On a stochastic differential equation arising in a price impact model

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Abstract

We provide sufficient conditions for the existence and uniqueness of solutions to a stochastic differential equation which arises in a price impact model from [2]. These conditions are stated as smoothness and boundedness requirements on utility functions or Malliavin differentiability of payoffs and endowments.

Keywords: Clark-Ocone formula, large investor, Malliavin derivative, Pareto allocation, price impact, Sobolev's embedding, stochastic differential equation.

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1 Introduction

In [1] and [2], we develop a financial model for a large investor who trades with market makers at their utility indifference prices. It is shown there how the evolution of this system can be described by a nonlinear stochastic differential equation; see (10) below.

It is the purpose of this paper to derive conditions for existence and uniqueness of solutions to this SDE. Standard results for SDEs, see, e.g., Kunita [5], formulate suitable Lipschitz and growth conditions in terms of the SDE's coefficients. For the financial model at hand these conditions are not easy to verify, though. We thus aim to provide readily verifiable criteria directly in terms of the model primitives, i.e., the market makers' utility functions, their endowments and the tradable contingent claims.

Our main results, stated in Section 3, yield such conditions for locally bounded order flows. Theorem 3.1 shows that if the market makers' risk aversions are bounded along with sufficiently many of their derivatives then there exist unique maximal local solutions. Its proof relies on Sobolev's embedding results for stochastic integrals due to Sznitman [7]. For the special case of exponential utilities Theorem 3.2 establishes existence of a unique global solution. Theorem 3.3 proves this under the alternative assumption that, in a Brownian framework, endowments and tradable contingent claims are Malliavin differentiable and risk aversions are bounded along with their first derivatives. The main tool here is the Clark-Ocone formula for $\mathbf{D}^{1,1}$ from Karatzas, Ocone, and Li [3].

2 Setup

Let $u_m = u_m(x)$, $m = 1, \dots, M$, be functions on the real line $\mathbf{R}$ satisfying

Assumption 2.1. Each u_m is strictly concave, strictly increasing, twice continuously differentiable,

$$\lim_{x \to \infty} u_m(x) = 0$$

and for some constant $c > 0$ the absolute risk aversion

$$\frac{1}{c} \leq a_m(x) \triangleq -\frac{u_m''(x)}{u_m'(x)} \leq c, \quad x \in \mathbf{R}. \tag{1}$$

Denote by $r = r(v, x)$ the v-weighted sup-convolution:

$$r(v, x) \triangleq \max_{x^1 + \cdots + x^M = x} \sum_{m=1}^{M} v^m u_m(x^m), \quad (v, x) \in (0, \infty)^M \times \mathbf{R}. \tag{2}$$

The main properties of the saddle function $r = r(v, x)$ are collected in [1, Section 4.1]. In particular, for $v \in (0, \infty)$, the function $r(v, \cdot)$ has the same properties as the functions u_m, $m = 1, \ldots, M$, of Assumption 2.1 and, for $c > 0$ from (1),

$$\frac{1}{c}\frac{\partial r}{\partial x}(v, x) \leq -M \frac{\partial^2 r}{\partial x^2}(v, x) \leq c \frac{\partial r}{\partial x}(v, x). \tag{3}$$

From (3) we deduce the exponential growth property

$$e^{-y^+ c/M + y^-/(cM)} \leq \frac{\frac{\partial r}{\partial x}(v, x + y)}{\frac{\partial r}{\partial x}(v, x)} \leq e^{-y^+/(cM) + y^- c/M}, \quad y \in \mathbf{R}, \tag{4}$$

where for real x we denote by $x^+ \triangleq \max(x, 0)$ and $x^- \triangleq (-x)^+$ the positive and negative parts of x. As $r(v, x) \to 0$ when $x \to \infty$, we also obtain the estimates

$$-\frac{1}{c} r(v, x) \leq M \frac{\partial r}{\partial x}(v, x) \leq -c r(v, x). \tag{5}$$

Let $(\Omega, \mathscr{F}_1, (\mathscr{F}_t)_{0 \leq t \leq 1}, \mathbb{P})$ be a complete filtered probability space satisfying

Assumption 2.2. There is a d-dimensional Brownian motion $B = (B^i)$ such that any local martingale M admits an integral representation

$$M_t = M_0 + \int_0^t H_u dB_u \triangleq M_0 + \sum_{i=1}^{d} \int_0^t H_u^i dB_u^i, \quad t \in [0, 1],$$

for some predictable process $H = (H^i)$ with values in $\mathbf{R}^d$.

Of course, this assumption holds if the filtration is generated by B. Note that under Assumption 2.2 any adapted process is automatically predictable; it will be called simply a process.

Let Σ_0 and $\psi = (\psi^j)_{j=1,\dots,J}$ be random variables. We denote

$$\Sigma(x,q) \triangleq \Sigma_0 + x + \langle q, \psi \rangle = \Sigma_0 + x + \sum_{j=1}^{J} q^j \psi^j, \quad (x,q) \in \mathbf{R} \times \mathbf{R}^J,$$

and assume that

$$\mathbb{E}[r(v, \Sigma(x,q))] > -\infty, \quad (v,x,q) \in \mathbf{A} \triangleq (0,\infty)^M \times \mathbf{R} \times \mathbf{R}^J. \tag{6}$$

From (4) and (5) we deduce that this integrability condition holds if

$$\mathbb{E}[e^{p|\psi| + c\Sigma_0^-/M}] < \infty, \quad p > 0. \tag{7}$$

For a metric space $\mathbf{X}$ denote by $\mathbf{C}(\mathbf{X}, [0,1])$ the space of continuous maps of $[0,1]$ to $\mathbf{X}$. For non-negative integers m and n and an open set $V \subset \mathbf{R}^d$ denote by $\mathbf{C}^m = \mathbf{C}^m(V, \mathbf{R}^n)$ the Fréchet space of m-times continuously differentiable functions $f : V \to \mathbf{R}^n$ with the topology generated by the semi-norms

$$\|f\|_{m,C} \triangleq \sum_{0 \le |\beta| \le m} \sup_{x \in C} |\partial^\beta f(x)|,$$

where C is a compact subset of V, $\beta = (\beta_1, \dots, \beta_d)$ is a multi-index of non-negative integers, $|\beta| \triangleq \sum_{i=1}^d \beta_i$, and

$$\partial^\beta \triangleq \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_d^{\beta_d}}.$$

In particular, for $m = 0$, ∂^0 is the identity operator and $\|f\|_C \triangleq \|f\|_{0,C} \triangleq \sup_{x \in C} |f(x)|$.

Under Assumptions 2.1, 2.2 and the integrability condition (6) the stochastic fields

$$\begin{aligned} F_t(v,x,q) &\triangleq \mathbb{E}[r(v, \Sigma(x,q)) | \mathscr{F}_t], \quad (v,x,q) \in \mathbf{A}, \\ G_t(u,y,q) &\triangleq \sup_{v \in (0,\infty)^M} \inf_{x \in \mathbf{R}} [\langle v, u \rangle + xy - F_t(v,x,q)], \\ &\qquad (u,y,q) \in \mathbf{B} \triangleq (-\infty, 0)^M \times (0,\infty) \times \mathbf{R}^J, \end{aligned}$$

are well-defined, have sample paths in $\mathbf{C}(\mathbf{C}^2(\mathbf{A}),[0,1])$ and $\mathbf{C}(\mathbf{C}^2(\mathbf{B}),[0,1])$, respectively, and for a multi-index $\beta=(\beta_1,\dots,\beta_{M+1+J})$ with $|\beta|\le 2$

$$\partial^\beta F_t(v,x,q)=\mathbb{E}[\partial^\beta r(v,\Sigma(x,q))|\mathscr{F}_t];$$

see Theorems 4.3 and 4.4 and Corollary 5.4 in [2]. In view of Assumption 2.2 the martingales $\partial^\beta F$, $|\beta|\le 2$, admit integral representations:

$$\partial^\beta F_t(v,x,q)=\mathbb{E}[\partial^\beta r(v,\Sigma(x,q))]+\int_0^t \partial^\beta H_s(v,x,q)dB_s \tag{8}$$

for some d-dimensional processes $\partial^\beta H(v,x,q)$, where the notation $\partial^\beta H$ can be justified using the concept of $\mathscr{L}$-derivatives from [4, Section 2.7]. Finally, for $u\in(-\infty,0)^M$ and $q\in\mathbf{R}^J$ define an $M\times d$-dimensional process $K(u,q)$ by

$$K_t^{mi}(u,q)\triangleq\frac{\partial H_t^i}{\partial v^m}(\frac{\partial G_t}{\partial u}(u,1,q),G_t(u,1,q),q),\quad t\in[0,1]. \tag{9}$$

The paper is concerned with the existence and uniqueness of a (strong) solution $U=(U^m)_{m=1,\dots,M}$ with values in $(-\infty,0)^M$ to the stochastic differential equation

$$U_t=U_0+\int_0^t K_s(U_s,Q_s)dB_s,\quad U_0\in(-\infty,0)^M, \tag{10}$$

parametrized by a process Q with values in $\mathbf{R}^J$. This equation appears in a price impact model of [2], where it describes the evolution of the expected utilities $U=(U^m)$ of M market makers who collectively buy $Q=(Q^j)$ stocks from a “large” investor. The functions $u_m=u_m(x)$, $m=1,\dots,M$, define the market makers' utilities for terminal wealth and Σ_0 stands for their total initial random endowment. The cumulative dividends paid by the stocks are given by $\psi=(\psi^j)$. According to [2, Theorem 5.8] a process $Q=(Q^j)$ is a (well-defined) *strategy* if and only if (10) has a unique solution U. In this case, the total cash amount received by the market makers (and paid by the investor) up to t is given by $G_t(U_t,1,Q_t)$.

Of course, it is easy to state standard conditions on the stochastic field K guaranteeing the existence and uniqueness of a solution U to (10); see Lemma 4.2 below. However, such criteria have little practical value, because, except in special cases such as Example 5.9 in [2], an explicit expression for K is not available. Instead, we look for easily verifiable conditions in terms of the model primitives: the functions (u_m) and the random variables Σ_0 and $\psi=(\psi^j)$.

3 Main results

Let Q be a process with values in $\mathbf{R}^J$ and τ be a stopping time with values in $(0,1] \cup \{\infty\}$. We remind the reader that a process U with values in $(-\infty,0)^M$ defined on $[0,\tau) \cap [0,1]$ is called a *maximal local solution* to (10) with the *explosion time* τ if for any stopping time σ with values in $[0,\tau) \cap [0,1]$ the process U satisfies (10) on $[0,\sigma]$ and

$$\lim_{t \uparrow \tau} \max_{m=1,\dots,M} U^m_t = 0 \text{ on } \{\tau < \infty\}.$$

Note that, since a negative local martingale is a submartingale, $\lim_{t\uparrow\tau} U^m_t$ exists and is finite.

Recall the notation $a_m = a_m(x)$ from (1) for the absolute risk-aversion of $u_m = u_m(x)$.

Theorem 3.1. *Let Assumptions 2.1 and 2.2 and condition (6) hold. Denote by l the smallest integer such that*

$$l > \frac{M+J}{2},$$

and suppose $u_m \in \mathbf{C}^{l+2}$ and

$$\sup_{x\in\mathbf{R}} |a^{(k)}_m(x)| < \infty, \quad k = 1,\dots,l, \quad m = 1,\dots,M. \tag{11}$$

Then for any locally bounded process Q with values in $\mathbf{R}^J$ there is a unique maximal local solution to (10).

Lemma 5.2 contains equivalent reformulations of (11). The proof of Theorem 3.1 as well as of Theorems 3.2 and 3.3 below will be given in Section 5.

Clearly, the conditions of Theorem 3.1 are satisfied for the exponential utilities:

$$u_m(x) = -\frac{b_m}{a_m} e^{-a_m x}, \quad x \in \mathbf{R}, \quad m = 1,\dots,M, \tag{12}$$

where a_m and b_m are positive numbers. Direct computations show that in this case

$$r(v,x) = -\frac{1}{a} e^{-ax} \prod_{m=1}^{M} (v_m b_m)^{\frac{a}{a_m}}, \quad (v,x) \in (0,\infty)^M \times \mathbf{R}, \tag{13}$$

where $a > 0$ is the harmonic mean of $(a_m)_{m=1,\dots,M}$:

$$\frac{1}{a} = \sum_{m=1}^{M} \frac{1}{a_m}. \tag{14}$$

The integrability condition (6) takes now the form:

$$\mathbb{E}[e^{-a\Sigma_0 + p|\psi|}] < \infty, \quad p > 0. \tag{15}$$

In fact, under (12) we have a stronger (global) result.

Theorem 3.2. *Let Assumption 2.2 and conditions (12) and (15) hold. Then for any locally bounded process Q with values in $\mathbf{R}^J$ there is a unique solution to (10).*

We conclude with conditions in terms of the Malliavin differentiability of Σ_0 and $\psi = (\psi^m)$. We refer the reader to [6] for an introduction to the Malliavin Calculus and the notation used in the sequel. Specifically, for $p \geq 1$ we denote by $\mathbf{D}^{1,p}$ the Banach space of random variables ξ with Malliavin derivatives $D\xi = (D_t\xi)_{t\in[0,1]}$ and the norm:

$$\|\xi\|_{\mathbf{D}^{1,p}} \triangleq \left(\mathbb{E}[|\xi|^p] + \mathbb{E}\left[\left(\int_0^1 |D_t\xi|^2 dt \right)^{p/2} \right] \right)^{1/p}.$$

Theorem 3.3. *In addition to Assumption 2.1 suppose $u_m \in \mathbf{C}^3$ and*

$$\sup_{x\in\mathbf{R}} |a'_m(x)| < \infty, \quad m = 1, \dots, M. \tag{16}$$

Assume also that the filtration is generated by a d-dimensional Brownian motion $B = (B^i)$ and that the random variables Σ_0 and $\psi = (\psi^j)$ belong to $\mathbf{D}^{1,2}$ and satisfy the integrability condition

$$\mathbb{E}[e^{p|\psi| + 2c\Sigma_0^-/M}] < \infty, \qquad p > 0, \tag{17}$$

with the constant $c > 0$ from (1). Then for any locally bounded process Q with values in $\mathbf{R}^J$ there is a unique solution to (10).

As a corollary of Theorem 3.3 we consider the case where Σ_0 and ψ are defined in terms of the solution X with values in $\mathbf{R}^N$ to the stochastic differential equation:

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad X_0 \in \mathbf{R}^N. \tag{18}$$

We assume that the functions $\mu:\ [0,1]\times\mathbf{R}^N\to\mathbf{R}^N$ and $\sigma:\ [0,1]\times\mathbf{R}^N\to\mathbf{R}^{N\times d}$ are Lipschitz-continuous with respect to x and bounded, that is, there is a constant $c>0$ such that for all $x,y\in\mathbf{R}^N$ and $t\in[0,1]$

$$
\begin{aligned}
|\sigma(t,x)-\sigma(t,y)|+|\mu(t,x)-\mu(t,y)| &\le c|x-y|,\\
|\sigma(t,x)|+|\mu(t,x)| &\le c.
\end{aligned} \tag{19}
$$

Corollary 3.4. *In addition to Assumption 2.1 suppose $u_m\in\mathbf{C}^3$ and (16) holds. Assume also that the filtration is generated by a d-dimensional Brownian motion $B=(B^i)$ and that the random variables Σ_0 and $\psi=(\psi^j)$ are of the form*

$$
\Sigma_0=g(X_1),\quad \psi^j=f^j(X_1),\quad j=1,\dots,J,
$$

for some Lipschitz-continuous functions g and $f=(f^j)$ on $\mathbf{R}^N$ and for the solution X to the stochastic differential equation (18) with coefficients satisfying (19). Then for any locally bounded process Q with values in $\mathbf{R}^J$ there is a unique solution to (10).

Proof. It is well-known, see Theorem 2.2.1 in [6], that under the stated assumptions $X_1\in\mathbf{D}^{1,2}$. By the chain rule of Malliavin calculus this also holds for any Lipschitz-continuous transformation of X_1; see Proposition 1.2.3 in [6]. As the coefficients μ and σ are bounded, X_1 and, therefore, Σ_0 and $\psi=(\psi^j)$ have finite exponential moments of any order. The proof now follows from Theorem 3.3. □

4 Conditions in terms of SDE-coefficients

In this section we state solvability criteria for (10) in terms of the stochastic fields K and H. These conditions will be used later in the proofs of the main theorems.

We shall make frequent use of the conjugacy relations between the stochastic fields F and G, which for $a=(v,x,q)\in\mathbf{A}$, $b=(u,1,q)\in\mathbf{B}$, and $t\in[0,1]$ state that

$$
v=\frac{\partial G_t}{\partial u}(\frac{\partial F_t}{\partial v}(a),1,q), \tag{20}
$$

$$
x=G_t(\frac{\partial F_t}{\partial v}(a),1,q), \tag{21}
$$

$$
u=\frac{\partial F_t}{\partial v}(\frac{\partial G_t}{\partial u}(b),G_t(b),q); \tag{22}
$$

see Corollary 4.14 in [2].

Lemma 4.1. *Suppose the process K defined in (9) has values in $\mathbf{C}^1((-\infty,0)^M \times \mathbf{R}^J, \mathbf{R}^{M\times d})$ and for any compact set $C \subset (-\infty,0)^M \times \mathbf{R}^J$*

$$\int_0^1 \|K_t\|_{1,C}^2 dt < \infty.$$

If Q is a process with values in $\mathbf{R}^J$ such that for any compact set $C \subset (-\infty,0)^M$

$$\int_0^1 \|K_t(\cdot, Q_t)\|_{1,C}^2 dt < \infty,$$

then there is a unique maximal local solution U to (10). In particular, such a solution exists for any locally bounded Q.

Proof. Follows from well-known criteria for maximal local solutions; see Theorem 3.4.5 in [5]. □

For vectors $x, y \in \mathbf{R}^M$ we shall write $x \geq y$ if $x^m \geq y^m$, $m = 1, \ldots, M$. Denote $\mathbf{1} \triangleq (1, \ldots, 1)$.

Lemma 4.2. *Let the processes K and Q satisfy the conditions of Lemma 4.1 and suppose for any constant $b > 0$ we have*

$$\int_0^1 \sup_{-b\mathbf{1}\leq u<0} L_t(u, Q_t) dt < \infty, \tag{23}$$

where, for $u \in (-\infty,0)^M$, $q \in \mathbf{R}^J$, and $t \in [0,1]$,

$$L_t(u,q) \triangleq \frac{1}{1+\sum_{m=1}^M |\log(-u^m)|} \sum_{m=1}^M \left| \frac{K_t^m(u,q)}{u^m} \right|^2. \tag{24}$$

Then (10) has a unique (global) solution.

Proof. In view of Lemma 4.1 we have to show that the explosion time τ for the maximal local solution U to (10) is infinite. By localization and accounting for the submartingale property of U we can assume without a loss in generality that $U^m \geq -b$ for some $b > 0$.

After the substitution $U^m = -\exp(Z^m)$, $m = 1, \ldots, M$, we can rewrite (10) as

$$Z_t^m = Z_0^m + \int_0^t A_s^m(Z_s) dB_s - \frac{1}{2} \int_0^t |A_s^m(Z_s)|^2 ds,$$

where

$$A_t^m(z) \triangleq -e^{-z^m} K_t^m(-e^{z^1}, \dots, -e^{z^M}, Q_t), \quad m = 1, \dots, M.$$

Each component of Z is bounded from above by $\ln b$ and

$$\lim_{t \uparrow \tau} \min_{m=1,\dots,M} Z_t^m = -\infty \text{ on } \{\tau < \infty\}.$$

It is well-known, see [5, Theorem 3.4.6], that $\tau = \infty$ if

$$\int_0^1 \sup_{z \leq \mathbf{1} \ln b} \frac{|A_t^m(z)|^2}{1 + |z|} dt < \infty, \quad m = 1, \dots, M,$$

which is equivalent to (23). ☐

When Q is locally bounded, we can state more convenient conditions in terms of the "primal" processes F and H. Recall first a result from [2], see Lemma 5.15 and Theorems 5.16 and 5.17.

Lemma 4.3 ([2])**.** *Let Assumptions 2.1 and 2.2 and condition (6) hold. Suppose the process H from (8) has values in $\mathbf{C}^2(\mathbf{A}, \mathbf{R}^d)$ and for any compact set $C \subset \mathbf{A}$*

$$\int_0^1 \|H_t\|_{2,C}^2 dt < \infty. \tag{25}$$

Then the process K from (9) satisfies the conditions of Lemma 4.1 and for any locally bounded Q with values in $\mathbf{R}^J$ there is a unique maximal local solution to (10).

Remark 4.4. As $H(bv, x, q) = bH(v, x, q)$ for any $b > 0$ and $(v, x, q) \in \mathbf{A}$, it is sufficient to verify (25) for compact sets $C \subset \widetilde{\mathbf{A}} \triangleq \mathbf{S}^M \times \mathbf{R} \times \mathbf{R}^J$, where $\mathbf{S}^M$ is the interior of the simplex in $\mathbf{R}^M$:

$$\mathbf{S}^M \triangleq \{w \in (0,1)^M : \sum_{m=1}^M w^m = 1\}.$$

Lemma 4.5. *Let Assumptions 2.1 and 2.2 and conditions (6) and (25) hold. Suppose for any constant $b > 0$*

$$\int_0^1 \sup_{a \in A_t(b)} M_t(a) dt < \infty, \tag{26}$$

where, for $t \in [0,1]$,

$$A_t(b) \triangleq \{a = (v,x,q) \in \mathbf{A} : \frac{\partial F_t}{\partial v}(a) \geq -b\mathbf{1},\ |q| \leq b\},$$

and, for $a = (v,x,q) \in \mathbf{A}$,

$$M_t(a) \triangleq \frac{1}{1+|x|} \sum_{m=1}^{M} \left| \frac{1}{\frac{\partial F_t}{\partial v^m}(a)} \frac{\partial H_t}{\partial v^m}(a) \right|^2 .$$

Then for any locally bounded Q *with values in* $\mathbf{R}^J$ *there is a unique (global) solution to* (10).

Remark 4.6. Since for all $(v,x) \in (0,\infty)^M \times \mathbf{R}$ and $m = 1,\dots,M$

$$\frac{1}{c}\frac{\partial r}{\partial x}(v,x) \leq -v^m \frac{\partial r}{\partial v^m}(v,x) \leq c\frac{\partial r}{\partial x}(v,x),$$

see Section 4.1 in [1], (26) is equivalent to

$$\int_0^1 \sup_{a \in \widetilde{A}_t(b)} \widetilde{M}_t(a)dt < \infty, \quad b > 0,$$

where

$$\widetilde{A}_t(b) \triangleq \{a = (v,x,q) \in \mathbf{A} : \frac{\partial F_t}{\partial x}(a)\mathbf{1} \leq bv,\ |q| \leq b\},$$

$$\widetilde{M}_t(a) \triangleq \frac{1}{(1+|x|)(\frac{\partial F_t}{\partial x}(a))^2} \sum_{m=1}^{M} \left| v^m \frac{\partial H_t}{\partial v^m}(a) \right|^2 .$$

Proof. Hereafter, $c > 0$ denotes the bound in (1). By Lemma 4.2 it is enough to show that for $L = L_t(u,q)$ defined in (24) and any $b \geq c$

$$\int_0^1 \sup_{-b\mathbf{1} \leq u < 0,\ |q| \leq b} L_t(u,q)dt < \infty.$$

By Corollary 5.4 in [2] the process $G = G_t(b)$ has trajectories in $\mathbf{C}(\widetilde{\mathbf{G}}^2(c),[0,1])$, where $\widetilde{\mathbf{G}}^2(c)$ is a linear subspace of saddle functions in $\mathbf{C}^2(\mathbf{B})$ defined and studied in Section 3 of [1]. Property (G7) of the elements of $\widetilde{\mathbf{G}}^2(c)$ states

$$\frac{1}{b} \leq \frac{1}{c} \leq -u^m \frac{\partial G_t}{\partial u^m}(u,1,q) \leq c \leq b, \quad m = 1,\dots,M,$$

which for $-b\mathbf{1} \leq u < 0$ implies

$$b \sum_{m=1}^{M} \log(-u^m/b) \leq G_t(-b\mathbf{1}, 1, q) - G_t(u, 1, q) \leq \frac{1}{b} \sum_{m=1}^{M} \log(-u^m/b). \tag{27}$$

For $n \geq 1$ define the stopping times

$$\sigma_n \triangleq \inf\{t \in [0,1] : \sup_{|q| \leq b} |G_t(-b\mathbf{1}, 1, q)| > n\},$$

where, by convention, $\inf \varnothing \triangleq \infty$. Since the sample paths of $G = G_t(b)$ belong to $\mathbf{C}(\mathbf{C}(\mathbf{B}), [0,1])$, we deduce $\sigma_n \to \infty$, $n \to \infty$. Hence, the result holds if

$$\int_0^{\sigma_n \wedge 1} \sup_{-b\mathbf{1} \leq u < 0,\ |q| \leq b} L_t(u, q) dt < \infty, \quad n \geq 1.$$

Observe now that (20)–(22) jointly with (27) and the construction of $K = K_t(u, q)$ in (9) imply that for $t \leq \sigma_n \wedge 1$

$$\sup_{-b\mathbf{1} < u < 0,\ |q| \leq b} L_t(u, q) \leq c(b, n) \sup_{a = (v,x,q) \in A_t(b)} M_t(a),$$

where the constant $c(b, n)$ depends only on b and n. The result now follows from (26). □

The conditions of Lemma 4.5 can be simplified further if instead of (6) we assume the integrability condition (7).

Lemma 4.7. *Let Assumptions 2.1 and 2.2 and conditions (7) and (25) hold. Suppose for any constant $b > 0$*

$$\int_0^1 \sup_{a \in B(b)} N_t(a) dt < \infty, \tag{28}$$

where

$$B(b) \triangleq \{a = (v, x, q) \in \mathbf{A} : \frac{\partial r}{\partial x}(v, x)\mathbf{1} \leq bv,\ |q| \leq b\},$$

and, for $a = (v, x, q) \in \mathbf{A}$,

$$N_t(a) \triangleq \frac{1}{(1 + |x|)(\frac{\partial r}{\partial x}(v, x))^2} \sum_{m=1}^{M} \left| v^m \frac{\partial H_t}{\partial v^m}(a) \right|^2.$$

Then for any locally bounded Q with values in $\mathbf{R}^J$ there is a unique solution to (10).

Proof. The result follows from Lemma 4.5 and Remark 4.6 if we can find strictly positive random variables η and ζ such that

$$\zeta \frac{\partial r}{\partial x}(v,x) \leq \frac{\partial F_t}{\partial x}(v,x,q) \leq \eta \frac{\partial r}{\partial x}(v,x),$$

for all $(v,x) \in (0,\infty)^M \times \mathbf{R}$, $|q| \leq b$, and $t \in [0,1]$.

From (4) we deduce for such (v,x,q)

$$e^{-c(\Sigma_0^+ + b|\psi|)/M} \frac{\partial r}{\partial x}(v,x) \leq \frac{\partial r}{\partial x}(v,\Sigma(x,q)) \leq e^{c(\Sigma_0^- + b|\psi|)/M} \frac{\partial r}{\partial x}(v,x).$$

The random variables ζ and η can now be chosen as

$$\zeta \triangleq \inf_{t\in[0,1]} \mathbb{E}[e^{-c(\Sigma_0^+ + b|\psi|)/M} | \mathscr{F}_t],$$

$$\eta \triangleq \sup_{t\in[0,1]} \mathbb{E}[e^{c(\Sigma_0^- + b|\psi|)/M} | \mathscr{F}_t].$$

Note that ζ is strictly positive and, in view of (7), η is finite. □

5 Proofs of the main results

5.1 Proof of Theorem 3.1

We divide the proof into a series of lemmas. A key role is played by the following direct corollary of Proposition 1 in [7].

Lemma 5.1. *Suppose Assumption 2.2 holds. Let V be an open set in $\mathbf{R}^n$ and m, j be non-negative integers satisfying*

$$m - \frac{n}{2} > j.$$

Consider a random field $\eta = (\eta(x))_{x\in V}$ with sample paths in $\mathbf{C}^m(V)$ such that for any compact set $C \subset V$

$$\mathbb{E}[\|\eta\|_{m,C}] < \infty.$$

Then there exists a process H with values in $\mathbf{C}^j(V,\mathbf{R}^d)$ such that for any $x \in V$ and $t \in [0,1]$ and any multi-index $\beta = (\beta_1,\dots,\beta_n)$ of order $0 \leq |\beta| \leq j$

$$\mathbb{E}[\partial^\beta \eta(x)|\mathscr{F}_t] = \mathbb{E}[\partial^\beta \eta(x)] + \int_0^t \partial^\beta H_u(x) dB_u.$$

Moreover, for any compact set $C \subset V$,

$$\int_0^1 \|H_t\|_{j,C}^2 dt < \infty.$$

Proof. Without restricting generality we can assume that V is a ball in $\mathbf{R}^d$ and

$$\mathbb{E}[\|\eta\|_{m,V}] < \infty.$$

Assumption 2.2 implies that

$$L_t \triangleq \mathbb{E}[\|\eta\|_{m,V} | \mathscr{F}_t], \quad t \in [0,1],$$

is a continuous martingale. Hence, there is a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\{\tau_n < 1\} \downarrow \emptyset$ and $|L| \leq n$ on $[0, \tau_n]$. By Lemma 4.5 in [2], the random field

$$\eta_n(x) \triangleq \mathbb{E}[\eta(x) | \mathscr{F}_{\tau_n}], \quad x \in V,$$

has a version with values in $\mathbf{C}^m$ and $\|\eta_n\|_{m,V} \leq L_{\tau_n} \leq n$.

These localization arguments imply that without any loss of generality we can assume that the random variable $\|\eta\|_{m,V}$ is bounded and, in particular,

$$\mathbb{E}[\|\eta\|_{m,V}^2] < \infty.$$

However, under this condition the assertion of the lemma is a special case of Proposition 1 in [7]. □

Lemma 5.2. *Let Assumption 2.1 hold and suppose each u_m is of class $\mathbf{C}^{l+2}$ for an integer $l \geq 0$. Then (11) is equivalent to the condition*

$$\sup_{x \in \mathbf{R}} \left(\frac{|u_m^{(k)}(x)|}{u_m'(x)} \right) < \infty, \quad k = 0, \dots, l+2, \quad m = 1, \dots, M,$$

and also to the condition

$$\sup_{x \in \mathbf{R}} |t_m^{(k)}(x)| < \infty, \quad k = 0, \dots, l, \quad m = 1, \dots, M, \tag{29}$$

where $t_m(x) \triangleq 1/a_m(x)$ is the absolute risk-tolerance of $u_m = u_m(x)$.

Proof. Follows from Assumption 2.1 by direct computations. □

Lemma 5.3. *Let Assumption 2.1 hold and suppose each $u_m \in \mathbf{C}^{l+2}$ and (11) holds for an integer $l \geq 0$. Then the function $r = r(v,x)$ is of class $\mathbf{C}^{l+2}$ and there is a constant $b > 0$ such that for any multi-index $\beta = (\beta_1, \ldots, \beta_M, \beta_{M+1})$ of non-negative integers with $|\beta| \leq l+2$*

$$|\mathbb{T}^\beta r(v,x)| < b \frac{\partial r}{\partial x}(v,x), \quad (v,x) \in (0,\infty)^M \times \mathbf{R}, \tag{30}$$

where $\mathbb{T}^\beta$ is the differential operator

$$\mathbb{T}^\beta \triangleq \left(\prod_{m=1}^M (v^m \frac{\partial}{\partial v^m})^{\beta_m} \right) \frac{\partial^{\beta_{M+1}}}{\partial x^{\beta_{M+1}}}.$$

Proof. If $|\beta| \leq 2$, (30) follows from Theorems 4.1 and 4.2 in [1] containing explicit expressions for $\mathbb{T}^\beta r(v,x)$ in this case. If $2 < |\beta| \leq l+2$, then elementary computations based on the formulas from Theorem 4.2 in [1] yield

$$\mathbb{T}^\beta r(v,x) = \frac{\partial r}{\partial x}(v,x) P_\beta(v,x) \frac{1}{(\sum_{m=1}^M t_m(\widehat{x}^m))^{2|\beta|}},$$

where $(\widehat{x}^m)_{m=1,\ldots,M}$ is the arg-max in (2) and $P_\beta = P_\beta(v,x)$ is a polynomial of $t_m^{(n)}(\widehat{x}^m)$, $n = 0, \ldots, |\beta|-2$, $m = 1, \ldots, M$. The inequality (30) now follows from (29) and Assumption 2.1. □

Lemma 5.4. *Assume the conditions of Lemma 5.3 and denote*

$$\xi(a) \triangleq r(v, \Sigma(x,q)), \quad a = (v,x,q) \in \mathbf{A}. \tag{31}$$

Then for any compact set $C \subset \mathbf{A}$ there are a constant $b > 0$ and a compact set $D \subset \mathbf{A}$ containing C such that

$$\|\xi\|_{l+2,C} \leq b \|\xi\|_D \triangleq b \sup_{a \in D} |\xi(a)|.$$

Proof. Lemma 5.3 implies the existence of a constant $b > 0$ such that

$$\|\xi\|_{l+2,C} \leq b \sup_{(v,x,q) \in C} \frac{\partial r}{\partial x}(v, \Sigma(x,q))(1 + |\psi|)^{l+2}. \tag{32}$$

In view of (4) the right side of (32) is dominated by $b_1 \|\xi\|_{1,E}$ for some constant $b_1 > 0$ and a compact set E in $\mathbf{A}$ containing C. Since the sample paths of $\xi = \xi(a)$ are saddle functions, $\|\xi\|_{1,E}$ is dominated by $b_2 \|\xi\|_D$ for some constant $b_2 > 0$ and a compact set $D \subset \mathbf{A}$ containing E; see Lemma 4.12 in [1]. □

The proof of Theorem 3.1 now follows from Lemma 4.3 and

Lemma 5.5. *Under the assumptions of Theorem 3.1 the process H from (8) has values in $\mathbf{C}^2(\mathbf{A}, \mathbf{R}^d)$ and (25) holds for any compact set $C \subset \mathbf{A}$.*

Proof. Recall the notation $\widetilde{\mathbf{A}}$ from Remark 4.4 and observe that the dimension of this set is $M+J$. In view of Lemmas 4.3 and 5.1 and Remark 4.4 it is enough to show that for any compact set $C \subset \widetilde{\mathbf{A}}$ the random field ξ of (31) satisfies

$$\mathbb{E}[\|\xi\|_{l+2,C}] < \infty.$$

This follows from Lemma 5.4 and the fact that (6) implies the integrability of $\|\xi\|_D$ for any compact set $D \subset \mathbf{A}$; see Lemma 4.12 in [1]. □

5.2 Proof of Theorem 3.2

It is enough to verify the growth condition (26) of Lemma 4.5. For $q \in \mathbf{R}^J$ define the processes $\widetilde{F}(q)$ and $\widetilde{H}(q)$ by

$$\widetilde{F}_t(q) \triangleq \mathbb{E}[e^{-a(\Sigma_0 + \langle q, \psi \rangle)} | \mathscr{F}_t] = \widetilde{F}_0(q) + \int_0^t \widetilde{H}_s(q) dB_s,$$

with the constant $a > 0$ from (14), and observe that, by (13),

$$\begin{aligned} F_t(v, x, q) &= r(v, x)\widetilde{F}_t(q), \\ H_t(v, x, q) &= r(v, x)\widetilde{H}_t(q). \end{aligned}$$

It follows that (26) holds if for any $b > 0$

$$\int_0^1 \sup_{|q| \leq b} \left(\frac{|\widetilde{H}_t(q)|}{\widetilde{F}_t(q)} \right)^2 dt < \infty.$$

This inequality holds since

$$\inf_{t \in [0,1]} \inf_{|q| \leq b} \widetilde{F}_t(q) \geq \inf_{t \in [0,1]} \mathbb{E}[\inf_{|q| \leq b} e^{-a(\Sigma_0 + \langle q, \psi \rangle)} | \mathscr{F}_t] = \inf_{t \in [0,1]} \mathbb{E}[e^{-a(\Sigma_0 + b|\psi|)} | \mathscr{F}_t] > 0,$$

and because, in view of Lemma 5.5,

$$\int_0^1 \sup_{|q| \leq b} |\widetilde{H}_t(q)|^2 dt < \infty.$$

This ends the proof of Theorem 3.2.

5.3 Proof of Theorem 3.3

The proof is divided into a series of lemmas, where we shall verify the assumptions of Lemma 4.7. Hereafter we denote

$$\xi(a) = r(v, \Sigma(x,q)), \quad a = (v,x,q) \in \mathbf{A}.$$

As usual $\mathbf{L}^p$ stands for the space of p-integrable random variables, $p \geq 1$.

Lemma 5.6. *Suppose Assumption 2.1 and conditions (16) and (17) hold. Then $\|\xi\|_{3,C} \in \mathbf{L}^2$ for any compact set $C \subset \mathbf{A}$.*

Proof. Lemma 5.4 and our boundedness assumptions on a_m and a'_m yield that $\|\xi\|_{3,C} \leq b_1 \|\xi\|_E$ for some constant $b_1 > 0$ and a compact set $E \subset \mathbf{A}$ containing C. From (4) and (5) we deduce the existence of a constant $b_2 > 0$ such that

$$\|\xi\|_E \leq b_2 e^{b_2|\psi| + c\Sigma_0^-/M}$$

and the result follows from (17). □

Lemma 5.7. *Under the conditions of Theorem 3.3 we have $\xi(a) \in \mathbf{D}^{1,1}$, $a \in \mathbf{A}$, with Malliavin derivative*

$$D\xi(a) = \frac{\partial r}{\partial x}(v, \Sigma(x,q))(D\Sigma_0 + \langle q, D\psi\rangle)$$

and for any compact set $C \subset \mathbf{A}$

$$\mathbb{E}\left[\left(\int_0^1 \|D_t\xi\|_{2,C}^2 dt\right)^{1/2}\right] < \infty. \tag{33}$$

Proof. Let $(f_n)_{n\geq 1}$ be a sequence of continuously differentiable functions on $(-\infty, 0)$ such that $f_n(x) = x$ on $(-n, 0)$ and $0 \leq f'_n(x) \leq -n/x$ on $(-\infty, -n]$. For example, we can take

$$f_n(x) = x 1_{\{-n<x<0\}} - (2n + \frac{n^2}{x})1_{\{x\leq -n\}}.$$

The function $f_n(r(v,.))$ is continuously differentiable and, in view of (5), has bounded derivatives. By the chain rule of Malliavin calculus, see Proposition 1.2.4 in [6], $\xi_n(a) \triangleq f_n(\xi(a))$ belongs to $\mathbf{D}^{1,2}$ with Malliavin derivative $D\xi_n(a) = f'_n(\xi(a))Z(a)$, where

$$Z_t(a) \triangleq \frac{\partial r}{\partial x}(v, \Sigma(x,q))(D_t\Sigma_0 + \langle q, D_t\psi\rangle), \quad t \in [0,1].$$

By the construction, $\xi(a) \leq \xi_n(a) < 0$ and $\xi_n(a) \to \xi(a)$ almost surely. This readily implies the convergence $\xi_n(a) \to \xi(a)$ in $\mathbf{L}^1$. Since $0 \leq f_n'(x) \leq 1$ and $f_n'(x) \uparrow 1$, the convergence $\xi_n(a) \to \xi(a)$ in $\mathbf{D}^{1,1}$ and the identity $D\xi(a) = Z(a)$ follow if we can show for any compact set $C \subset \mathbf{A}$

$$\mathbb{E}\left[\left(\int_0^1 \|Z_t\|_{2,C}^2 dt\right)^{1/2}\right] < \infty.$$

Note that in this case we also prove the finite bound (33).

We have

$$\int_0^1 \|Z_t\|_{2,C}^2 dt \leq \|\xi\|_{3,C}^2 \int_0^1 (|D_t\Sigma_0| + b|D_t\psi|)^2 dt,$$

where $b \triangleq \sup_{(v,x,q)\in C} |q|$, and then, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^1 \|Z_t\|_{2,C}^2 dt\right)^{1/2}\right] &\leq \mathbb{E}\left[\|\xi\|_{3,C}\left(\int_0^1 (|D_t\Sigma_0| + b|D_t\psi|)^2 dt\right)^{1/2}\right] \\ &\leq \left(\mathbb{E}\left[\|\xi\|_{3,C}^2\right]\right)^{1/2}\left(\mathbb{E}\left[\int_0^1 (|D_t\Sigma_0| + b|D_t\psi|)^2 dt\right]\right)^{1/2},\end{aligned}$$

which is finite because of Lemma 5.6 and because $\Sigma_0, \psi \in \mathbf{D}^{1,2}$. □

Lemma 5.8. *Under the conditions of Theorem 3.3 the process $H = H_t(a)$ from (8) has values in $\mathbf{C}^2(\mathbf{A}, \mathbf{R}^d)$ and (25) holds for any compact set $C \subset \mathbf{A}$. Moreover, for any multi-index $\beta = (\beta^1, \ldots, \beta^{J+M+1})$ with $|\beta| \leq 2$*

$$(34)\quad \partial^\beta H_t(a) = \mathbb{E}\left[\partial^\beta\left(\frac{\partial r}{\partial x}(v, \Sigma(x,q))(D_t\Sigma_0 + \langle q, D_t\psi\rangle)\right)\middle|\mathscr{F}_t\right],\ t \in [0,1].$$

Proof. Fix a compact set $C \subset \mathbf{A}$. Since, by Lemma 5.7, $\xi(a) \in \mathbf{D}^{1,1}$ the Clark-Ocone formula from [3] yields

$$H_t(a) = \mathbb{E}[D_t\xi(a)|\mathscr{F}_t], \quad a \in \mathbf{A},\ t \in [0,1],$$

or, equivalently,

$$H(a) \triangleq \widetilde{\mathbb{E}}[D\xi(a)|\mathscr{P}], \quad a \in \mathbf{A},$$

where $\widetilde{\mathbb{P}} \triangleq \mathbb{P}\otimes dt$ and $\mathscr{P}$ is the σ-algebra on $\Omega\times[0,1]$ generated by predictable processes. From Lemma 5.7 we deduce

$$\widetilde{\mathbb{E}}[\|D\xi\|_{2,C}] \triangleq \mathbb{E}[\int_0^1 \|D_t\xi\|_{2,C}dt] \leq \mathbb{E}\left[\left(\int_0^1 \|D_t\xi\|_{2,C}^2 dt\right)^{1/2}\right] < \infty.$$

As a result, see Lemma 6.5 of [2], the random field $H = H(a)$ can be chosen with sample paths in $\mathbf{C}^2(\mathbf{A},\mathbf{R}^d)$ and, for a multi-index $\beta = (\beta^1,\dots,\beta^{J+M+1})$ with $|\beta| \leq 2$

$$\partial^\beta H(a) = \widetilde{\mathbb{E}}[\partial^\beta D\xi(a)|\mathscr{P}], \quad a \in \mathbf{A},$$

which is just a reformulation of (34).

It only remains to verify (25). From (34) we obtain

$$\begin{aligned}\|H_t\|_{2,C}^2 &\leq (\mathbb{E}\left[\|\xi\|_{3,C}(|D_t\Sigma_0| + b|D_t\psi|)|\mathscr{F}_t\right])^2\\ &\leq \mathbb{E}\left[\|\xi\|_{3,C}^2|\mathscr{F}_t\right]\mathbb{E}\left[(|D_t\Sigma_0| + b|D_t\psi|)^2|\mathscr{F}_t\right],\end{aligned}$$

for some constant b depending on C. The result now follows because, in view of Lemma 5.6, $\mathbb{E}[\|\xi\|_{3,C}^2|\mathscr{F}_t]$, $t\in[0,1]$, is a martingale and thus has bounded paths and because

$$\mathbb{E}[\int_0^1 (|D_t\Sigma_0|^2 + |D_t\psi|^2)dt] < \infty \tag{35}$$

as $\Sigma_0, \psi \in \mathbf{D}^{1,2}$. □

To conclude the proof of Theorem 3.3 it only remains to verify the growth condition (28) of Lemma 4.7. This is accomplished in

Lemma 5.9. *Under the conditions of Theorem 3.3 for any $b > 0$*

$$\int_0^1 \sup_{(v,x,q)\in C(b)} \frac{1}{(\frac{\partial r}{\partial x}(v,x))^2}\left|v^m\frac{\partial H_t}{\partial v^m}(v,x,q)\right|^2 dt < \infty, \quad m = 1,\dots,M, \tag{36}$$

where

$$C(b) \triangleq \{(v,x,q)\in\mathbf{A} : \ |q| \leq b\}.$$

Proof. For $(v,x,q)\in\mathbf{A}$ we already know from Lemma 5.8 that

$$\frac{\partial H_t}{\partial v_m}(v,x,q) = \mathbb{E}[\frac{\partial^2 r}{\partial x\partial v_m}(v,\Sigma(x,q))(D_t\Sigma_0 + \langle q, D_t\psi\rangle)|\mathscr{F}_t].$$

According to Theorem 4.2 in [1] there is a constant $b_1 > 0$ such that for all $(v, x) \in (0, \infty)^M \times \mathbf{R}$ and $m = 1, \dots, M$

$$\frac{1}{b_1} \frac{\partial r}{\partial x}(v, x) \leq v^m \frac{\partial^2 r}{\partial v^m \partial x}(v, x) \leq b_1 \frac{\partial r}{\partial x}(v, x).$$

Accounting for (4) we deduce that for $(v, x, q) \in C(b)$

$$0 < v^m \frac{\partial^2 r}{\partial v^m \partial x}(v, \Sigma(x, q)) \leq b_1 \frac{\partial r}{\partial x}(v, \Sigma(x, q)) \leq b_1 \frac{\partial r}{\partial x}(v, x) e^{c(\Sigma_0^- + b|\psi|)/M}.$$

It follows that

$$\begin{aligned} \left(\frac{v^m}{\frac{\partial r}{\partial x}(v, x)} \frac{\partial H_t}{\partial v^m}(v, x, q) \right)^2 &\leq b_1^2 \left(\mathbb{E}[e^{c(\Sigma_0^- + b|\psi|)/M} (|D_t \Sigma_0| + b|D_t \psi|) | \mathscr{F}_t] \right)^2 \\ &\leq b_1^2 \mathbb{E}[e^{2c(\Sigma_0^- + b|\psi|)/M} | \mathscr{F}_t] \, \mathbb{E}[(|D_t \Sigma_0| + b|D_t \psi|)^2 | \mathscr{F}_t], \end{aligned}$$

which proves (36) because, in view of (17), the martingale $\mathbb{E}[e^{2c(\Sigma_0^- + b|\psi|)/M} | \mathscr{F}_t]$, $t \in [0, 1]$ has bounded paths and because of (35). $\square$

The proof of Theorem 3.3 is completed.

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