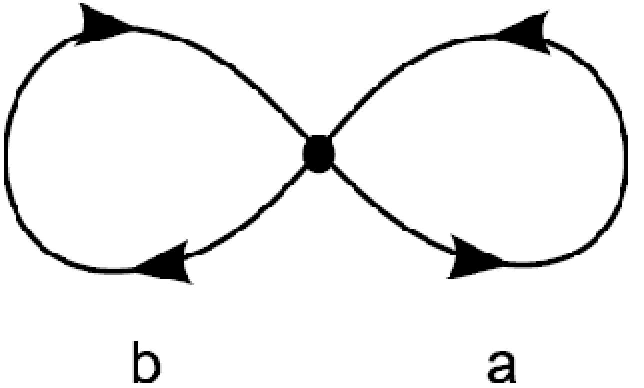
Regular Coverings of the Figure Eight Space



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Regular Coverings of the Figure Eight Space

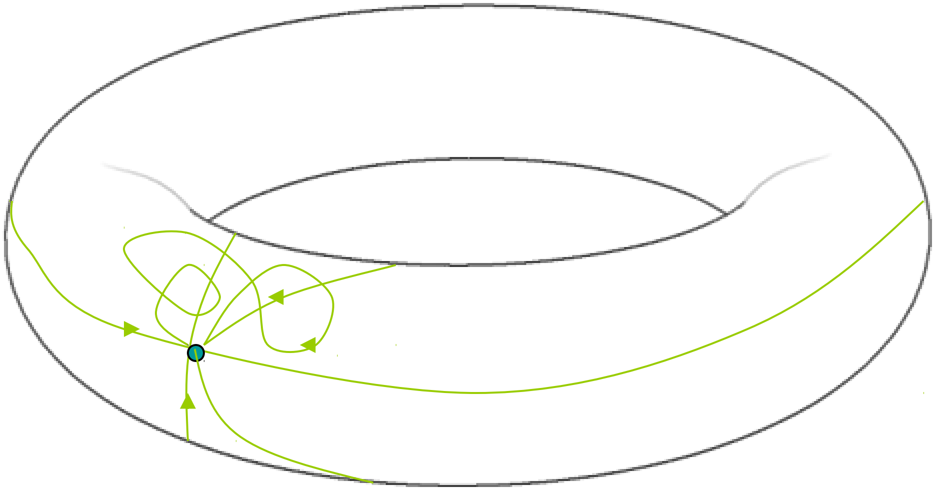
Plan of action

- Review some topological intuition
- Discuss motivating algebraic results
- Consider a method for finding “desirable” covers for certain types of words
- Look at what remains unsolved

Fundamental group

Intuition

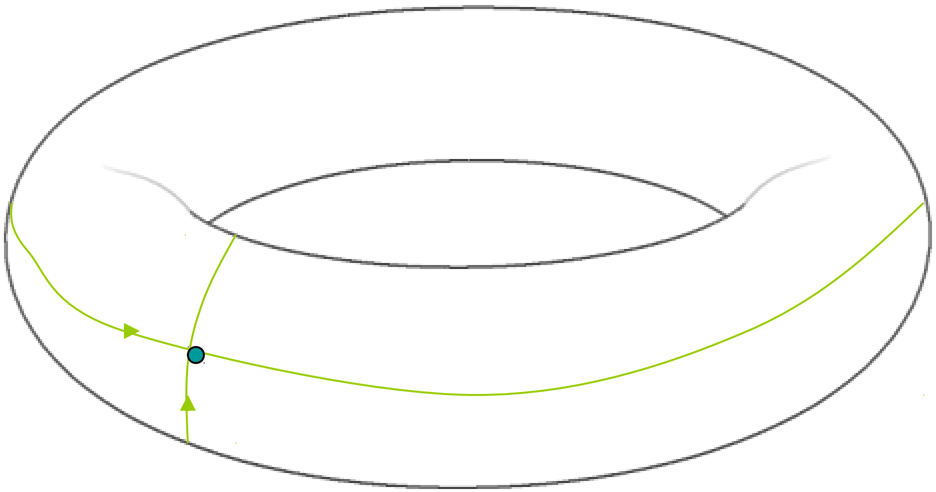
Take a “space,” a point in it, and all loops at this point.



Fundamental group

Intuition

Two loops are **topologically equivalent** if one can be deformed into the other without breaking.

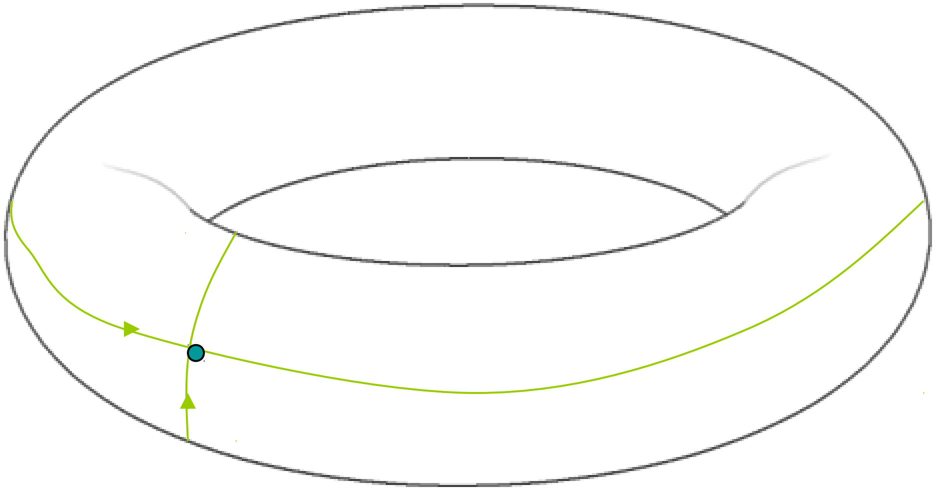


Fundamental group

Intuition

Two loops can be combined: traverse the first loop, then the second.

The set of all the loops together with the operation of combining them this way is the **fundamental group** of this space at the chosen base point.



Covering spaces

A **covering space** of X is a space C together with a continuous surjective map $p : C \to X$ such that for every x in X there exists an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in C each of which is mapped homeomorphically onto U by *p*.

Intuition

A covering space "covers" another space by a **covering map** (a surjective local homeomorphism) from it to this other space.

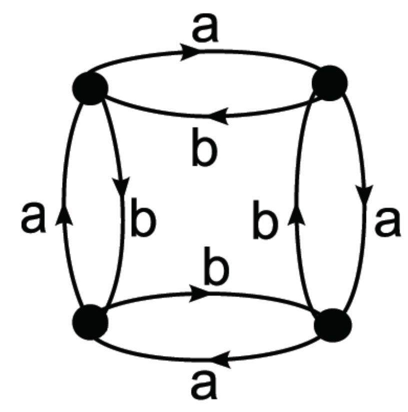
Regular covers

A cover of the figure-8 space is **regular** if its fundamental group is the same from each base vertex.

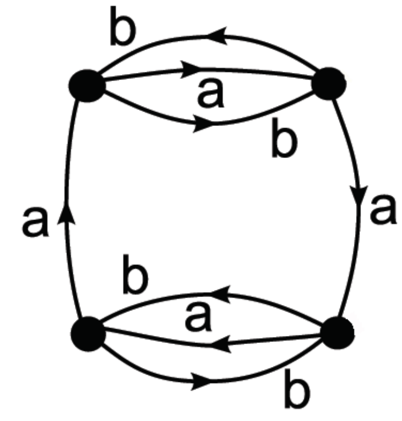
This means

- A word forms a loop from one vertex if and only if it forms a loop from all vertices.

- Regular covers have an inherent "symmetry."



Ex: Regular cover.

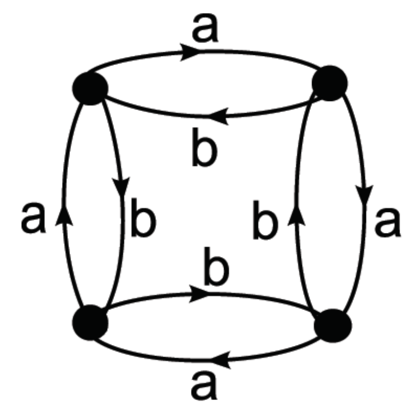


Ex: Non-regular cover; the word *ab* only forms a loop at some vertices.

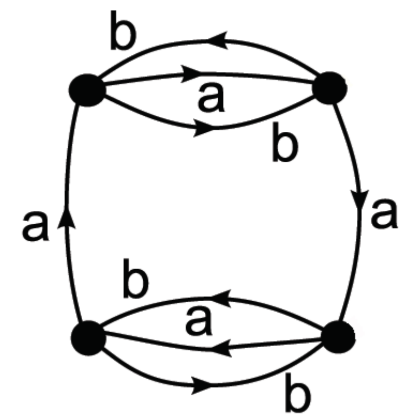
Regular covers

Fact

The normal subgroups of a space's fundamental group correspond to its set of regular covers.



Ex: Regular cover.



Ex: Non-regular cover; the word *ab* only forms a loop at some vertices.

Free group on r generators, $F(r)$

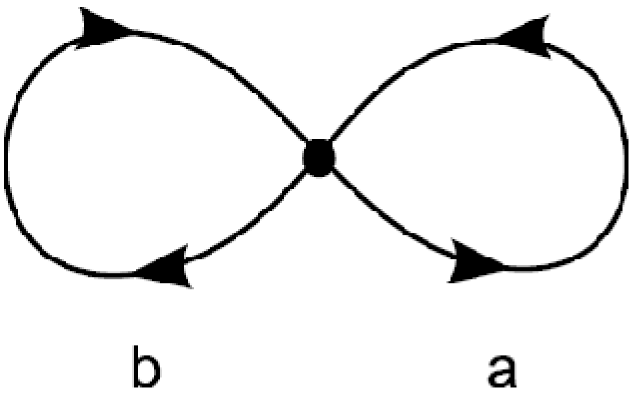
$F(r)$ corresponds to the wedge of r circles.

$F(r) = \langle a_1, a_2, \ldots, a_r \mid \quad \rangle$ (the generators are independent of each other).

Group operation: concatenation

Group elements: 'words' in the alphabet consisting of the r generators.

(e.g. $g = a_1 a_2^5 a_4^{-1} a_r^2$).



The free group on 2 generators corresponds to the wedge of 2 circles, or the figure-eight space.

Algebraic motivation

K. Iwasawa (1943) & M. Hall (1948)

Given $g \in F(r)$, $\exists H \lhd F(r)$ such that $g \notin H$.

In other words, given g, a word on r generators, there exists a normal subgroup (of finite index) of the free group not containing g.

Topological consequence

For any word, there exists a regular cover of the wedge of r circles in which it does not lift to a loop.

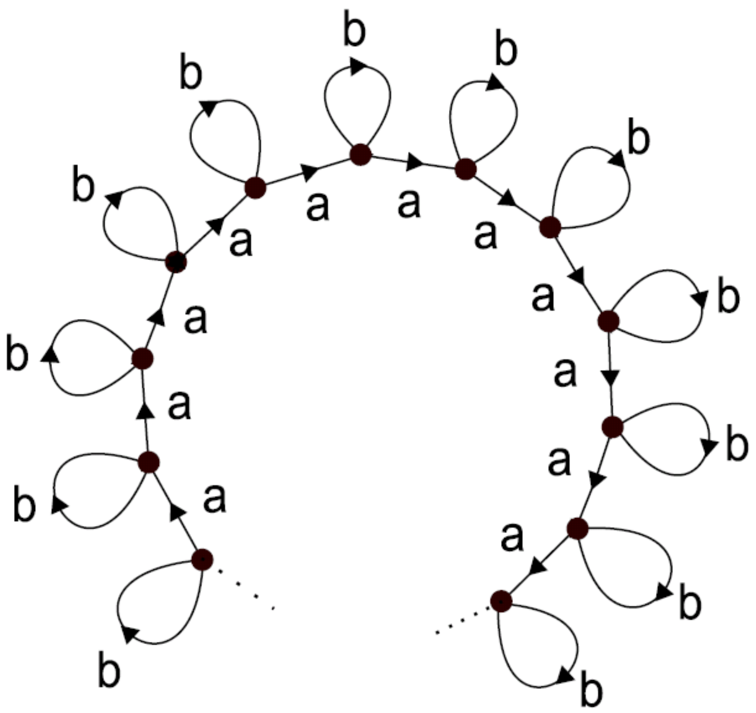
Our objective

1. Provide a method for finding a regular cover in which a given word g does not lift to a loop.

   (This is equivalent to finding a normal subgroup of the free group on two generators that doesn't contain *g*.)

2. Show that the covers produced by this method have minimal fold, corresponding to a subgroup of minimal index.

Simple *n*-gons



Simple *n* – gons correspond to Z_n, with group presentation $<a, b \mid a^n , b>$

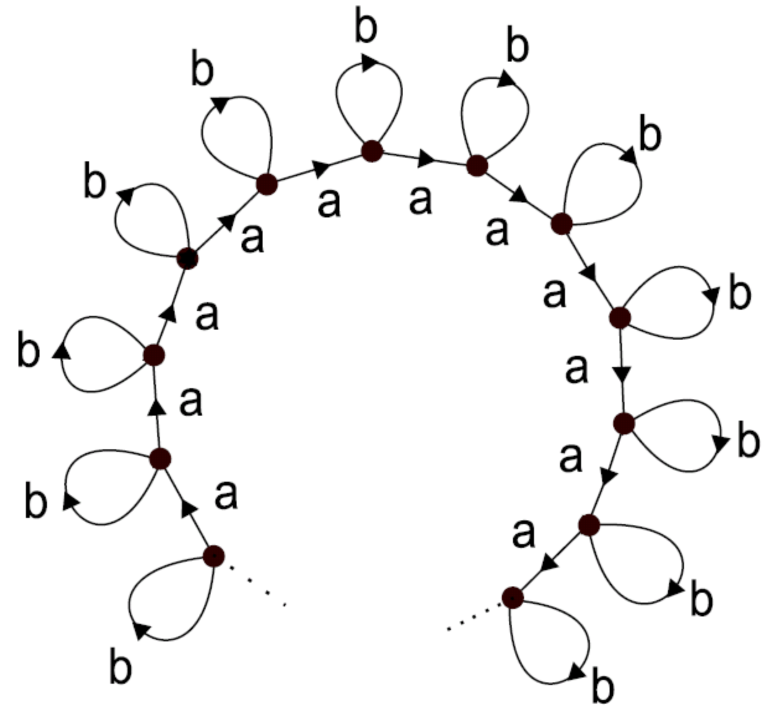
Simple *n*-gons

Definitions

For a given word in *F(2)*,

$$w = a^{i_1} b^{j_2} \cdots a^{i_r} b^{j_s}$$

define the ***a*-length** to be the sum of the powers of *a*, and denote it by *p*. Define the ***b*-length** similarly, and denote it by *q*.



Simple *n* – gons correspond to Z_n, with group presentation $<a, b \mid a^n, b>$

Simple *n*-gons

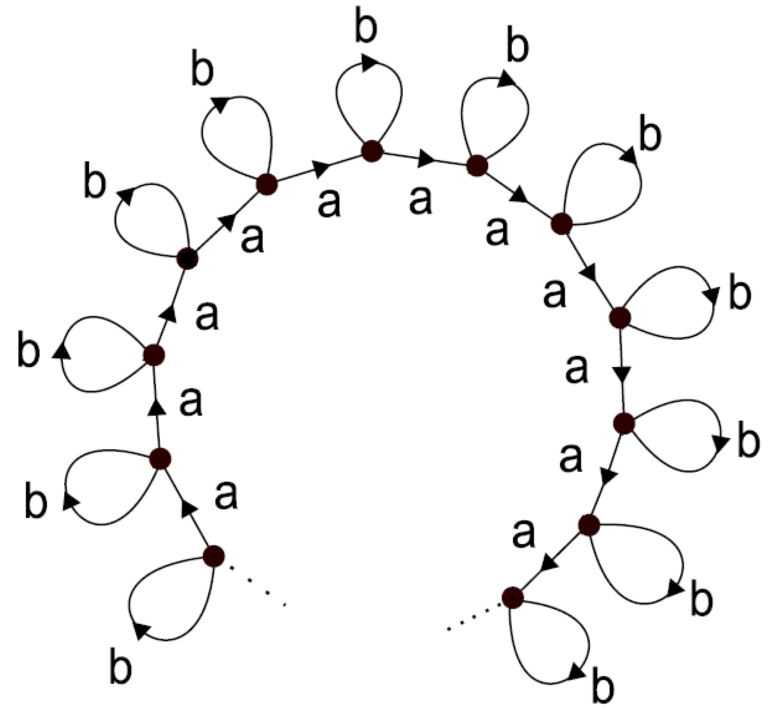
We will first consider $g \in F(2)$ with p, q not both zero.

Let $n \le m$ be the smallest **positive non-divisors** of *p* and *q*, respectively.

(In other words, for all $r < m$, $s < n$, $r|p$ and $s|q$.)

Lemma

g will not lift to a loop in the simple n-gon.



Simple *n* – gons correspond to Z_n, with group presentation $<a, b \mid a^n, b>$

Simple n-gons

Main result for this class of words

Theorem

The simple n-gon is the minimal cover in which g does not lift to a loop for all words g in which p and q are not both multiples of 60.

Simple *n*-gons

Intuition

We proved that any smaller index abelian symmetry group (than the group corresponding to the symmetries of the simple *n*-gon) will contain *g.*

The smallest non-abelian group is of order 6.

So for any *g* with *p* and *q* which are not both multiples of 2, 3, 4, and 5, this will be the smallest indexed normal subgroup corresponding to an abelian symmetry group not containing *g.*

Simple *n*-gons

Corollary

For $n = 2, 3, 4,$ or 5, the simple n-gon is the minimal regular cover in which g does not lift to a loop.

Covers for 0-0 words

If a word has *p = 0* and *q = 0*, there are covers in which the word may not lift to a loop based on certain properties. These covers correspond to **non-abelian symmetry groups**.

We were able to find covers for a large class of words using the generalized quaternion group, the dihedral group, and simple groups such as A_5, the alternating group on five symbols.

The remaining set of "hell words" consists of words that lift to loops in *all* of the covers corresponding to A_5 and $SL(2,5)$, and is given by

$$\left\langle\left\langle a^2\bar{b}^3, \bar{b}^3a^2, \bar{b}^3(ab)^5, (ab)^5\bar{b}^3, \bar{a}^2(ab)^5, (ab)^5\bar{a}^2 \right\rangle\right\rangle \cap \left\langle\left\langle b^2\bar{a}^3, \bar{a}^3b^2, \bar{a}^3(ba)^5, (ba)^5\bar{a}^3, \bar{b}^2(ba)^5, (ba)^5\bar{b}^2 \right\rangle\right\rangle \cap \left[F(2), F(2)\right]$$