Math 228: Homework 6

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Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. Suppose that the vertices of a complete graph are points on a map, and the cost of each edge is the distance between the corresponding vertices in Euclidean space. Prove that a cheapest Hamiltonian cycle on this graph cannot intersect itself when drawn in Euclidean space.

2. (*) Suppose K_n is a weighted complete graph, in which all the edge weights are different. Prove that there exists only one minimal spanning tree.

3. (*) How could you find a spanning tree in a weighted K_n to minimize the product of the edge costs (instead of the sum)?

4. Here is a different algorithm to find a minimal spanning tree in a weighted K_n, called Prim's algorithm.

   (a) Pick a vertex $v \in V(K_n)$. Call this vertex "used" and all other vertices "unused."

   (b) Look at all the edges connecting "used" vertices to "unused" vertices. Find one that is the smallest, and add it to your tree. Move the unused vertex that touches this edge to the used vertex set.

   (c) Keep doing this until all vertices have been used.

   Note that Prim's algorithm is still a greedy algorithm, like Kruskal's algorithm, but now forces that the vertices added in each stage are connected, so that at every partial stage, you have a connected tree.

   Prove that Prim's algorithm also produces a minimal spanning tree in K_n.

5. (*) Prove that if G is a bipartite graph in which every vertex has the same degree, say k, then G contains a perfect matching.

6. Show, by example, that problem 5 is not true if we allow G to be a nonbipartite graph in which every vertex has degree k.

7. Complete problem 10.4.13 on page 178.

8. Complete problem 10.4.14 on page 178.

9. (*) Complete problem 10.4.15 on page 178.

10. (*) Suppose G is a bipartite graph, with partite sets V_1 and V_2, such that $|V_1| = |V_2|$. Suppose also that if $A \subset V_1$, $A \neq \emptyset$, then A has at least $|A|+1$ neighbors in V_2. Note that by Hall's Theorem, G will contain a perfect matching.

    Prove the stronger result that for every edge $e \in E(G)$, there is a perfect matching in G that contains the edge e.