dz.
\end{aligned}$$

Since $\lim_{n\to\infty} x_{\varepsilon,n} = 0$, we obtain that

$$\lim_{n\to\infty} \int_{(Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n})\cap\{z:0<z_N<-\varepsilon(x_{\varepsilon,n})_N\}} |b-a|dz = 0,$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \int_{(Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n})\cap \ z:|z_N|\le \frac{\varepsilon_n^{\mathcal{R}} T_k}{2}} C\left(1 + \left|v_k\left(\frac{z}{\varepsilon_n^{\mathcal{R}} T_k}\right)\right|\right) dz \\
&= \lim_{n\to\infty} \int_{-\frac{\varepsilon_n^{\mathcal{R}} T_k}{2}}^{\frac{\varepsilon_n^{\mathcal{R}} T_k}{2}} \int_{(Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n})'} C\left(1 + \left|v_k\left(\frac{z'}{\varepsilon_n^{\mathcal{R}} T_k}, \frac{z_N}{\varepsilon_n^{\mathcal{R}} T_k}\right)\right|\right) dz' dz_N \\
&= \lim_{n\to\infty} \int_{-\frac{T_k}{2}}^{\frac{T_k}{2}} \int_{(Q(0,\varepsilon)-\varepsilon x_{\varepsilon,n})'} C\varepsilon_n^{\mathcal{R}}\left(1 + \left|v_k\left(\frac{z'}{\varepsilon_n^{\mathcal{R}} T_k}, \frac{z_N}{T_k}\right)\right|\right) dz' dz_N = 0.
\end{aligned}$$

Thus, in view of (4.21), we deduce that (4.20) holds.
Consequently,

$$\frac{\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;Q(x_0,\varepsilon))}{\varepsilon^{N-1}} \le \liminf_{n\to\infty} \frac{1}{\varepsilon^{N-1}} \int_{Q(x_0,\varepsilon)} \left(\frac{1}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x}{\varepsilon_n^{\mathcal{R}}}, u_{n,\varepsilon,k}(x)\right) + \varepsilon_n^{\mathcal{R}} |\nabla u_{n,\varepsilon,k}(x)|^2\right) dx$$

$$= \liminf_{n\to\infty} \int_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x_0+\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, u_{n,\varepsilon,k}(x_0+\varepsilon x) \right) + \varepsilon\varepsilon_n^{\mathcal{R}} |\nabla u_{n,\varepsilon,k}(x_0+\varepsilon x)|^2 \right) dx$$

$$= \liminf_{n\to\infty} \int_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x_0+\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x-x_{\varepsilon,n}) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x-x_{\varepsilon,n})|^2 \right) dx. \tag{4.22}$$

Changing variables, we have by (H1),

$$\int_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x_0+\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x-x_{\varepsilon,n}) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x-x_{\varepsilon,n})|^2 \right) dx$$

$$= \int_{-x_{\varepsilon,n}+Q} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x_0+\varepsilon(x+x_{\varepsilon,n})}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx$$

$$= \int_{-x_{\varepsilon,n}+Q} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(m_n + \frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx$$

$$= \int_{-x_{\varepsilon,n}+Q} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx. \tag{4.23}$$

Thus,

$$\int_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{x_0+\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x-x_{\varepsilon,n}) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x-x_{\varepsilon,n})|^2 \right) dx \tag{4.24}$$

$$\leq \int_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx$$

$$+ \int_{(-x_{n,\varepsilon}+Q)\setminus Q} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx.$$

We claim that

$$\limsup_{k\to\infty} \limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} \int_{(-x_{n,\varepsilon}+Q)\setminus Q} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx = 0. \tag{4.25}$$

After changing variables, we have

$$\int_{(-x_{n,\varepsilon}+Q)\setminus Q} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx$$

$$= \int_{((-x_{n,\varepsilon}+Q)\setminus Q)\cap \ x:|x_N|\leq \frac{\varepsilon_n^{\mathcal{R}} T_k}{2\varepsilon}} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x) \right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{((-x_{n,\varepsilon}+Q)\setminus Q)'} \left(T_k W\left(\left(\frac{T_k \varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k}, y_N T_k \right), v_k\left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k}, y_N \right) \right) + \frac{1}{T_k} \left| \nabla v_k\left(\frac{\varepsilon x'}{\varepsilon_n^{\mathcal{R}} T_k}, y_N \right) \right|^2 \right) dy' dy_N.$$

For each $\varepsilon > 0$, take $n(\varepsilon) \in \mathbb{N}$ such that $|x_{\varepsilon,n}| < \varepsilon$ for all $n \geq n(\varepsilon)$. In particular, we have $(-x_{n,\varepsilon} + Q) \setminus Q \subset (1+\varepsilon)Q \setminus Q$ and, in view of the Riemann-Lebesgue Lemma,

$$\begin{aligned}&\limsup_{n\to\infty} \int\limits_{(-x_{n,\varepsilon}+Q)\setminus Q} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x)\right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx \\ &\leq O(\varepsilon) \int\limits_Q \left(T_k W\left(T_k y, v_k(y)\right) + \frac{1}{T_k} \left|\nabla v_k\right|^2 \right) dy,\end{aligned}$$

thus asserting (4.25). Taking into account (4.17), (4.22), (4.23), (4.24), and (4.25), we obtain that

$$\begin{aligned}\frac{d\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\, \partial^* A_0}(x_0) &\leq \limsup_{k\to\infty} \limsup_{\varepsilon\to 0^+} \left(\limsup_{n\to\infty} \int\limits_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x)\right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx \right. \\ &\qquad \left. + \limsup_{n\to\infty} \int\limits_{(-x_{\varepsilon,n}+Q)\setminus Q} \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x)\right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx \right) \\ &= \limsup_{k\to\infty} \limsup_{\varepsilon\to 0^+} \limsup_{n\to\infty} \int\limits_Q \left(\frac{\varepsilon}{\varepsilon_n^{\mathcal{R}}} W\left(\frac{\varepsilon x}{\varepsilon_n^{\mathcal{R}}}, v_{n,k}^{(\varepsilon)}(x)\right) + \frac{\varepsilon_n^{\mathcal{R}}}{\varepsilon} |\nabla v_{n,k}^{(\varepsilon)}(x)|^2 \right) dx = K_1,\end{aligned}$$

where the last equality follows by (4.19). Thus, (4.16) holds. In view of (4.7) and (4.16), we have

$$\begin{aligned}\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) &= \int\limits_\Omega \frac{d\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\, \partial^* A_0}(x) d\mathcal{H}^{N-1}\lfloor\, \partial^* A_0(x) = \int\limits_{\Omega\cap\partial^* A_0} \frac{d\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor\, \partial^* A_0}(x) d\mathcal{H}^{N-1}(x) \\ &\leq \int\limits_{\Omega\cap\partial^* A_0} K_1(\nu(x)) d\mathcal{H}^{N-1}(x).\end{aligned}$$

By Proposition 3.3, we deduce that, in fact,

$$\mathcal{W}_{\{\varepsilon_n^{\mathcal{R}}\}}(u;\Omega) = \int\limits_{\Omega\cap\partial^* A_0} K_1(\nu(x)) d\mathcal{H}^{N-1}(x),$$

and the conclusion follows by a diagonalization argument.

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References

[1] G. Alberti, Variational models for phase transitions, an approach via Γ-convergence, *Quaderni del Dipartimento di Matematica "U. Dini", Universitá degli Studi di Pisa* (1998).

[2] L. Ambrosio, Metric space valued functions of bounded variation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **17** (1990), 439-478.

[3] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, Oxford, 2000.

[4] S. Baldo, Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids, *Ann. Inst. H. Poincaré Analyse Non Linéaire*, **7** (1990), 67-90.

[5] A. C. Barroso and I. Fonseca, Anisotropic singular perturbations–the vectorial case, *Proc. Roy. Soc. Edinburgh Sect. A* **124** (1994), 527-571.

[6] A. Braides and A. Defranceschi, Homogenization of multiple integrals, Clarendon Press, Oxford, 1998.

[7] A. Braides, I. Fonseca and G. Francfort, 3D-2D asymptotic analysis for inhomogeneous thin films, *Indiana Univ. Math. J.* **49** (2000), 1367-1404.

[8] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.* **28** (1958), 258-267.

[9] J. Carr, M. Gurtin and M. Slemrod, Structured phase transitions on a finite interval, *Arch. Rational Mech. Anal.* **86** (1984), 317-351.

[10] G. Dal Maso, An Introduction to Γ-convergence, Progress In Nonlinear Differential Equations and Their Applications 8, Birkäuser, Boston, 1993.

[11] G. Dal Maso and L. Modica, Nonlinear stochastic homogenization, *Ann. Mat. Pura. Appl.* **144** (1986), 347-389.

[12] E. De Giorgi, Sulla convergenza di alcune succesioni di integrali del tipo dell'area, *Rend. Mat.* **8** (1975), 277-294.

[13] L.C. Evans and R.F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC, Boca Raton, FL, 1992.

[14] I. Fonseca and J. Malý, Relaxation of multiple integrals below the growth exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), 309-338.

[15] I. Fonseca and S. Müller, Relaxation of Quasiconvex Functionals in $BV(\Omega, \mathbb{R}^d)$ for Integrands $f(x, u, \nabla u)$, *Arch. Rational Mech. Anal.* **123** (1993), 1-49.

[16] I. Fonseca and L. Tartar, The gradient theory of phase transitions for systems with two potential wells, *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), 89-102.

[17] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Birkäuser, Boston, 1984.

[18] M.E. Gurtin, Two-phase deformations of elastic solids, *Arch. Rational Mech. Anal.* **84** (1983), 1-29.

[19] M.E. Gurtin, Some results and conjectures in the gradient theory of phase transitions, *Preprint IMA* 156 (1985).

[20] L. Modica, The Gradient Theory of Phase Transitions and the Minimal Interface Criterion, *Arch. Rational Mech. Anal.* **98** (1987), 123-142.

[21] L. Modica and S. Mortola, Un esempio di Γ-convergenza, *Boll. Un. Mat. Ital. B* **14** (1977), 285-299.

[22] P. Sternberg, The effect of a singular perturbation on nonconvex variational problems, *Arch. Rational Mech. Anal.* **101** (1988), 209-260.

[23] P. Sternberg, Vector-valued local minimizers of nonconvex variational problems, *Rocky Mountain J. Math.* **21** (1991), 799-807.