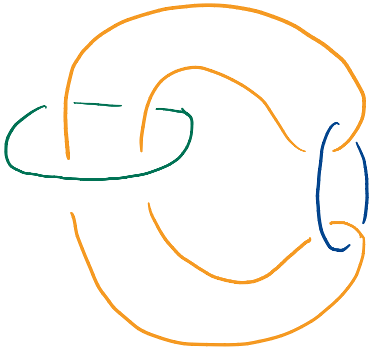
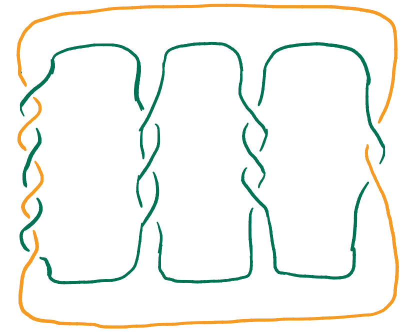
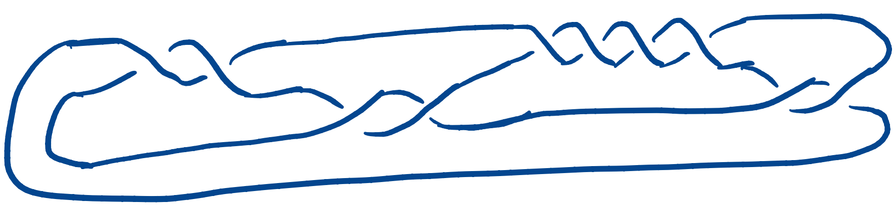




X-Slice links

Jon Simone

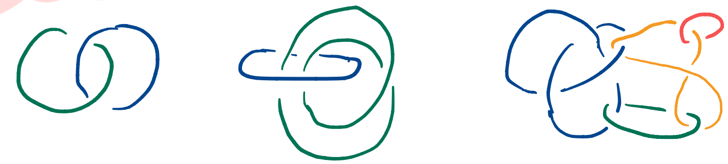
Georgia Tech



Knots: Knotted circles in $\mathbb{R}^3$

(unknot) (trefoil) (figure-eight)

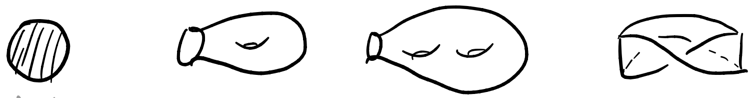
Links: Collection of knots in $\mathbb{R}^3$



Surfaces:

sphere torus genus 2 surface genus g surface

(without boundary)



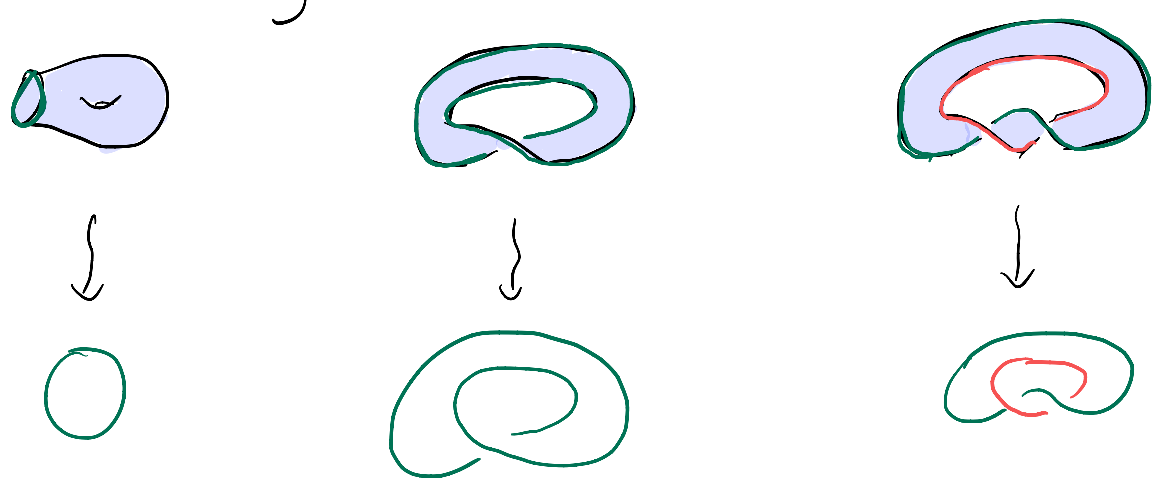
disk Mobius band

(with one boundary component)

etc.

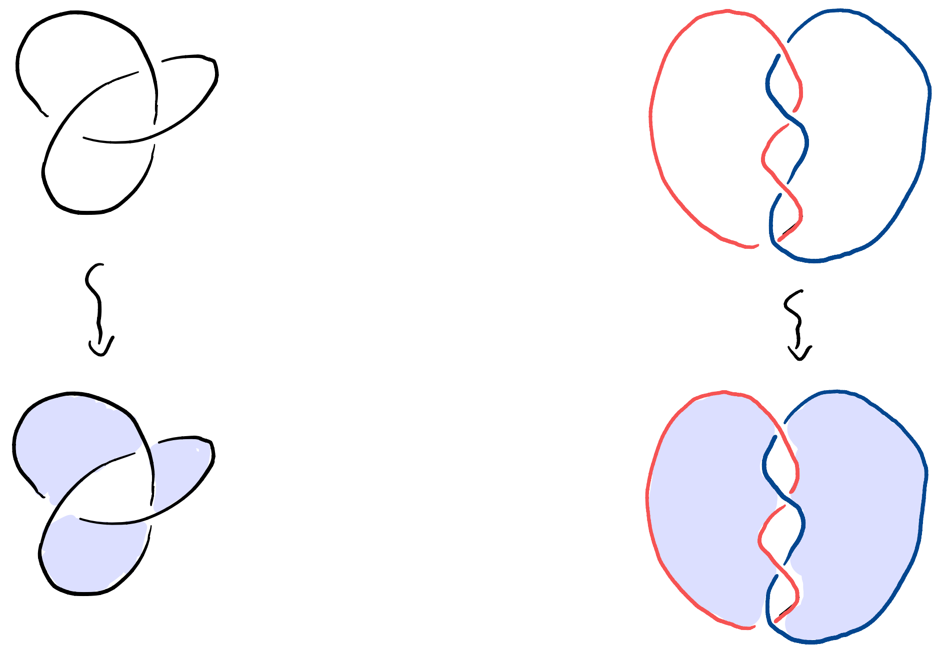
(In topology, everything is made of rubber)

Note: The boundary of a surface in $\mathbb{R}^3$ is a link in $\mathbb{R}^3$

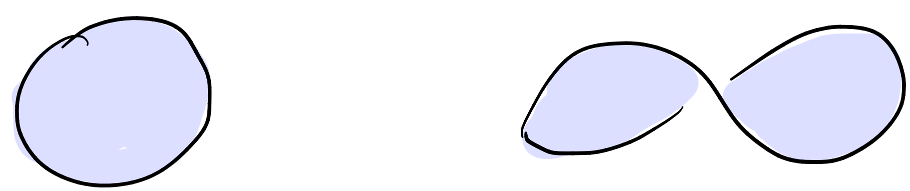


Fact: Every link in $\mathbb{R}^3$ is the boundary of a surface in $\mathbb{R}^3$

(proved by Frankl-Pontryagin in 1930 and an algorithm to construct such surfaces was given by Seifert in 1934)

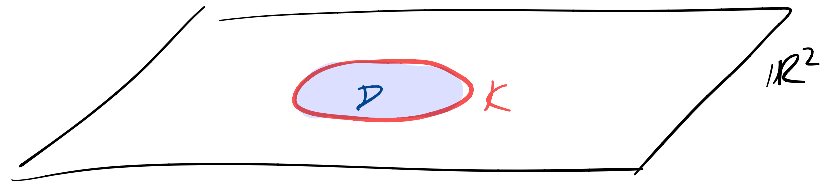


The only knot in $\mathbb{R}^3$ that bounds a disk in $\mathbb{R}^3$ is the unknot.



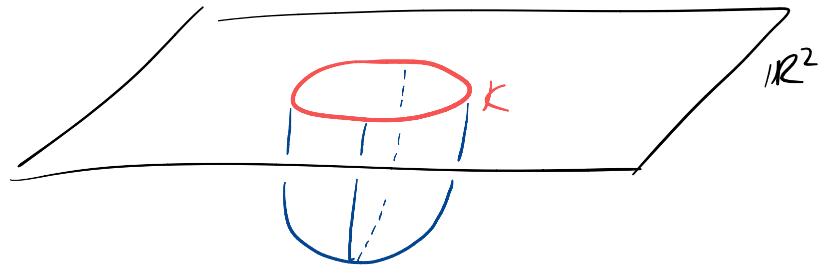
Motivating Example

Let K be a knot in $\mathbb{R}^2$
(it must be the unknot)

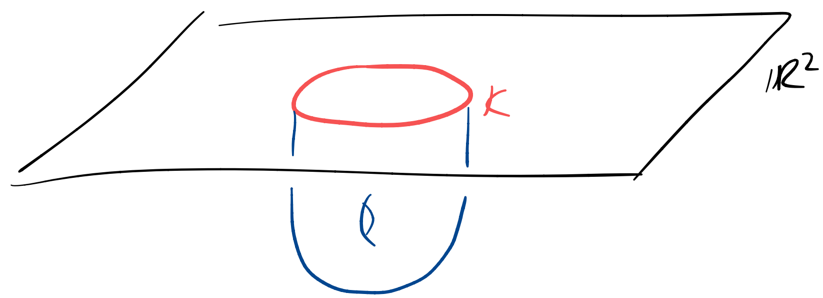


Let D be a disk in $\mathbb{R}^2$ bounded by K

Then D can be pushed into $\mathbb{R}^3$ giving a disk in $\mathbb{R}^3$ bounded by K, which is still in $\mathbb{R}^2$



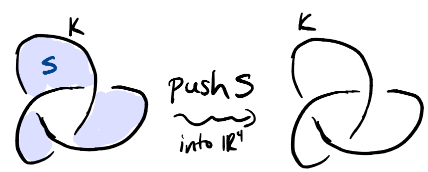
Moreover, K can bound other kinds of surfaces in $\mathbb{R}^3$ that it cannot bound in $\mathbb{R}^2$



The same is true for knots & links in $\mathbb{R}^3$:

Any surface bounded by a link L in $\mathbb{R}^3$ can be pushed into $\mathbb{R}^4$

Moreover, L can potentially bound more kinds of Surfaces in $\mathbb{R}^4$ than in $\mathbb{R}^3$



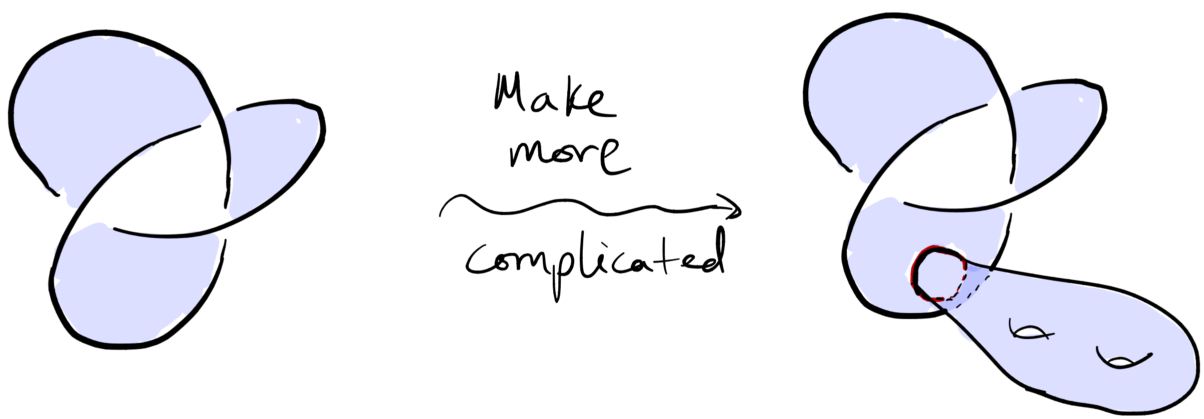
Def: A knot in $\mathbb{R}^3$ is called **slice** if it bounds a disk in $\mathbb{R}^4$

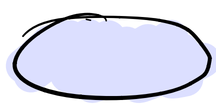
Why a disk?

The simplest surface a knot can bound is a disk.

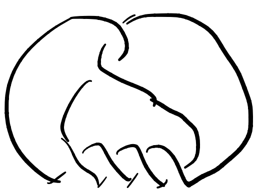
It is hard to bound simple surfaces

It is easy to bound more complicated surfaces.



Ex: The unknot is slice 

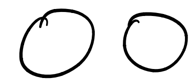
since it bounds a disk in $\mathbb{R}^3$ that we can push into $\mathbb{R}^4$

Ex:  is slice even though it does not bound a disk in $\mathbb{R}^3$ (since it is not the unknot).

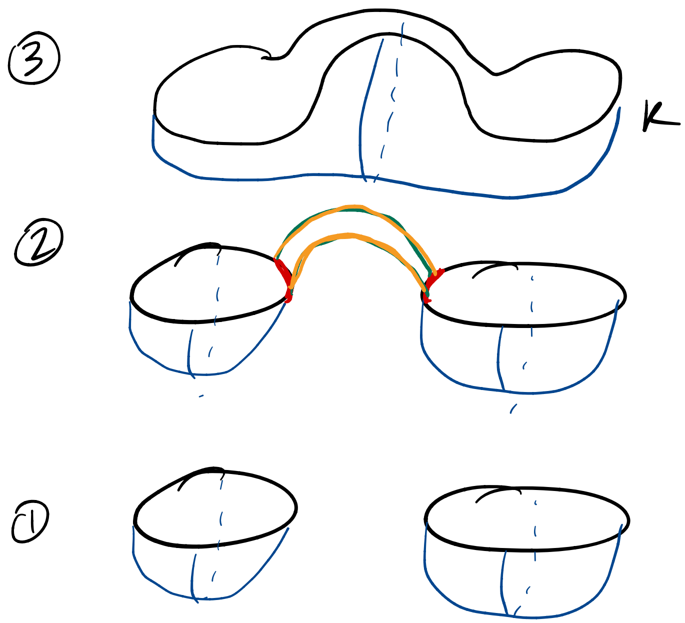
--- how can we tell?

Band Moves

pick two arcs on K

Fact: If we perform one band move to K and get  then K is slice.

Why?



The result is a disk in $\mathbb{R}^4$ with boundary K

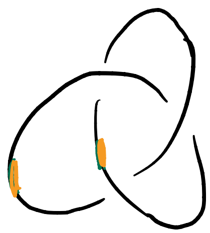
Redraw the arcs that were erased to recover K.
Fill in the arcs with a rectangle (band)

These unknots bound disjoint disks in $\mathbb{R}^4$.

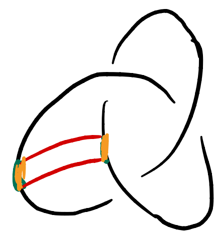
More generally, if we perform n band moves and get  then K is slice

Band Moves

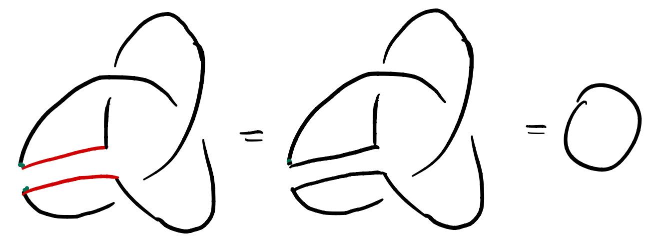
pick two arcs on K



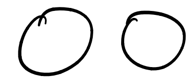
Draw two arcs between the endpoints of the arcs you drew on K



Erase the arcs on K

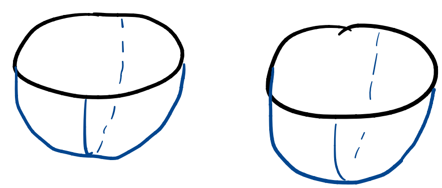


Fact: If we perform one band move to K and get

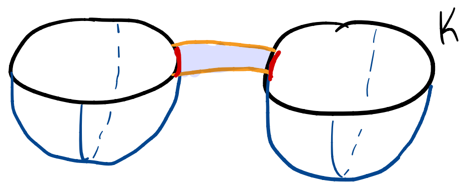


then K is slice

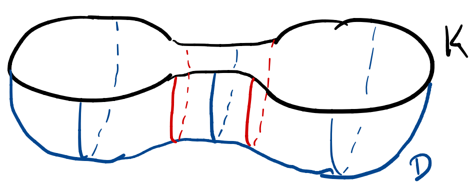
Why?



These unknots bound disjoint disks in $\mathbb{R}^4$.

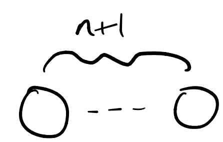


Redraw the arcs that were erased to recover K.
Fill in the arcs with a rectangle (band)



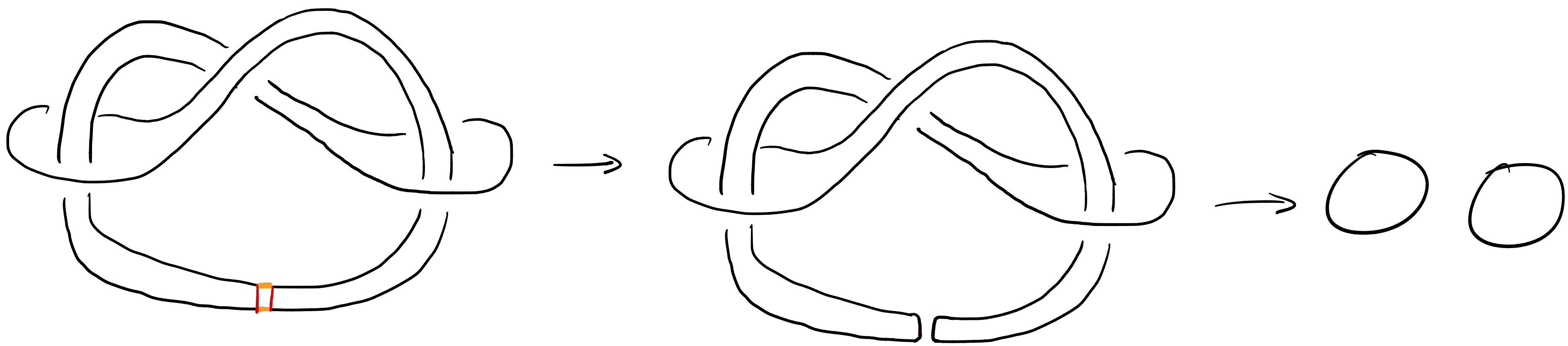
The result is a disk in $\mathbb{R}^4$ with boundary K

More generally, if we perform n band moves and get

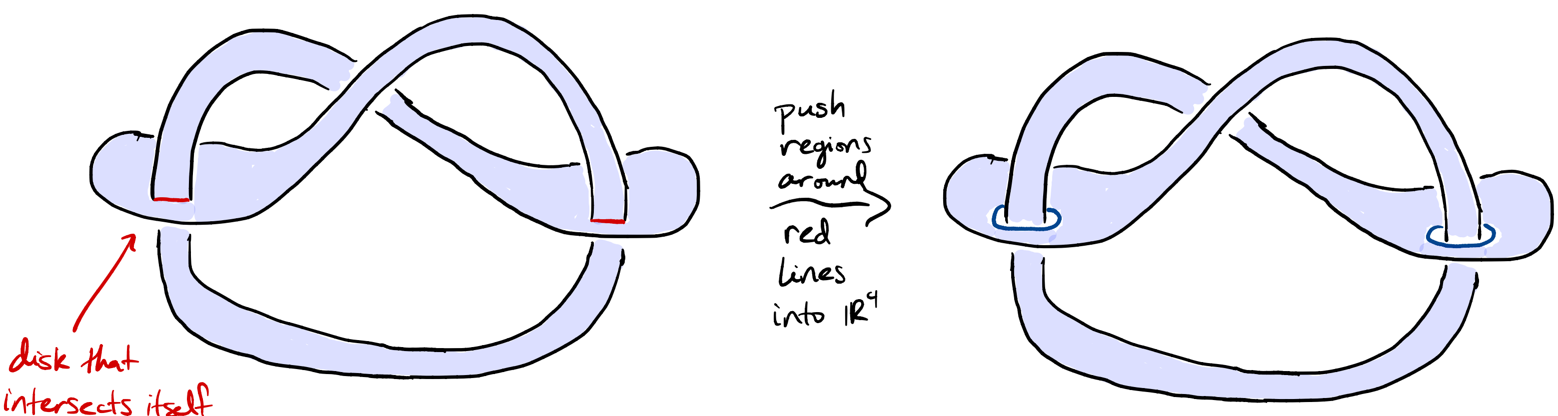


then K is slice

Ex

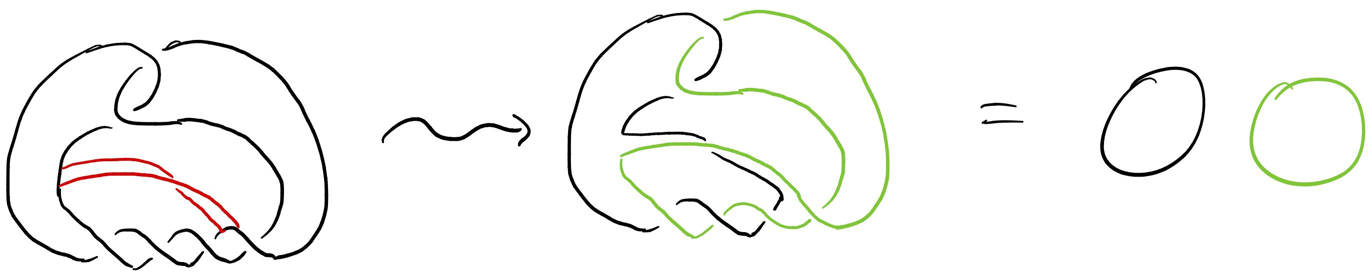


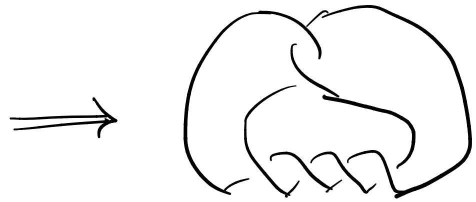
We can also see the disk



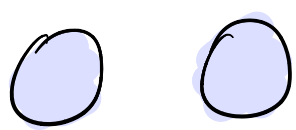
Now disk is in $\mathbb{R}^4$ & does not intersect itself.

Ex:



 is slice!

Def: A link L in $\mathbb{R}^3$ is slice if each component of L bounds a disk in $\mathbb{R}^4$ and the disks are disjoint.

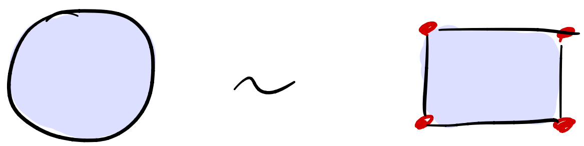
Ex: 

This is a "classical" notion of sliceness for links that many researchers have studied.

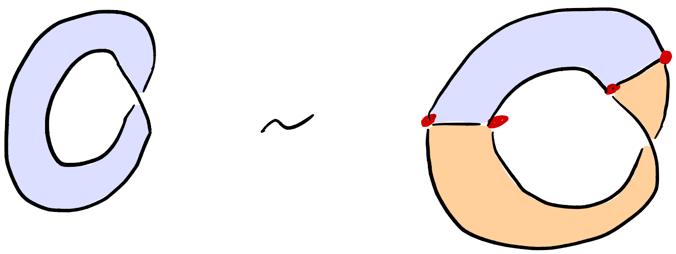
Def (2012): A link L in $\mathbb{R}^3$ is χ-slice if L bounds a surface in $\mathbb{R}^4$ with $\underbrace{\text{no closed components}}_{\text{each surface has boundary}}$ and Euler characteristic 1

The Euler characteristic of a surface S is

$$\chi(S) = \#\text{vertices} - \#\text{edges} + \#\text{faces}$$



$$\chi = 4-4+1 = 1$$



$$\chi = 4-6+2 = 0$$

Note: If L has one component, then χ-slice $=$ slice

(since the only $\chi=1$ surface w/ one boundary component is the disk)

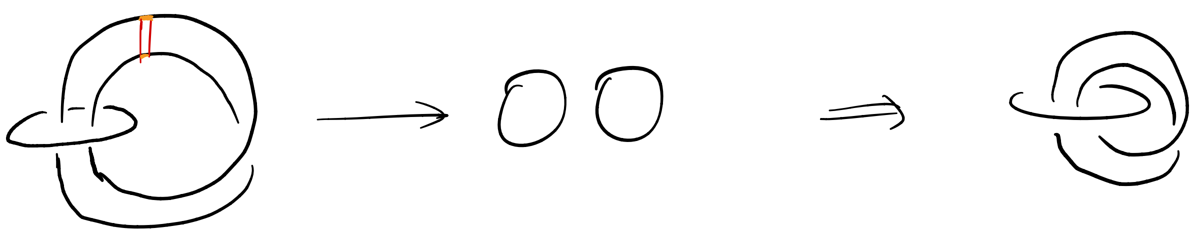
Why look at χ-slice?

- As with knots, the simplest surfaces that certain links can bound are those with $\chi = 1$.

  those with "nonzero determinant"

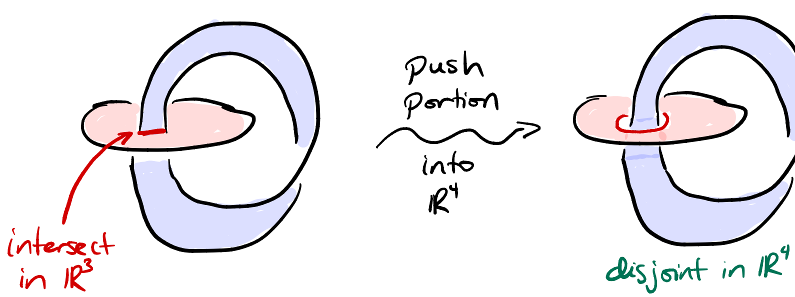
- χ-slice links are also useful for constructing 3-dimensional and 4-dimensional objects that topologists are interested in studying.

To show a link is χ-slice: use band moves as with knots



is χ-slice

We can also see the $\chi = 1$ surface in this example:



L bounds Disk $\sqcup$ Mobius band

$\Rightarrow \chi = 1 + 0 = 1$

$\Rightarrow L$ is χ-slice

Summer REU: Determine which pretzel links are χ-slice

with:

Hannah Turner (co-mentor)

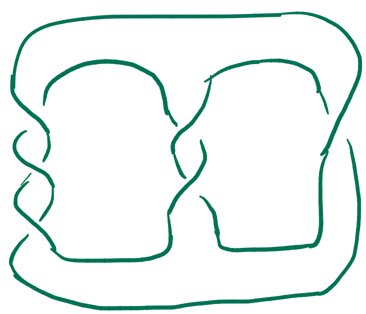
Weizhe Shen (grad TA)

Students:

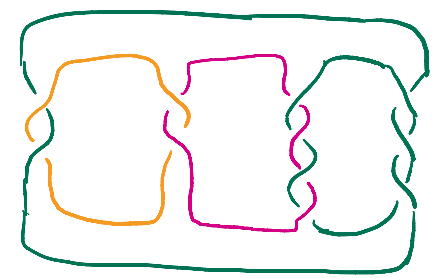
Sophia Fanelle (Barnard)

Ben Heunemann (Utah)

Evan Huang (Rice)

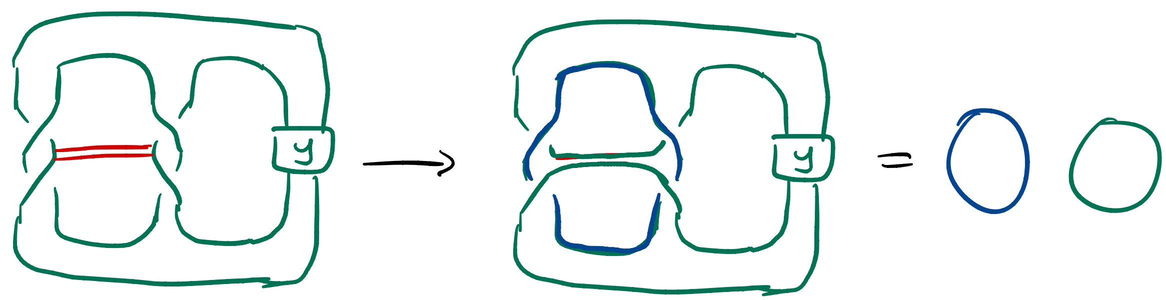


$P(-3,2,1)$



$P(2,-2,4,-3)$

Ex: $P(x,-x,y)$ is χ-slice $\forall\ x,y$



Much is known about which pretzel knots are slice

The goal of the REU was to extend these results to links and understand which pretzel links are χ-slice

Tools

- **Construction**

  Show certain infinite families of pretzel links are χ-slice by using band moves

- **Obstruction** (this is the hard part)

  Show "most" pretzel links are not χ-slice using algebraic tools

  e.g. Donaldson's Diagonalization Theorem,
  Lattice Embeddings
  } more or less understandable if you know linear algebra

  Heegaard Floer Homology d-invariants – More advanced

(The majority of the Summer was spent understanding and applying these obstructive tools)

Some Results

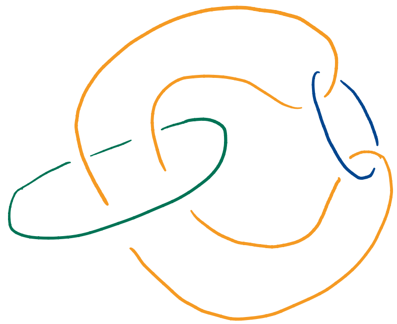
- If $p_1, p_2, \dots, p_k > 0$, then $P(p_1, \dots, p_k)$ is χ-slice if and only if
  $(p_1, p_2, \dots, p_k) = (k-3, 1, \dots, 1)$ or $(p_1, p_2, \dots, p_k) = (m+1, 1, \dots, 1)$

- The following 4-stranded pretzel links are χ-slice:
  $P(p, 1, -2, -2)$, $P(p, q, -q, -p-1)$, $P(p, q, -q, -p-4)$ (there are more)

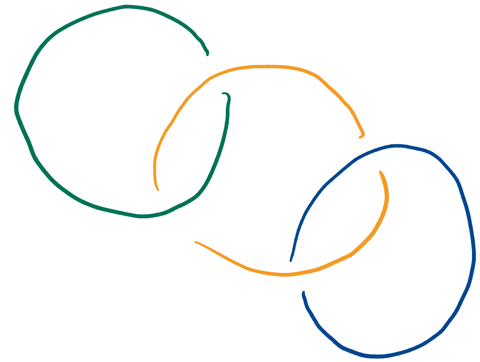
- If p, q, r, s satisfy some restrictive algebraic conditions, then $P(p, q, r, s)$ is not χ-slice

Open Problem: finish the classification of χ-slice 4-stranded pretzel links

(slice 4-stranded knots were classified by Lecuona in 2013)



(Borromean Rings)



Thanks!

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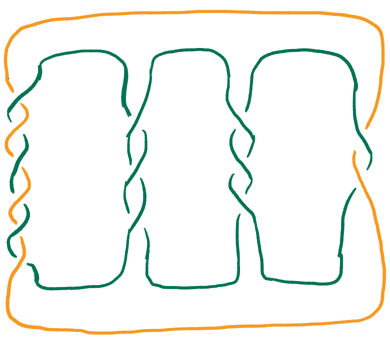
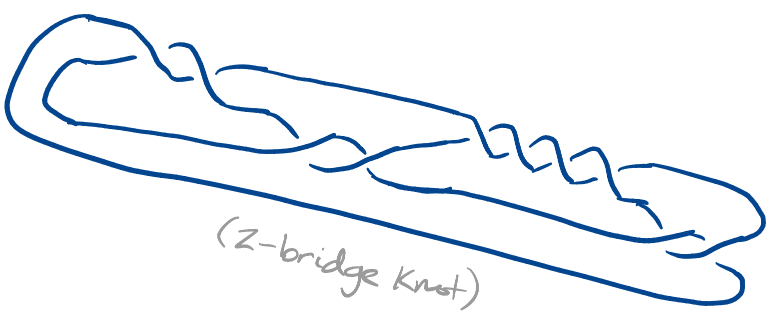
jsimone7@gatech.edu



(4-stranded pretzel link)



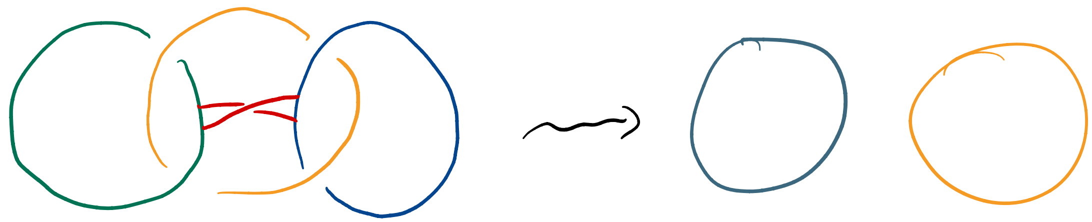
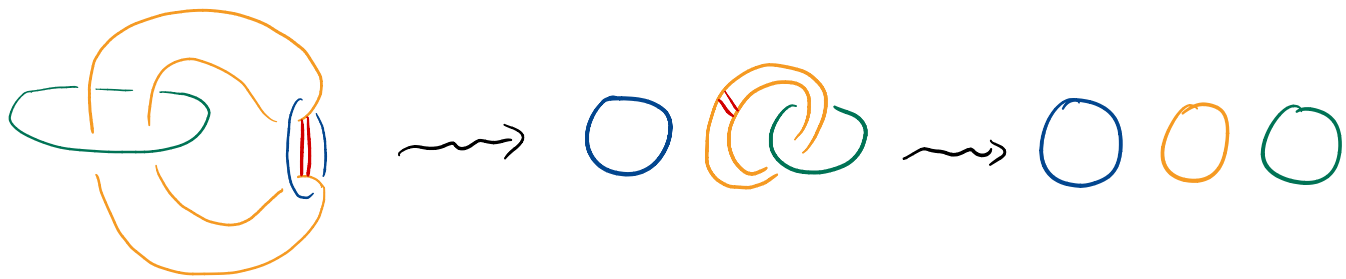
(2-bridge knot)

Challenge!

All links on this page are χ-slice.

Can you find band moves to prove it?

Answers:

- 
- 
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