

# Math 301: Homework 2

Due Friday September 14 at noon on Canvas

1. Let  $\pi(n)$  denote the number of primes less than or equal to  $n$ . Show that  $\pi(n) = O(n/\log n)$ . (Hint: first show that for  $m$  a natural number, the product of the primes between  $m$  and  $2m$  is bounded above by  $\binom{2m}{m}$ ).
2. How many natural numbers are there up to  $10^{30}$  that are either perfect squares, perfect cubes, or perfect fifth powers?
3.  $n$  married couples sit at a long table (of length  $n$  with 2 sides). How many ways can they sit so that no couple sits across from each other? How many ways can they sit such that exactly  $k$  couples sit across from each other?
4. (\*) Let  $n$  be large and choose two numbers from  $[n]$  uniformly at random. What is the probability that they are coprime? For each fixed  $n$  one may compute the probability  $p_n$ , and I am asking for the limit of  $p_n$  as  $n \rightarrow \infty$ . (Hint: Use inclusion-exclusion and then a useful function is the Möbius function  $\mu(n)$  which is  $-1$  if  $n$  is the product of an odd number of distinct primes,  $1$  if  $n$  is the product of an even number of distinct primes, and  $0$  if a square divides  $n$ .)