Math 301: Homework 9

Due Friday Dec 7 at noon on Canvas

1. Show that in any 2-coloring of the edge set of K_n, there are $\Omega(n^s)$ monochromatic copies of K_s.
2. Show that there is a 2-coloring of the natural numbers that contains no infinite arithmetic progression.
3. The *Kneser graph* $\mathrm{KG}(n,k)$ is the graph whose vertex set is the k-element subsets of $[n]$ where A and B are adjacent if and only if they are disjoint. It is known that for $0 \leq j \leq k$, $\mathrm{KG}(n,k)$ has eigenvalue $(-1)^j\binom{n-k-j}{k-j}$ with multiplicity $\binom{n}{j} - \binom{n}{j-1}$ for $j > 0$ and 1 for $j = 0$. Use the Hoffman Ratio Bound to prove the Erdős-Ko-Rado theorem.
4. The purpose of this problem is to show that any regular graph can be partitioned into parts such that between parts the graph is almost biregular. The constants 1/4, 1/4 and 1/2 may obviously be changed depending on the situation. For a vertex v we denote its neighbors by $\Gamma(v)$. Show that for any $\epsilon > 0$ there exists a D_0 such that for any $d > D_0$, any d regular graph has a vertex partition into three parts A, B, C so that for any vertex v
$$\left(\frac{1}{4} - \epsilon\right) d \leq |\Gamma(v) \cap A| \leq \left(\frac{1}{4} + \epsilon\right) d$$
$$\left(\frac{1}{4} - \epsilon\right) d \leq |\Gamma(v) \cap B| \leq \left(\frac{1}{4} + \epsilon\right) d$$
$$\left(\frac{1}{2} - \epsilon\right) d \leq |\Gamma(v) \cap C| \leq \left(\frac{1}{2} + \epsilon\right) d$$
   (a) For each vertex, independently put it in A with probability 1/4, into B with probability 1/4 and into C with probability 1/2. For each v, denote by A_v the event that either $|\Gamma(v) \cap A| < (1/4 - \epsilon)d$ or $|\Gamma(v) \cap A| > (1/4 + \epsilon)d$. Define events B_v and C_v similarly.
   (b) Show that the probability of each of the events A_v, B_v, C_v is exponentially small as a function of d. (Use the Chernoff Bound)
   (c) Let D be a dependency graph for the events A_v, B_v, C_v. Show that the maximum degree of D is $O\left(d^2\right)$.
   (d) Use the Lovász Local Lemma to prove the theorem.