

6. Inequalities

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1 Classical results

Smoothing principle. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Then if $x + y = x' + y'$ but x' and y' are closer together, we have

$$f(x') + f(y') \leq f(x) + f(y).$$

Furthermore, if f is strictly convex, then the inequality is strict.

Jensen. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Then for any $a_1, a_2, \dots, a_n \in \mathbb{R}$,

$$f\left(\frac{a_1 + \dots + a_n}{n}\right) \leq \frac{f(a_1) + \dots + f(a_n)}{n}.$$

Compactness. If D is a compact set and $f : D \rightarrow \mathbb{R}$ is continuous, then f achieves a maximum on D , i.e., there is at point $x \in D$ such that for all $y \in D$, $f(x) \geq f(y)$.

AM-GM. Let a_1, a_2, \dots, a_n be non-negative real numbers. Then

$$(a_1 a_2 \dots a_n)^{1/n} \leq \frac{a_1 + \dots + a_n}{n},$$

with equality if and only if all a_i are equal.

Cauchy-Schwarz. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be real numbers. Then

$$\left(\sum_i a_i b_i\right)^2 \leq \left(\sum_i a_i^2\right) \left(\sum_i b_i^2\right),$$

with equality only if the sequences (a_1, \dots, a_n) and (b_1, \dots, b_n) are proportional.

Dirichlet approximation. For any real number r and any positive integer N , there are integers a and b with $1 \leq b \leq N$ which satisfy

$$\left|r - \frac{a}{b}\right| < \frac{1}{b^2}.$$

2 Problems

1. Let P_1, P_2, \dots, P_n be points on a line, not necessarily distinct. Which points P on the line minimize the sum of distances $\sum_i |PP_i|$?
2. Let P_1, P_2, \dots, P_n be points in the plane, not necessarily distinct. Which point P minimizes the sum of squared distances $\sum_i |PP_i|^2$?

3. Prove that for all positive real numbers a, b, c , the following holds:

$$\frac{9}{a+b+c} \leq 2 \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right).$$

4. Given $n > 8$, let $a = \sqrt{n}$ and $b = \sqrt{n+1}$. Which is greater, a^b or b^a ?
5. Show that $\log\left(1 + \frac{1}{x}\right) > \frac{1}{1+x}$ for $x > 0$.
6. Let $p(x)$ be a real polynomial of degree at most 2, which satisfies $|p(x)| \leq 1$ for all $-1 \leq x \leq 1$. Show that $|p'(x)| \leq 4$ for all $-1 \leq x \leq 1$.
7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuously differentiable function satisfying $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x \in \mathbb{R}$. Show that f is constant.
8. Let C be a closed plane curve with the property that every pair of points in C are at distance at most 1 apart. Show that we can find a disk of radius $\frac{1}{\sqrt{3}}$ which contains C .
9. Let O be the origin $(0, 0)$, and let C be the line segment $\{(x, y) : x \in [1, 3], y = 1\}$. Let K be the curve $\{P : \text{for some } Q \in C, P \text{ lies on } OQ \text{ and } PQ = 0.01\}$. Let k be the length of the curve K . Is k greater or less than 2?
10. Show that for any rational $0 < \frac{a}{b} < 1$, we have $\left| \frac{a}{b} - \frac{1}{\sqrt{2}} \right| > \frac{1}{4b^2}$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.