# Math 301: Homework 6

Due Wednesday Friday October 19 at noon

1. The goal of this problem is to prove the famous Szemerédi-Trotter Theorem. Given a set of points and lines in the Euclidean plane, an *incidence* is a point-line pair such that the point is on the line.

   **Theorem 1.** *Given $n$ points and $m$ lines in the plane, the number of point-line incidences is*
   $$O\left(n^{2/3}m^{2/3} + n + m\right).$$

   First we need to prove a lemma. Given a graph $G$, the *crossing number* of $G$ is the minimum number of edge-crossings possible among all drawings of the graph in the plane with the edges as straight line segments. The crossing number of a graph $G$ is denoted by $\mathrm{cr}(G)$.

   **Lemma 1.** *Let $G$ be a graph with $e$ edges and $n$ vertices. Then*
   $$\mathrm{cr}(G) \geq \frac{e^3}{64n^2} - n.$$

   a Show that the lemma is trivially true if $e < 4n$, so we may assume $e \geq 4n$.

   b Assume that $G$ is drawn in the plane so that it has $\mathrm{cr}(G)$ crossings.

   c Select a subset of vertices $S \subset V(G)$ independently with probability $p$, and let $H$ be the subgraph induced by the selected vertices. Define random variables $X = |S|$, $Y = |E(H)|$. Define $c_S$ to be the number of crossings that are left in the drawing after $S$ is selected. Note that $c_S$ and $\mathrm{cr}(H)$ are random variables.

   d We need an easy bound on the crossing number of any graph. Let $F$ be a graph and let $F'$ be a planar subgraph of $F$ with the maximum number of edges. Euler's formula says that $|E(F')| \leq 3|V(F)| - 6$. Since $F'$ is maximal, adding any additional edge will create at least one crossing. Deduce that $\mathrm{cr}(F) \geq |E(F)| - 3|V(F)|$ for any graph $F$.

   e Deduce that
   $$Y - 3X \leq \mathrm{cr}(H) \leq c_S.$$

   f From this, calculate the expected value of $X$, $Y$, and $c_S$ and deduce that
   $$p^2 e - 3pn \leq p^4 \mathrm{cr}(G).$$

g Choose $p = \frac{4n}{e}$ (why is this a legitimate probability?) to finish the proof of the lemma.

Now we will prove the Szemerédi-Trotter Theorem.

a Let $P$ and $L$ be a set of $n$ points and $m$ lines. Construct a graph $G$ with $V(G) = P$. Define adjacency in $G$ by letting two points be adjacent if and only if they are consecutive on some line in $L$.

b Prove that $\mathrm{cr}(G) < m^2$.

c Let $x$ be the number of incidences between $P$ and $L$. Prove that the number of edges in $G$ is at least $(x - m)$.

d Deduce that

$$m^2 > \frac{(x-m)^3}{64n^2} - n$$

and show that this implies the theorem.

Give a construction that shows that the Szemerédi-Trotter Theorem is best possible up to the implied constant.

2. Let $G$ be a random graph on $n$ vertices where each edge is selected independently with probability $p$. Let $\omega(n)$ be a function that tends to infinity with $n$ arbitrarily slowly.

a Use Markov's inequality to show that if $p \leq \frac{1}{\omega(n)n^{2/3}}$ then $G$ does not contain a $K_4$ with probability tending to 1.

b Use Chebyshev's inequality to show that if $p \geq \frac{\omega(n)}{n^{2/3}}$ then $G$ contains a $K_4$ with probability tending to 1.

3. Let $G$ be a random graph on $n$ vertices with edge probability $1/2$. Let $\epsilon > 0$ be arbitrary and let $k = (2 + \epsilon)\ln n$.

(a) Use the Chernoff Bound to give an upper bound on the probability that any fixed set of $k$ vertices forms an independent set.

(b) Use part (a) to show that $\alpha(G) \leq k$ with probability tending to 1.