21-465 Topology and Geometry **Final Exam Questions** **D. Handron**

The exam is open book and open notes. If you wish to use any other resource, you should check with me before doing so. Please feel free to ask me any questions you may have, particularly about confusing wording or typo's.

You have 24 hours to work on this exam. You may choose any 24 hour period during the exam period. The exam must be turned in to me or put in my mailbox by 5:00pm on May 16.

Good Luck.

1. Let X be a Hausdorff space such that each point of X has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^n$. Show that if X is compact, then X is an n-manifold.

2. If M is an n-manifold, a continuous function $f : U_f \to M$ from an open subset U_f of $\mathbb{R}^n$ into M is called a *coordinate chart* if it is a homeomorphism onto its image. A collection $\{f_\alpha : U_\alpha \to M\}$ is called an *atlas* if the open sets $f_\alpha(U_\alpha)$ cover M.

   (a) If $p \in M$ is contained in the image of two coordinate charts, $f_\alpha(U_\alpha) \cap f_\beta(U_\beta)$, show that the map $f_\beta^{-1} \circ f_\alpha$ determines a homeomorphism from a neighborhood of $f_\alpha^{-1}(p)$ in $\mathbb{R}^n$ to a neighborhood of $f_\beta^{-1}(p)$ in $\mathbb{R}^n$. The functions $f_\gamma^{-1} \circ f_\delta$ are called the *transition functions* for this atlas.

   (b) It is possible to provide a manifold with additional structure by placing additional requirements on the transition functions.

   Suppose that $\mathcal{A} = \{f_\alpha : U_\alpha \to \text{M}\}$ is an atlas for M such that the transition functions $f_\gamma^{-1} \circ f_\delta$ are all differentiable with differentiable inverse. Then we can make the following definition: A function $h : M \to \mathbb{R}$ is differentiable at $p \in f_\alpha(U_\alpha)$ if $h \circ f_\alpha$ is differentiable at $f_\alpha^{-1}(p)$.

   Show that this definition does not depend on the choice of coordinate chart.

3. Let w be the proper labeling scheme

$$w = (a_1 b_1 a_1^{-1} b_1) \cdots (a_m b_m a_m^{-1} b_m).$$

   Show that w is equivalent to the labeling scheme

$$(c_1 c_1) \cdots (c_{2m} c_{2m}).$$

   Note: This shows that a connected sum of m Klein bottles is homeomorphic to a connected sum of $2m$ projective planes.

4. (a) Show that a wedge sum of spheres (copies of S^2) has trivial fundamental group.

   (b) For $r \in \mathbb{R}$ and $x \in \mathbb{R}^3$, let $S_r(x)$ denote the sphere with radius r and center x. Let $X = \bigcup_{n \in \mathbb{Z}_+} S_n(n, 0)$ and $Y = \bigcup_{n \in \mathbb{Z}_+} S_{\frac{1}{n}}(\frac{1}{n}, 0)$. Show that these two spaces are topologically distinct. Are either of these homeomorphic to a wedge of spheres?

5. Let $X = B^3$. Define an equivalence relation on X by setting $x \sim y$ if $x, y \in Bd(X)$ and $x = -y$. Let Y be the resulting quotient space. Let $y_0 \in Y$.

   What is $\pi_1(Y, y_0)$?

6. Let $X = S^1$, $B_1 = B^2$, and $B_2 = B^2$. Choose $x_0 \in X$. Let maps $f_i : Bd(B_i) \to X$ be given by

$$f_1(z) = z^n, \quad f_2(z) = z^m,$$

where S^1 and each $Bd(B_i)$ are identified with the unit circle in $\mathbb{C}$, and m and n are positive integers.

Let Y be the space $Y = X \cup_{f_1} B_1 \cup_{f_2} B_2$, i.e. B_1 and B_2 are attached to X by the attaching maps f_1 and f_2.

What can you say about $\pi_1(Y, x_0)$ in terms of n and m?