Math 301: Homework 4

Due Friday September 28 at noon on Canvas

1. Let G be a graph with adjacency eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Prove that

$$\lambda_n = \min_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

2. Prove that $\lambda_1 \geq |\lambda_n|$.

3. Assume that G is connected. Prove that the eigenvector corresponding to λ_1 can be chosen to have all non-negative entries.

4. Prove that G is bipartite if and only if $\lambda_1 = -\lambda_n$.

5. Let T be a tree on $t + 1$ vertices.

    (a) Assume that n is divisible by t. Show that $\mathrm{ex}(n, T) \geq \frac{t-1}{2}n$. (Hint: K_t cannot contain a copy of T).

    (b) Use a problem in Homework 3 to show that $\mathrm{ex}(n, T) \leq (t - 1)n$.