Series: Convergence and Divergence

Here is a compilation of convergence/divergence tests.

- Series that we know about:

  Geometric Series: A geometric series is a series of the form $\sum_{n=0}^{\infty} ar^n$. The series converges if $|r| < 1$ and diverges otherwise[1]. If $|r| < 1$, the sum of the entire series is $\frac{a_1}{1-r}$ where a is the first term of the series and r is the common ratio.

  p-Series Test: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise[2].

- *Nth Term Test for Divergence:* If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

  Note: If $\lim_{n\to\infty} a_n = 0$ we know nothing. It is possible that the series converges but it is possible that the series diverges.

Comparison Tests:

These tests can be applied only to series in which all the terms are positive. (They are based on the idea of bounding the monotonic sequence of partial sums.)

- *Direct Comparison Test:* If a series $\sum_{n=1}^{\infty} a_n$ has all positive terms, and all of its terms are eventually bigger than those in a series that is known to be divergent, then it is also divergent. The reverse is also true–if all the terms are eventually smaller than those of some convergent series, then the series is convergent.

  That is, suppose $\sum a_n$, $\sum b_n$ and $\sum c_n$ are all series with positive terms and $a_n \leq b_n \leq c_n$ for all n sufficiently large, then

  if $\sum c_n$ converges, then $\sum b_n$ does as well

  if $\sum a_n$ diverges, then $\sum b_n$ does as well.

  (This is a good test to use with rational functions. Specifically, if the degree of the denominator is more than 1 greater than the degree of the numerator, try to prove that the series converges (compare with a p-series). In other cases, including when the difference in degree is exactly 1, prove that it diverges).

- *Limit Comparison Test:* Use this when you know what you want to compare to but the inequalities go the wrong way. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is finite and non-zero, then either both the series converge or both the series diverge.

- *Integral Test:* Let $a_n = f(n)$. Then, if f is continuous, decreasing, and positive on $[1, \infty)$, we have that $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

This test was instrumental in proving the p-series result.

This week we introduced the Alternating Series Test and the Ratio Test in addition to the tests of the other side of this sheet. Unlike the Comparison Tests and Integral Test these tests are intrinsic.

- *Alternating Series Test:* If the terms in a series are (i) alternating in sign, (ii) decreasing in absolute value and (iii) approaching 0, then the series converges. This is a test for convergence only. Don't use it and conclude a series diverges!

- *Ratio Test:* Given $\sum_{n=1}^{\infty} a_n$, look at $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$. If this ratio is less than 1, the series converges absolutely. If this ratio is greater than 1, the series diverges. If it equals 1, no conclusion can be drawn (Note that if a series converges conditionally then the Ratio Test will be inconclusive. The Ratio Test works well for series involving factorials and where the n is in an exponent. It is always the test to use when trying to determine a radius or interval of convergence).

So, given a series, how do we know what to think about: p-series, geometric series, Nth Term Test for Divergence, Direct Comparison, Limit Comparison, Integral Test, Ratio Test, Alternating Series Test .... There is no definitive answer, but the following guidelines may be useful.

[1] We proved this by writing the partial sums in closed form and computing a limit.

[2] We proved this using the Integral Test

Guidelines for using the tests:

- Is it a familiar series?

  A geometric series is a series of the form $\sum_{n=0}^{\infty} ar^n$; it converges if and only if $|r| < 1$.

  A $p-$Series Test: is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$; it converges if and only if $p > 1$.

- If you can see easily that $\lim_{n\to\infty} a_n \neq 0$, then by the Nth Term Test for Divergence the series diverges and you're done.

- If the series is neither geometric nor a $p-$ series but looks similar to one of these *and* the terms of the series are eventually all positive then use the Direct Comparison Test or the Limit Comparison Test.

  If the terms are not eventually all positive (i.e. $a_n > 0$ for all n large enough) then you can look at $\sum |a_n|$ and test for absolute convergence. If the series converges absolutely, then it converges. Before doing this however, see if the series is alternating, and if so, try the alternating series test.

- If the series is alternating, see if you can apply the Alternating Series Test.

- Series involving factorials or constants raised to a variable power readily lend themselves to be tested using the Ratio Test. (If the series is not initially composed of positive terms (or terms that eventually are positive) then the ratio test is used to test for absolute convergence.) Keep in mind that the Ratio Test will fail for all $p-$series and therefore for series that are rational functions of n.

- If $a_n = f(n)$ where f is continuous, decreasing, and positive on $[a, \infty)$, and we can evaluate $\int_a^{\infty} f(x)\, dx$, then the Integral Test can be applied.

Notice that the Nth Term Test for Divergence can be applied to any series, and likewise with the Ratio Test. You cannot, however, invoke the Comparison or Limit Comparison Tests if the terms are not eventually positive. If the series is not an alternating series then you can try to deal with it by looking at $\sum |a_n|$ and seeing if the series converges absolutely.

When using the convergence tests, think of this as an exercise in making a clear and logical argument. After you figure out what test you'll use, make it clear by "calling upon the test". It should be clear to the reader what test you are using.