Velocity enhancement of reaction-diffusion fronts by a line of fast diffusion.

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Model under study

- Unknowns $u(t,x), v(t,x,y)$

$$\partial_t u - D\partial_{xx} u = v(x,0) - \mu u$$

$$d\partial_y v = \mu u - v(x,0)$$

$$\partial_t v - d\Delta v = f(v)$$

$$\partial_y v = 0 \tag{1}$$

- Travelling waves $u(t,x) = \phi(x+ct), v(t,x,y) = \psi(x+ct,y)$ connecting $(0,0)$ and $(1/\mu, 1)$?

- Unknowns $c > 0$, $\phi(x), \psi(x,y)$:

$$0 \leftarrow \phi \qquad -D\phi'' + c\phi' = \psi(x,0) - \mu\phi \qquad \phi \to 1/\mu$$

$$d\partial_y \psi = \mu\phi - \psi(x,0)$$

$$0 \leftarrow \psi \qquad -d\Delta\psi + c\partial_x\psi = f(\psi) \qquad \psi \to 1$$

$$\partial_y \psi = 0 \tag{2}$$

- Existence ? Influence of D on the velocity c ?

Motivation : initial model

- Proposed by Berestycki, Roquejoffre, Rossi :

$$\partial_t u - D\partial_{xx}^2 u = v(t,x,0) - \mu u$$

$$d\partial_y v = \mu u - v$$

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- Comparison principle.
- Ecological motivation : transportation networks increase the speed of biological invasions.

- Ex. : the pine processionary moth. Thought to move northwards because of climate change, but roads also thought to play a role.

Figure: Pine processionary from Auray (Britain). Source : Wikipédia

Questions

- Long-time behaviour of u, v ?
- Influence of the road ?

Comparison : the homogeneous case

Theorem-definition (Aronson-Weinberger 1975)

Let $u_t - \Delta u = u(1-u)$ with $u_0 \in \mathcal{C}_c^\infty$, $0 \leq u_0 \leq 1$, $u_0 \not\equiv 0$. Then

- For all $c > 2$, $\lim_{t\to+\infty} \sup_{|x|\geq ct} u(t,x) = 0$
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$$c_*(d) = 2\sqrt{\mathbf{d}f'(0)}$$

- What is the influence of D on the **propagation speed** in direction e_1 in our model ?

With a line of fast diffusion

Theorem (Berestycki, Roquejoffre, Rossi 2012)

Let $f(v)$ be such that $f(v) \leq f'(0)v$ (KPP assumption). There is a propagation speed $c^*(D) > 0$ in direction e_1 that satisfies :

- If $D \leq 2d$, $c^* = c_{KPP} = 2\sqrt{df'(0)}$.

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Remark

Thus we observe a propagation enhancement phenomenon in the direction of the road.

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Does this phenomenon persist in more general situations ?

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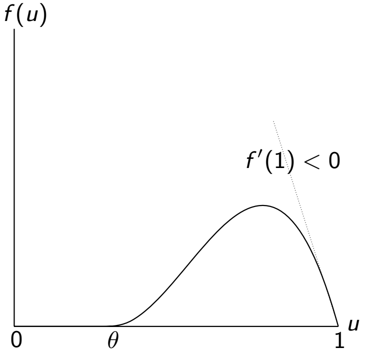

Figure: Example $f = \mathbf{1}_{u>\theta}(u-\theta)^2(1-u)$

This is a non trivial question since :

- The Fisher-KPP assumption enables to reduce the question to algebraic computations.

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- The Fisher-KPP assumption enables to reduce the question to algebraic computations.
- It could be necessary for the enhancement to happen : for example

$$u_t + (-\Delta)^{\alpha} u = f(u)$$

propagates initially c.c. datum at exponential speed (Cabré, Coulon, Roquejoffre), but if f has a threshold then propagation stays linear in time (Metllet, Roquejoffre, Sire).

Point of view : the travelling waves

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$\rightarrow$ We are led to the study of (2).

Results

Theorem 1 (D., 2013) : existence of travelling fronts

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Results

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- There exists (c, ϕ, ψ) a solution of (2)
- $0 < \phi < \frac{1}{\mu}$, $0 < \psi < 1$, and $\partial_x \phi, \partial_x \psi > 0$.
- If $(\underline{c}, \overline{\phi}, \overline{\psi})$ solves (2), then $\underline{c} = c$ and there exists $r \in \mathbb{R}$ s.t. $\overline{\phi}(\cdot + r) = \phi(\cdot)$ and $\overline{\psi}(\cdot + r) = \psi(\cdot)$.

Continuation to

$$-d\psi'' + c\psi' = f(\psi),\ \psi(-\infty) = 0, \psi(+\infty) = 1$$

$0 \leftarrow \phi$ $\qquad -D\phi'' + c\phi' = \ \psi(x,0) - \mu\phi$ $\qquad \phi \rightarrow 1/\mu$

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Step 1 : impose $\mu\phi = \psi$ on the road via $\varepsilon \in (0,1)$.

$$d\partial_y \psi = \frac{D}{\mu}\partial_{xx}\psi - \frac{c}{\mu}\partial_x\psi$$

$0 \leftarrow \psi$ $$-d\Delta\psi + c\partial_x\psi = f(\psi)$$ $\psi \rightarrow 1$

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Step 2 : vary D with $s \in (0,1)$.

$$d\partial_y\psi + \frac{c}{\mu}\partial_x\psi = 0$$

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Step 3 : vary $\frac{1}{\mu}$ with $t \in (0,1)$.

$$d\partial_y \psi = 0$$

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Interpretation : the road becomes a fence

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Interpretation : the road becomes a fence

Theorem : Kanel '69, Berestycki-Nirenberg '90

This problem has a unique solution : the planar wave.

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- Moreover, c_∞ is the unique admissible velocity for the following renormalised limiting model, which admits a unique travelling front $(x \leftarrow x\sqrt{D}$ and $c \leftarrow \frac{c}{\sqrt{D}})$ as $D \to +\infty$:

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- Despite the non-standard diffusion, the limiting model is well-posed : this solution can be obtained by a direct method without using the "regularisation" $-\frac{d}{D}\partial_{xx}$.

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- Despite the non-standard diffusion, the limiting model is well-posed : this solution can be obtained by a direct method without using the "regularisation" $-\frac{d}{D}\partial_{xx}$.
- Thus there is a regularisation effect in x due to the road and the term $c\partial_x v$: this has to be seen in the light of regularity in kinetic equations.
- When $c = 0$ one can show that there are only discontinuous solutions : hence the $c\partial_x v$ term is necessary.

Parallel : speed-up of a front by a shear flow

Model :

$$\partial_t v + A\alpha(y)\partial_x v = \Delta v + f(v), \qquad t \in \mathbb{R}, (x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \tag{4}$$

$A > 1$ large, $\alpha(y)$ smooth and $(1, \cdots, 1)$-periodic and Hörmander cd :

$$\exists r \in \mathbb{N}^* \text{ s.t. } \sum_{1 \leq |\zeta| \leq r} |D^\zeta \alpha(y)| > 0$$

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T.W. equation $c > 0$, $v(t,x) = u(x - ct, y)$:

$$\begin{cases} \Delta u + (c - A\alpha(y))\partial_x u + f(u) = 0, & (x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \\ \lim_{x \to +\infty} u(x,y) \equiv 0 \text{ uniformly in } \mathbb{T}^{N-1} \\ \lim_{x \to -\infty} u(x,y) \equiv 1 \text{ uniformly in } \mathbb{T}^{N-1} \end{cases}$$

Theorem (Hamel-Zlatoš 2013)

There exists $\gamma^* \geq \int_{\mathbb{T}^{N-1}} \alpha(y)dy$ s.t. the speed c of travelling fronts of (4) satisfies

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Moreover γ^* is the unique admissible velocity for the following degenerate system

$$\begin{cases} \Delta_y U + (\gamma - \alpha(y))\partial_x U + f(U) = 0 \text{ in } D'(\mathbb{R} \times \mathbb{T}^{N-1}) \\ 0 \leq U \leq 1 \text{ a.e. in } \mathbb{R} \times \mathbb{T}^{N-1} \\ \lim_{x \to +\infty} U(x,y) \equiv 0 \text{ uniformly in } \mathbb{T}^{N-1} \\ \lim_{x \to -\infty} U(x,y) \equiv 1 \text{ uniformly in } \mathbb{T}^{N-1} \end{cases} \quad (5)$$

Main idea

Renormalise (2) with $x \leftarrow x\sqrt{D}$ and $c \leftarrow c/\sqrt{D}$

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Show :

- $\exists m, M > 0$ such that $m \leq c(D) \leq M$.
- Uniqueness of the limiting point of $c(D)$.

Upper bound : exponential supersolution

- Study exponential solutions : $\exists \lambda > c$ and $C_1, C_2 > 0$ s.t. on $x \leq 0$:

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- Sliding argument : impossible.

$$c(D) \leq \sqrt{\frac{D}{D-d} \text{Lip} f} \sim_{D \to +\infty} \sqrt{\text{Lip} f}$$

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$$c_n = \frac{1}{L + 1/\mu} \int_{\Omega_L} f(\psi_n) \to 0$$

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(6)$\times\psi_n$ and IBP :

$$\frac{d}{D_n} \int_{\Omega_L} \partial_x \psi_n^2 + d \int_{\Omega_L} \partial_y \psi_n^2 + \int_{\mathbb{R}} \phi_n' \partial_x \psi_n(\cdot, 0) + c_n \int_{\mathbb{R}} \phi_n' \psi_n(\cdot, 0) + \frac{c_n L}{2} = \int_{\Omega_L} f(\psi_n)\psi_n \tag{7}$$

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so that

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so that $\left(1+\frac{L}{\mu}\right) c_n = \int_{\Omega_L} f(\psi_n) \geq \int_{J_n \times [-L,0]} f(\psi_n) \geq L \inf_{((1-\delta)\theta_1,(1+\delta)\theta_1)} f.$

Contradiction for small δ.

Uniqueness of the limiting point

- Compactness : any (ϕ_n, ψ_n) with $D_n \to \infty$ and $c_n \to c > 0$ is bounded in H^3_{loc} (use of Gagliardo-Nirenberg and Ladyzhenskaya ineq.)

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- Treating x as time : extract (c, ϕ, ψ) that solves (6) with $D = +\infty$ (Jensen ineq. and heat-semigroup regularisation).

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- Treating x as time : extract (c, ϕ, ψ) that solves (6) with $D = +\infty$ (Jensen ineq. and heat-semigroup regularisation).
- Uniqueness of c for such a problem using a parabolic sliding : if (c, ϕ, ψ) and $(\bar{c}, \underline{\phi}, \underline{\psi})$ solutions with $\bar{c} > c$: call $U = \phi - \underline{\phi}$, $V = \psi - \underline{\psi}$.

- Choose $a > 0$ large enough (dashed line) and :

$0 \leftarrow U$ $\qquad$ $-U'' + cU' = V - \mu U$ $\qquad$ $V \rightarrow 0$

$$d\partial_y V + V = \mu U$$

$0 \leftarrow V$ $\qquad$ $c\partial_x V - d\partial_{yy} V - \frac{f(\psi) - f(\underline{\psi})}{\psi - \underline{\psi}} V \geq 0$ $\qquad$ $V \rightarrow 0$

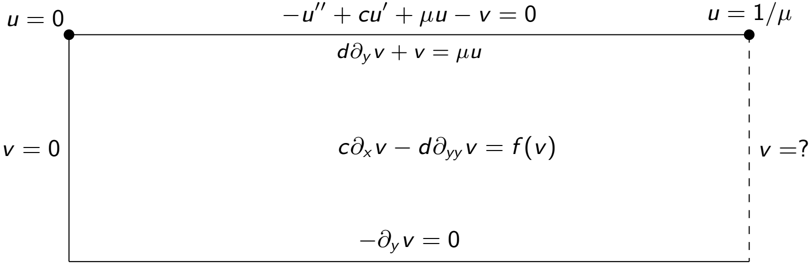
$$\partial_y V = 0$$

$$\mu\phi, \psi > \mu\underline{\phi}, \underline{\psi} > 1 - \varepsilon$$

- As $x \rightarrow -\infty$: use monotonicity and continuity of λ, get a comparison on some $x < X$.
- Contradiction.

Direct method for the rescaled limiting problem

Mixed elliptic-parabolic theory : works well for studying



(8)

and sending length to infinity : we recover the preceding limiting solution.

Extensions

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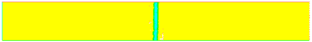

Thank you for your attention !

Happy birthday Alessandro !