2. Polynomials

Po-Shen Loh

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1 Classical results

1. Let $P(x)$ and $Q(x)$ be polynomials with real coefficients such that $P(x) = Q(x)$ for all real values of x. Prove that $P(x) = Q(x)$ for all complex values of x.

2. Find a nice expression for the derivative of the polynomial $(x-1)(x-2)(x-3)^2$.

3. Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial which satisfies $p(-x) = p(x)$ for every real x. Prove that $a_i = 0$ for every odd i.

4. Suppose the real polynomial $P(x)$ satisfies $P(x) \geq 0$ for all x. Prove that there exist real polynomials $A(x)$ and $B(x)$ such that $P(x) = A(x)^2 + B(x)^2$.

2 Problems

1. Show that the real polynomial $\sum_0^n a_i x^i$ has at least one real root if $\sum \frac{a_i}{i+1} = 0$.

2. Prove that we can find a real polynomial $p(y)$ such that $p(x - 1/x) = x^n - 1/x^n$ (where n is a positive integer) iff n is odd.

3. $p(z) = z^2 + az + b$ has complex coefficients. $|p(z)| = 1$ on the unit circle $|z| = 1$. Show that $a = b = 0$.

4. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients can be written as the average of two monic polynomials of degree n with n real roots.

5. Let a, b, c be positive integers. Prove that if there exist coprime polynomials P, Q, R with complex coefficients such that
$$P^a + Q^b = R^c$$
then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. Corollary: Fermat's Last Theorem for polynomials.

6. The roots of $x^3 + ax^2 + bx + c = 0$ are α, β, and γ. Find the cubic whose roots are α^3, β^3, and γ^3.

7. Let $p(x)$ be a polynomial with real coefficients, and let $r(x)$ be the polynomial defined by the derivative $r(x) = p'(x)$. Suppose that there are positive integers a and b for which $r^a(x)$ divides $p^b(x)$ as polynomials. Prove that for some real numbers A and α, and for some integer n, we have $p(x) = A(x-\alpha)^n$.

8. Let $p(z)$ be a polynomial of degree n with complex coefficients. Its roots (in the complex plane) can be covered by a disk of radius r. Show that for any complex k, the roots of $np(z) - kp'(z)$ can be covered by a disk of radius $r + |k|$.

3 Homework

Please write up solutions to two of the problems, to turn in at next week's meeting. One of them may be a problem that we discussed in class.