# Solving First Order Linear Differential Equations

**Example 1.** A 20-quart juice dispenser in a cafeteria is filled with a juice mixture that is 10% cranberry and 90% orange juice. A pineapple-orange blend (40% pineapple and 60% orange) is entering the dispenser at a rate of 4 quarts an hour and the well-stirred mixture leaves at a rate of 5 quarts an hour. Model the situation with a differential equation whose solution is the amount of orange juice in the container at time $t$.

Let $y = y(t)$ be the amount of orange juice in the container at time $t$.

Notice that unlike previous problems we've done, the rate in and rate out are not the same. Each hour the number of gallons of juice mixture decreases by 1 quart.

$\frac{dy}{dt}$ = rate in of orange juice - rate out of orange juice

$$\text{rate in} = 4\,\frac{\text{qts mixture}}{\text{hr}} \cdot (0.6)\,\frac{\text{qts orange}}{\text{qts mixture}} = 2.4\,\frac{\text{qts orange}}{\text{hr}}$$
$$\text{rate out} = 5\,\frac{\text{qts mixture}}{\text{hr}} \cdot \frac{y \text{ qts orange}}{(20-t)\text{qts mixture}} = \frac{5y}{20-t}\,\frac{\text{qts orange}}{\text{hr}}$$

The differential equation modelling the situation is

$$\frac{dy}{dt} = 2.4 - \frac{5y}{20-t}$$

where $y(0) = 18$.

This differential equation is not separable. But it is a first order linear differential equation and by the end of this handout you should be able to solve it.

**Definition:** A **first order linear differential equation** is a differential equation that can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We will refer to this as 'standard form' for such a differential equation.

The goal of this handout is to show you how to systematically solve such a differential equation. The next example serves to motivate the method.

**Example 2.** Solve $\frac{dy}{dx} + \frac{1}{x}y = x^2$.

First verify for yourself that this differential equation is not separable. Having established that, the prospect of solving might look grim. However, suppose we were to multiply both sides of the equation by $x$.

$$\begin{aligned} \frac{dy}{dx} + \frac{1}{x}y &= x^2 \\ x\frac{dy}{dx} + y &= x^3 \end{aligned}$$

If you look long and hard at the left hand side of the equation above you might start to see the product rule in there and recognize that the left hand side of this equation is the derivative of $xy$ with respect to $x$. That observation proves to be very handy because we can then integrate with respect to $x$ and solve for $y$ as follows:

$$\begin{aligned}
\frac{d}{dx}(xy) &= x^3 \\
\int \frac{d}{dx}(xy)\ dx &= \int x^3\, dx \\
xy &= \frac{x^4}{4} + C \\
y &= \frac{x^3}{4} + \frac{C}{x}
\end{aligned}$$

Note that we cannot replace $\frac{C}{x}$ by a constant $C_1$ because $x$ is not a constant! What saved the day was the idea of multiplying both sides of the differential equation by some function so that the left hand side of the differential equation was just the derivative of a product. Let's try to do this in general. (The rewards are great – we'll end up with a recipe for solving this type of differential equation.)

**General Case:** We begin with

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Our goal is to find a function $v(x)$ such that when we multiply both sides by $v(x)$ the left hand side looks like a product. Let $v(x)$ be a positive function. Multiplying by $v(x)$ gives

$$v(x)\frac{dy}{dx} + v(x)P(x)y = v(x)Q(x). \quad (1)$$

Thinking about the product rule we see that we should aim for a function $v(x)$ such that the left hand side of (1) is $\frac{d}{dx}(vy)$. In other words, we want to find $v(x)$ such that

$$v(x)\frac{dy}{dx} + v(x)P(x)y = \frac{d}{dx}\,(vy). \qquad (2)$$

From the product rule we know that $\frac{d}{dx}(vy) = v(x)\frac{dy}{dx} + \frac{dv}{dx}\,y$ so our goal is to find $v(x)$ such that

$$v(x)\frac{dy}{dx} + v(x)P(x)y = v(x)\frac{dy}{dx} + \frac{dv}{dx}y.$$

This amounts to finding $v(x)$ such that

$$v(x)P(x)y = \frac{dv}{dx}y, \qquad \text{or equivalently} \qquad v(x)P(x) = \frac{dv}{dx}.$$

The latter is a separable differential equation. We solve for $v(x)$ as shown below. Keep in mind that in any particular problem $P(x)$ is known.

$$\begin{aligned} v(x)P(x) &= \frac{dv}{dx} \\ P(x)\,dx &= \frac{1}{v}\,dv \\ \int P(x)\,dx &= \int \frac{1}{v}\,dv \\ \int P(x)\,dx &= \ln v \\ v &= e^{\int P(x)\,dx} \end{aligned}$$

$v = e^{\int P(x)\,dx}$ is called the **integrating factor**. We now know that we can indeed find $v$ such that the left hand side of (1) is just $\frac{d}{dx}(vy)$ we can proceed as follows.

$$\begin{aligned} v(x)\frac{dy}{dx} + v(x)P(x)y &= v(x)Q(x) \\ \frac{d}{dx}vy &= v(x)Q(x) \\ vy &= \int v(x)Q(x)\,dx + C \\ y &= \frac{1}{v}\left[\int v(x)Q(x)\,dx + C\right] \end{aligned}$$

where $v(x) = e^{\int P(x)\,dx}$.
We have a systematic way to solve first order linear differential equations.

- Step 1: Put the equation into standard form. Identify $P(x)$ and $Q(x)$.
- Step 2: Find the integrating factor $v(x) = e^{\int P(x)\,dx}$ and simplify. [1]
- Step 3: Multiplying by $v$ and integrating both sides gives

$$v(x)y = \int v(x)Q(x)\,dx + C.$$

If you really want a formula, here it is:

$$y = \frac{1}{v}\left[\int v(x)Q(x)\,dx + C\right].$$

[1] When you integrate $P(x)$ you can let the constant be zero as you're just looking for one integrating factor.

If the equation is actually separable, it is advisable to simply separate variables!

**Example 3.** Solve $\frac{dy}{dx} = e^{2x} + 3y$. Find the general solution and then the particular solution with $y(0) = 3$.

This is not separable, so we put it in standard form: $\frac{dy}{dx} - 3y = e^{2x}$.
$P(x) = -3$ and $Q(x) = e^{2x}$
$v(x) = e^{\int P(x)\,dx} = e^{\int -3\,dx} = e^{-3x}$. Multiplying by $v(x)$ gives

$$\begin{aligned} e^{-3x}\frac{dy}{dx} - 3e^{-3x}y &= e^{-x} \\ \frac{d}{dx}(e^{-3x}y) &= e^{-x} \\ e^{-3x}y &= \int e^{-x}\,dx + C \\ e^{-3x}y &= -e^{-x} + C \\ y &= -e^{2x} + Ce^{3x} \end{aligned}$$

If $y(0) = 3$, then $3 = -e^0 + Ce^0 = -1 + C$, so $C = 4$ and $y = -e^{2x} + 4e^{3x}$.
Note that if you really wanted to, after identifying $P$, $Q$, and $v$, you could skip directly to $\frac{d}{dx}(e^{-3x}y) = e^{-x}\,dx + C$ and solve.

**Example 4.** Solve $(x^2+1)y' + 3xy = 6x$.

This is not separable, so we put it in standard form: $y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$.
$P(x) = \frac{3x}{x^2+1}$ and $Q(x) = \frac{6x}{x^2+1}$.
$v(x) = e^{\int P(x)\,dx} = e^{\int \frac{3x}{x^2+1}\,dx} = e^{3/2\ln(x^2+1)} = (x^2+1)^{3/2}$.

$$\begin{aligned} (x^2+1)^{3/2}y &= \int (x^2+1)^{3/2}\frac{6x}{x^2+1}\,dx + C \\ (x^2+1)^{3/2}y &= \int 6x(x^2+1)^{1/2}\,dx + C \\ (x^2+1)^{3/2}y &= 2(x^2+1)^{3/2} + C \\ y &= 2(x^2+1)^{-3/2}[(x^2+1)^{3/2} + C] \\ y &= 2 + \frac{C}{(x^2+1)^{3/2}} \end{aligned}$$

Now you are ready to go back and solve the equation from Example 1.

**Homework problems**: Solve the following:

1. $y' - 4xy = x$
2. $y' - 3\frac{y}{x} + x^3 - x = 0$
3. $y' + y = e^x$ where $y(0) = 6$.