Math 301: Homework 7

Due Thursday November 1 at noon on Canvas

1. (a) Modify the proof of the upper bound for $\mathrm{ex}(n, K_{2,t})$ that we did in class to prove the Kővari-Sós-Turán Theorem. For $2 \le s \le t$ there exists a constant c such that $\mathrm{ex}(n, K_{s,t}) \le cn^{2-1/s}$.

   (b) Give the best lower bound you can for $\mathrm{ex}(n, K_{s,t})$.

2. (a) Let G be a graph where $V(G)$ consists of n points in the Euclidean plane and two points are adjacent if and only if they are at distance 1 from each other. Show that no matter how the points are placed, the number of edges in the graph is $O(n^{3/2})$.

   (b) Make a construction of n points in the plane that has as many pairs at unit distance as you can. How many edges are in the graph?

3. Let k be fixed. Show that there is a constant c so that $\mathrm{ex}(n, \{C_3, C_4, \cdots, C_{2k}\}) \le cn^{1+1/k}$. (Hint: you need to show that if G has more than $cn^{1+1/k}$ edges then it must contain a cycle of length at most $2k$. Assume G has this many edges and use a previous homework problem to start with a graph of minimum degree $c'n^{1/k}$. Do a breadth first search and show that you must find your cycle).

4. In this problem, we will "smash together" the two partite sets of the incidence graph of a projective plane and give an asymptotic formula for $\mathrm{ex}(n, C_4)$. Let V be a 3-dimensional vector space over a finite field $\mathbb{F}_q$. We define a graph G_π^o where $V(G_\pi^o)$ is the set of one-dimensional subspaces of V. There are $q^2 + q + 1$ of these (to see this, think of a vector in V having 3 coordinates, and then for each subspace it is defined by a vector which you can normalize so that the first non-zero coordinate is 1). Two vertices are adjacent if and only if the subspaces are orthogonal to each other (note that angles don't really make sense in this setting. I say that two vectors are orthogonal if their dot product is 0. Note that this means that a vector can be orthogonal to itself!).

   (a) Show that each vertex has degree $q + 1$ (Hint: V is a 3-dimensional vector space. Given a fixed 1-dimensional subspace, the set of vectors orthogonal to it is 2-dimensional. How many 1-dimensional subspaces are in a 2-dimensional vector space over $\mathbb{F}_q$?)

   (b) Show that every pair of vertices has exactly one path of length 2 between them (Hint: this is *much* easier to do geometrically than algebraically).

(c) Show that there are loops in the graph (you may allow q to be of any form that is convenient for you).

(d) It is known that there are $q+1$ loops in this graph. Let G_π be the graph with the loops removed. Then G_π is a graph on q^2+q+1 vertices with q^2 vertices of degree $q+1$ and $q+1$ vertices of degree q.

(e) Use part (b) and (d) to count the number of triangles in G_π.

(f) It is known that for any $\epsilon > 0$, there is an M such that for $m \geq M$, there is a prime number in the interval $[m, (1+\epsilon)m]$. Use this to show that $\mathrm{ex}(n, C_4) \sim \frac{1}{2}n^{3/2}$.