Course Notes for Mathematical Finance (21-270, 21-370)

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CHAPTER 9

AMERICAN DERIVATIVE SECURITIES, RANDOM MATURITY, AND STOPPING TIMES IN THE MULTIPERIOD BINOMIAL MODEL

American Derivative Securities: A First Look

European put and call options have only one possible exercise date (also called the expiration date) and this date is specified when an option is purchased. Although the payoff amount of such an option is *random*, the date at which the payoff will be made is *deterministic*.

American puts and calls have expiration dates, but can be exercised at any time up to and including the expiration date. If all option parameters are the same, an American option must cost at least as much as the European counterpart. Indeed, if an American option were selling for strictly less than the European counterpart an investor could purchase the American option, sell the corresponding European option, invest the difference in the bank, store the American option in a drawer until the expiration date and then use the American option to cover any liability created by sale of the European option, thus creating an arbitrage.

The act of exercising an American option prior to the expiration date is known as *early exercise*. The difference in price between an American option and the European counterpart can be thought of as an *early exercise premium*. Assuming that the stock does not pay dividends, we shall prove that the price of an American call is the same as the price of a European call having the same strike price and expiration date; in other words, there is no early exercise premium in this case. On the other hand, we shall see that American puts can be worth strictly more than European puts having the same strike price and expiration date.

Remark 9.1: In situations where the stock pays dividends, there is sometimes an early exercise premium for American calls. The reason for this is that the stock price drops when a dividend is paid, and consequently the holder of a call option may be able to "capture a dividend payment" by exercising a call shortly before a dividend is paid. The issue will be explored in Problem (...).

There is also the class of so-called *Bermudan Options* that have more than one possible exercise date, but also have dates on which they cannot be exercised. In the N-period binomial model, a Bermudan put or call option on the stock will have a strike price K and a set $\mathcal{E}$ of possible exercise dates. The set $\mathcal{E}$ is a proper subset of $\{0, 1, \cdots, N\}$ having at least two elements. Assuming that $N \in \mathcal{E}$, the initial price of a Bermudan put with strike price K must lie between the initial prices of a European put and an American put with strike price K and expiration date N. Bermudan options will be treated in the exercises.

There are also much more general types of American derivative securities. Before we begin a treatment of such securities it seems worthwhile to analyze a simple numerical example.

Example 9.2: Consider the binomial model with $N = 2$, $u = 2$, $d = \frac{1}{2}$, $r = .25$, and $S_0 = 4$. Let U be a European put option on the stock with expiration date 2 and strike price $K = 5$, and let V be an American put option on the stock having the same expiration and strike.

We begin by computing the values $(U_n)_{0 \le n \le 2}$:

$$U_2(H,H) = (5-16)^+ = 0, \;\; U_2(H,T) = U_2(T,H) = (5-4)^+ = 1, \;\; U_2(T,T) = (5-1)^+ = 4,$$

$$U_1(H) = \frac{4}{5}\left[\frac{1}{2}(0) + \frac{1}{2}(1)\right] = .40, \quad U_1(T) = \frac{4}{5}\left[\frac{1}{2}(1) + \frac{1}{2}(4)\right] = 2.00,$$

$$U_0 = \frac{4}{5}\left[\frac{1}{2}(.40) + \frac{1}{2}(2)\right] = .96.$$

We can see immediately that $V_0 > U_0$, because if the American put is exercised at $t = 0$, the holder will collect \$1, so we must have $V_0 \ge 1 > .96 = U_0$.

In analyzing the American put, we will always need to be aware of how much could be collected if the option is exercised at the present time. At each time $n \in \{0, 1, \cdots, N\}$, the amount that the holder could collect by exercising the option immediately is known as the *intrinsic value* of the option at time n and will be denoted by G_n.

The American put option with strike price $K = 5$ has intrinsic values

$$G_n(\omega) = (5 - S_n(\omega))^+.$$

A simple computation shows that

$$G_0 = 1$$

$$G_1(H) = 0, \quad G_1(T) = 3,$$

$$G_2(H,H) = 0, \quad G_2(H,T) = 1, \quad G_2(T,H) = 1, \quad G_2(T,T) = 4.$$

Let us determine the values $(V_n)_{0\leq n\leq 2}$ of the option. (When we speak of the value V_n of the option at time n, we mean the value assuming that the option has not been exercised yet.) If we arrive at time 2, and the option has not previously been exercised, then $V_2(\omega) = G_2(\omega)$, i.e.

$$V_2(H,H) = 0,\ V_2(H,T) = 1,\ V_2(T,H) = 1,\ V_2(T,T) = 4.$$

Consider the situation at time 1.

1. Suppose the first toss is heads. The intrinsic value of the option is $G_1(H) = 0$, so there is no point in exercising immediately and the value of the option is

$$V_1(H) = \frac{1}{1+r}\left[\tilde{p}V_2(H,H) + \tilde{q}V_2(H,T)\right] = \frac{4}{5}\left[\frac{1}{2}(0) + \frac{1}{2}(1)\right] = .40.$$

2. Suppose the first toss is tails. The holder can either exercise the option now and receive $G_1(T) = 3$, or wait for the second coin toss. The value at time 1 of the potential payments at time 2 is

$$\frac{1}{1+r}\left[\tilde{p}V_2(T,H) + \tilde{q}V_2(T,T)\right] = \frac{4}{5}\left(\frac{1}{2}(1) + \frac{1}{2}(4)\right) = 2.$$

   Therefore, the holder should exercise the option immediately and collect $G_1(T) = 3$. We see that $V_1(T) = 3$. Even if the holder believes that the stock is very likely to go down and that she will most likely receive \$4 by waiting, she should still exercise the option at $t = 1$. (Indeed, using only \$2 at time 1, she can replicate a payoff at time 2 of \$1 if the second toss is heads and \$4 if the second toss is tails.)

To find V_0, we compute

$$\frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right] = \frac{4}{5}\left[\frac{1}{2}(.4) + \frac{1}{2}(3)\right] = 1.36.$$

The intrinsic value of the option at time 0 is only $G_0 = 1$, so the holder should not exercise at $t = 0$. The arbitrage-free price of the option is

$$V_0 = 1.36.$$

(Actually, some care must be used in defining precisely what we mean by the arbitrage-free price of an American option. This issue will be addressed carefully in Section (...).)

It is important to observe that $(V_n)_{0\le n\le 2}$ are not the capitals of a self-financing strategy. Indeed let us consider a broker who sells one American put for \$1.36 and wishes to hedge his short position. He needs to be able to pay \$.40 at $t = 1$ if the first toss is heads and \$3.00 at $t = 1$ if the first toss results in tails. Therefore he should purchase

$$\Delta_0 = \frac{.4 - 3}{8 - 2} = -\frac{13}{30}$$

shares of stock (i.e., sell short $\frac{13}{30}$ shares) at $t = 0$ and invest the proceeds of the short sale plus \$1.36 in the bank. Therefore at $t = 0$, the broker will hold $-\frac{13}{30}$ shares of stock and will have $\$\frac{232}{75}$ in the bank.

(i) If the first toss is heads, the broker's portfolio will be worth

$$\left(\frac{232}{75}\right)\left(\frac{5}{4}\right) - \left(\frac{13}{30}\right) 8 = .40$$

at $t = 1$. He should readjust his portfolio to be worthless at $t = 2$ if the second toss is heads and to have capital \$1 if the second toss is tails. The reader should verify that this can be accomplished by holding $-\frac{1}{12}$ shares of stock between $t = 1$ and $t = 2$ and having $\$\frac{16}{15}$ in the bank at $t = 1$.

(ii) If the first toss results in tails, the broker's portfolio will be worth

$$\left(\frac{232}{75}\right)\left(\frac{5}{4}\right) - \left(\frac{13}{30}\right) 2 = 3.$$

If the holder of the option chooses to exercise at $t = 1$, then the broker simply pays out \$3 and the transaction will be complete. If the option holder decides not to exercise at $t = 1$, then the broker must adjust his portfolio so that it will be worth \$1 at $t = 2$ if the second toss is heads and worth \$4 if the second toss is tails. However, the broker needs only \$2 at $t = 1$ in order to create the required capitals at $t = 2$. Consequently the broker can safely consume \$1 at $t = 2$.

□

We see from the example that hedging strategies for short positions on American options allow for the possibility of consumption if the holder of the option fails to exercise in an optimal fashion!

The mathematical analysis of American derivative securities presents a number of interesting challenges. The time at which the payoff will be made is not known initially. To complicate matters even more, if a broker sells the same option to more than one client, the clients may choose different exercise dates. In other words, the payment date depends not only on the result of the coin tosses, but also on the decision process (or exercise policy) adopted by the holder. In order to understand American options properly, it is very useful to first analyze a class of derivative securities in which the payment date is random, but for which all holders of a given security will receive exactly the same amount on precisely the same date (i.e., the holder of the security has no influence on the payoff.) We shall refer to such securities as *derivative securities with random maturity*. (There does not seem to be a standard name for the class of securities described above. Nevertheless, these securities will play a central role in our treatment of American options and it will be very convenient to have a name for them. The term "derivative securities with random maturity" is admittedly not ideal. An important point is that the randomness manifests itself only through the coin tossing and not through the holder of the security. If anyone has a better suggestion, please let me know.)

Derivative Securities with Random Maturity

We begin with an example.

Example 9.3: Consider a binomial model with $N = 3$, $u = 2$, $d = \frac{3}{4}$, $r = \frac{3}{8}$, and $S_0 = 8$. Let V denote a *rebate option* that pays \$1 at the first time the stock price hits or crosses the upper barrier $K = \$9$, and expires worthless at time 3 if the stock price is always below the barrier. More precisely:

(i) If $S_n(\omega) < 9$ for all $n = 0, 1, 2, 3$ then the holder of the option receives nothing.

(ii) If $\max\limits_{0\leq n\leq 3} S_n(\omega) \geq 9$ then the holder of the option receives \$1 at the smallest time n such that $S_n(\omega) \geq 9$.

Roughly speaking, by the *maturity* of such a security, we mean the smallest time at which it is known for sure that the holder of the security will not receive a payment at a future date. (A precise definition will be based on a type of random variable called a *stopping time*.) Here the maturity (which depends on ω) will be the smallest n such such that $S_n(\omega) = 9$, if such an n exists, and will be 3 if no such n exists.

We begin by recording the possible stock prices.

$$S_0 = 8,$$

$$S_1(H) = 16, \quad S_1(T) = 6,$$

$$S_2(H,H) = 32, \; S_2(H,T) = 12, \; S_2(T,H) = 12, \; S_2(T,T) = 4.5,$$

$$S_3(H,H,H) = 64,\ S_3(H,H,T) = 24,\ S_3(H,T,H) = 24,\ S_3(H,T,T) = 9,$$

$$S_3(T,H,H) = 24,\ S_3(T,H,T) = 9,\ S_3(T,T,H) = 9,\ S_3(T,T,T) = 3.375.$$

1. Suppose that the first toss is heads. Since $S_0 = 8 < 9$ and $S_1(H) = 16 > 9$, the holder of the option receives \$1 at time 1 and the options expires. (The holder is not entitled to any further payments from this option.)
2. Suppose that the first toss is tails. Since $S_0 = 8 < 9$ and $S_1 = 6 < 9$, the holder receives nothing at time 1 and must await the results of subsequent coin tosses.

   2.1: Suppose that the second toss is heads, i.e. (T,H) occurs. Then, since $S_0 < 9$, $S_1 < 9$, and $S_2 = 12 > 9$, the holder of the options receives \$1 at time 2 and the option expires.

   2.2: Suppose that the second toss is tails, i.e. (T,T) occurs. Then, since $S_0 < 9$, $S_1 < 9$, and $S_2 < 9$, the holder receives nothing at time 2 and must await the result of the third coin toss.

3. Suppose that the first two tosses are both tails. Then the following situations can occur at time 3.

   3.1: (T,T,H) Holder receives \$1 at time 3 and the option expires.

   3.2: (T,T,T) Holder receives nothing at time 3 and the option expires worthless.

We can summarize the option payment possibilities as follows.

$$V_1(H) = 1$$

$$V_2(T,H) = 1$$

$$V_3(T,T,H) = 1,\ V_3(T,T,T) = 0.$$

In order to determine the arbitrage-free price of the option at time 0, we shall try to find a replicating strategy X. An important observation arises here: The capital of a replicating strategy is not well defined at times that are (strictly) greater than the time at which the option matures, unless we make an assumption about what the option holder does with the payment.

In this example, we can expect to determine

$$X_0$$

$$\begin{array}{ll} X_1(H), & X_1(T) \\ X_2(T,H), & X_2(T,T), \\ X_3(T,T,H), & X_3(T,T,T). \end{array}$$

Notice that we must have

$$X_3(T,T,H) = 1, \quad X_3(T,T,T) = 0 \tag{1}$$

$$X_2(T,H) = 1, \tag{2}$$

$$X_1(H) = 1 \tag{3}$$

in order to reflect the option payments. We determine $X_2(T,T)$, $X_1(T)$, and X_0 by backward recursion. Observe that

$$\tilde{p} = \frac{1 + \frac{3}{8} - \frac{3}{4}}{2 - \frac{3}{4}} = \frac{1}{2}, \quad \tilde{q} = 1 - \tilde{p} = \frac{1}{2}.$$

We find that

$$X_2(T,T) = \frac{1}{1+r}\left[\tilde{p}X_3(T,T,H) + \tilde{q}X_3(T,T,T)\right] = \frac{8}{11}\left[\frac{1}{2}(1) + \frac{1}{2}(0)\right] = .3636 \tag{4}$$

$$X_1(T) = \frac{1}{1+r}\left[\tilde{p}X_2(T,H) + \tilde{q}X_2(T,T)\right] = \frac{8}{11}\left[\frac{1}{2}(1) + \frac{1}{2}(.3636)\right] = .4959 \tag{5}$$

$$X_0 = \frac{1}{1+r}\left[\tilde{p}X_1(H) + \tilde{q}X_1(T)\right] = \frac{8}{11}\left[\frac{1}{2}(1) + \frac{1}{2}(.4959)\right] = .5440. \tag{6}$$

It follows that the arbitrage-free price of the option is .5440.

Capitals such as $X_1(H,H)$ are not uniquely determined because we don't know how the \$1 payment received at time 1 will be invested (or if this payment will even be invested at all).

A very natural way to obtain values for $X_2(H,H)$, $X_2(H,T)$, $X_3(H,H,H)$, $X_3(H,H,T)$, $X_3(H,T,H)$, $X_3(H,T,T)$, $X_3(T,H,H)$, $X_3(T,H,T)$ is to assume that when a payment is received, it is put into the bank and left there until time 3. This leads to the following values:

$$X_2(H,H) = X_2(H,T) = \frac{11}{8}, \tag{7}$$

$$X_3(H,H,H) = X_3(H,H,T) = X_3(H,T,H) = X_3(H,T,T) = \left(\frac{11}{8}\right)^2, \tag{8}$$

$$X_3(T,H,H) = X_3(T,H,T) = \frac{11}{8}. \tag{9}$$

This gives us a self-financing strategy whose capitals at the option payment times are consistent with the option payments. However, it must be emphasized that there are numerous other strategies having this property. Only the capitals expressed by (1) through (6) are uniquely determined. □

We now formalize some ideas from Example 9.3. We need a mathematical way to express the time at which the securities matures. In other words we want a random variable τ such that $\tau(\omega)$ expresses the maturity of the option as a function of the outcome ω of the coin tosses. Such a random variable will be called a *stopping time* or a *stopping rule.*

Before giving a formal definition, let us write the stopping rule for Example 9.3. Let

$$A(\omega) = \{n = 0,1,2,3 : S_n(\omega) \geq 9\}.$$

If $A(\omega) = \emptyset$, we put $\tau(\omega) = 3$. If $A(\omega) \neq \emptyset$, we put

$$\tau(\omega) = \min\, A(\omega),$$

i.e. $\tau(\omega)$ is the smallest element of $A(\omega)$. Observe that the values of τ are given by

ω	$\tau(\omega)$
(H, H, H)	1
(H, H, T)	1
(H, T, H)	1
(H, T, T)	1
(T, H, H)	2
(T, H, T)	2
(T, T, H)	3
(T, T, T)	3

.

In the N-period binomial model, a stopping rule τ must take values in $\{0, 1, 2, \ldots, N\}$ and must have the property that whether or not $\tau(\omega) = n$ depends only on the outcome of the first n coin tosses, i.e. the decision on whether or not to stop at time n must be based solely on the information available at time n.

Definition 9.4: By a *stopping rule* or *stopping time* in the N-period binomial model we mean a random variable τ on Ω such that

(i) $\tau(\omega) \in \{0, 1, 2, \ldots, N\}$ for all $\omega \in \Omega$, and

(ii) For every $n \in \{0, 1, 2, \ldots, N\}$ and every $\omega_1, \omega_2, \ldots, \omega_N \in \{H, T\}$, if $\tau(\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_N) = n$ then

$$\tau(\omega_1, \omega_2, \ldots, \omega_n, \hat{\omega}_{n+1}, \ldots, \hat{\omega}_N) = n \quad \text{for all } \hat{\omega}_{n+1}, \hat{\omega}_{n+2}, \ldots, \hat{\omega}_N \in \{H, T\}.$$

Remark 9.5: Although the term "stopping time" is more commonly used in practice, we shall generally use the term "stopping rule", because it is really a procedure or rule used to determine the maturity of the option, rather than a single maturity time.

Remark 9.6: Observe that if there exist $\omega^* \in \Omega$ such that $\tau(\omega^*) = 0$, then $\tau(\omega) = 0$ for all $\omega \in \Omega$. This situation is, of course, not very interesting. Nevertheless, it is permitted by the definition.

In the N-period binomial model, a *derivative security with random maturity* is characterized by a stopping rule τ and a payment function $V_* : \Omega \to \mathbb{R}$. The holder of the security receives the amount $V_*(\omega)$ at time $\tau(\omega)$. It is assumed that V_* satisfies the following property: For every $n \in \{0, 1, \ldots, N\}$ and every $\omega_1, \omega_2, \ldots, \omega_N \in \{H, T\}$, if

$$\tau(\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_N) = n \tag{10}$$

then we have

$$V_*(\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_N) = V_*(\omega_1, \omega_2, \ldots, \omega_n, \hat{\omega}_{n+1}, \ldots, \hat{\omega}_N) \tag{11}$$

for all $\hat{\omega}_{n+1}, \hat{\omega}_{n+2}, \ldots, \hat{\omega}_N \in \{H, T\}$.

(In other words, the payment function cannot look into the future. Payments made at time n can depend only on the first n coin tosses.)

Before making some general observations about derivative securities with random maturity, let us describe the function V_* corresponding to the rebate option in Example 9.3. There is only one path for which the holder of the option receives nothing, namely (T, T, T). Along each other path the holder of the security receives \$1 (although for different paths, the payment is received at different times). We summarize the values of V_* and τ for Example 9.3 in the table below.

ω	$V_*(\omega)$	$\tau(\omega)$
(H, H, H)	1	1
(H, H, T)	1	1
(H, T, H)	1	1
(H, T, T)	1	1
(T, H, H)	1	2
(T, H, T)	1	2
(T, T, H)	1	3
(T, T, T)	0	3

Remark 9.7: Observe that the random variable τ defined by

$$\tau(\omega) = N \quad \text{for all } \omega \in \Omega$$

satisfies the conditions of a stopping rule. In this case, any random variable V_* on Ω satisfies the condition described by (10),(11). Consequently, a derivative security with (deterministic) maturity N is a special case of a derivative security with random maturity.

In order to write a replication algorithm for derivative securities with random maturity, it is convenient to define

$$V_{\tau(\omega)}(\omega) = V_*(\omega) \quad \text{for all } \omega \in \Omega. \tag{12}$$

By a replicating strategy for such a security we mean a (self-financing) strategy with capitals $(X_n)_{0 \le n \le N}$ such that

$$X_{\tau(\omega)}(\omega) = V_{\tau(\omega)}(\omega) \quad \text{for all } \omega \in \Omega. \tag{13}$$

In general, such a replicating strategy will not be unique. However, for a given security, all replicating strategies will have the same initial capital. The arbitrage-free price of the security at time 0 is defined to be the initial capital of any replicating strategy.

Proposition 9.8: Let V be a derivative security with random maturity described by the stopping rule τ and payment function V_* in the N-period binomial model. Then there is at least one replicating strategy. For $n \le \tau(\omega)$, the capital $X_n(\omega)$ is uniquely determined by the following backward induction algorithm:

(i) $X_N(\omega) = V_N(\omega) \quad \text{if } \tau(\omega) = N$

(ii) For each $n \in \{0, 1, \dots, N-1\}$,

$$X_n(\omega_1, \dots, \omega_n) = \begin{cases} V_n(\omega_1, \dots, \omega_n) \;\; \text{if } \tau(\omega) = n. \\ \\ \frac{1}{1+r}\left[\tilde{p} X_{n+1}(\omega_1, \dots, \omega_n, H) + \tilde{q} X_{n+1}(\omega_1, \dots, \omega_n, T)\right] \\ \\ \qquad\qquad \text{if } \tau(\omega) > n. \end{cases}$$

It is very useful to be able to express the initial capital X_0 in a replicating strategy (and hence the arbitrage-free initial price) as a risk-neutral expected value. We can do so by constructing a terminal capital $X_N : \Omega \to \mathbb{R}$ of a replicating strategy. As mentioned previously, terminal capitals generally are not unique; however, all terminal capitals of replicating strategies for a given derivative security with random maturity will have the same risk-neutral expected value (because the initial capital is uniquely determined).

Consider a derivative security with random maturity described by the stopping rule τ and the payment function V_*. For each path ω, we can think of taking the payment $V_*(\omega)$ and depositing it in the bank when it is received at time $\tau(\omega)$. The value of this deposit at time N will be $V_*(\omega)(1+r)^{N-\tau(\omega)}$. Therefore we define $X_N : \Omega \to \mathbb{R}$ by

$$X_N(\omega) = V_*(\omega)(1+r)^{N-\tau(\omega)} \tag{14}$$

for all $\omega \in \Omega$. It follows that the arbitrage-free price of the security is given by

$$V_0 = \frac{1}{(1+r)^N}\tilde{\mathbb{E}}[(V_*)(1+r)^{N-\tau}] = \tilde{\mathbb{E}}\left[\frac{V_*}{(1+r)^\tau}\right]. \tag{15}$$

Equation (15) will play a central role in pricing American derivative securities.

Example 9.9: Consider the binomial model and the rebate option described in Example 9.3. Observe that

$$\tilde{\mathbb{P}}(\omega) = \frac{1}{8} \quad \text{for all } \omega \in \Omega.$$

There are four paths ω along which the option pays \$1 at $t = 1$, two paths along which the option pays \$1 at $t = 2$, one path along which the option pays \$1 at $t = 3$, and one path along which the option pays nothing. Employing (15), we find that

$$V_0 = \frac{1}{8}\left[4\left(\frac{8}{11}\right) + 2\left(\frac{8}{11}\right)^2 + \left(\frac{8}{11}\right)^3\right] = .5440,$$

which agrees with the price determined in Example 9.3 using backward induction. □

General American Derivative Securities

An *American option*, or an *American derivative security*, in the N-period binomial model is characterized by its payment process $(G_n)_{0\le n\le N}$ which we assume to be a nonnegative adapted process ($G_n(\omega) \ge 0$ for all $\omega \in \Omega$ and all $n = 0, 1, \dots, N$). The random variable G_n is called the *intrinsic value* of the option at time n.

The holder of the option may choose any time $n \in \{0, 1, \ldots, N\}$ at which to exercise. If the holder exercises at time n, then she receives the amount $G_n(\omega)$ and the option expires. (Once the option has been exercised, the holder is not entitled to any future payments.) Of course, the decision of whether or not to exercise the option at time n must be based solely on the information that is available at time n, i.e. the outcomes of the first n coin tosses. In other words, the procedure used by the holder to determine the exercise time must be a stopping rule.

The most popular American options are calls and puts whose intrinsic values are given as follows.

1. Standard American call option with strike price K:

$$G_n(\omega) = (S_n(\omega) - K)^+.$$

2. Standard American put option with strike price K:

$$G_n(\omega) = (K - S_n(\omega))^+.$$

Two central questions concerning American options are:

1. How should the exercise time be chosen?
2. What is the arbitrage-free price of the option at time 0?

Observe that for a given stopping rule τ, an American option is the same to the holder as the derivative security with random maturity have stopping rule τ and payment function V_* given by

$$V_*(\omega) = G_{\tau(\omega)}(\omega) \quad \text{for all } \omega \in \Omega. \tag{16}$$

It is standard to use the notation G_τ to denote the random variable defined in (16). We denote by $\mathcal{P}_0(\tau)$ the corresponding arbitrage-free price of this security, i.e.

$$\mathcal{P}_0(\tau) = \tilde{\mathbb{E}}\left[\frac{G_\tau}{(1+r)^\tau}\right]. \tag{17}$$

(The reader should verify that the payment function in (**??**) satisfies the condition described by (10) and (11.)

An *optimal exercise policy*, or *optimal exercise rule*, for an American option is defined to be a stopping rule τ^* such that

$$\mathcal{P}_0(\tau) \leq \mathcal{P}_0(\tau^*) \quad \text{for all stopping rules } \tau.$$

In other words, an optimal exercise policy is a stopping rule that gives the largest possible initial value to the corresponding option with random maturity. The arbitrage-free price of the option is defined to be

$$\mathcal{P}_0^* = \mathcal{P}_0(\tau^*),$$

where τ^* is any optimal exercise policy.

Remark 9.10:

(i) Since the set of stopping rules is nonempty and finite, it is clear that an optimal exercise policy always exists.

(ii) It is not difficult to give examples in which there is more than one optimal exercise policy.

If we denote by $\mathcal{S}$ the collection of all stopping rules, we can express the arbitrage-free price by the formula

$$\mathcal{P}_0^* = \max_{\tau \in \mathcal{S}} \tilde{\mathbb{E}} \left[\frac{G_\tau}{(1+r)^\tau} \right]. \tag{18}$$

We do not want to call the arbitrage-free price V_0 yet because we want to reserve the symbol V_0 to denote the value at $t = 0$ of a process defined by the American backward recursion procedure. We shall then prove that $\mathcal{P}_0^* = V_0$.

Remark 9.11: We shall show later that if the stock does not pay dividends then $\tau^*(\omega) = N$ is always an optimal exercise policy for an American call option on the stock. In other words, within the context of the N-period binomial model there is no early exercise premium for an American call option. For this reason we will not devote much attention now to American calls.

It is generally not possible to replicate an American derivative security with a single (self-financing) strategy because different holders may use different exercise policies. It is possible to replicate an American derivative security together with a specified exercise policy. However, for different exercise policies, the initial capitals of the replicating strategies may be different. A broker who sells an American option and wants to hedge her short position must have enough capital to cover all possible exercise policies, i.e. she needs the maximum of the initial capitals of the replicating strategies corresponding to all exercise policies. This is the idea behind the definition of $\mathcal{P}_0^*$.

To understand the idea behind the American backward recursion algorithm, it is useful to think about Example 9.2. If we arrive at time N and the option has not yet been exercised, then clearly $V_N = G_N$. At a time $n \in \{0, 1, \cdots, N-1\}$, if the option has not yet been exercised and V_{n+1} has been determined then the value of the option at time n should be the maximum of the intrinsic value and the value at time n of a payment of V_{n+1} that will be received at time $n+1$.

Proposition 9.12: Consider an American option with payment process $(G_n)_{0 \leq n \leq N}$. Define the adapted process $(V_n)_{0 \leq n \leq N}$ by

(i) $V_N(\omega) = G_N(\omega)$

(ii) For $n < N$,

$$V_n(\omega) = \max\left\{G_n(\omega),\ \frac{1}{1+r}\left[\tilde{p}V_{n+1}(\omega_1, \ldots, \omega_n, H) + \tilde{q}V_{n+1}(\omega_1, \ldots, \omega_n, T)\right]\right\}$$

Then V_0 is the arbitrage-free price of the option at time 0 and the random variable τ^* defined by

$$\tau^*(\omega) = \min\left\{n = 0, 1, \ldots, N : G_n(\omega) = V_n(\omega)\right\}$$

is an optimal exercise policy.

Remark 9.13: It is important to understand that $(V_n)_{0 \leq n}$ need not be the capitals of a self-financing strategy. It is also important to observe

$$V_n = \max\left\{G_n, \frac{1}{1+r}\tilde{\mathbb{E}}_n\left[V_{n+1}\right]\right\} \geq \frac{1}{1+r}\tilde{\mathbb{E}}_n\left[V_{n+1}\right]$$

for all $n = 0, 1, \cdots, N-1$, which shows that the process $\left(\frac{V_n}{(1+r)^n}\right)_{0 \leq n \leq N}$ is a supermartingale under $\tilde{\mathbb{P}}$.

Before proving Proposition 9.12, we look at several examples. We begin with a 3-period version of Example 9.3.

Example 9.14: Consider the binomial model with $N = 3$, $u = 2$, $d = \frac{1}{2}$, $r = .25$, and $S_0 = 4$. An American put option on the stock with $K = 5$ has intrinsic value

$$G_n(\omega) = (5 - S_n(\omega))^+.$$

In particular, it is easy to check that

$$G_3(H,H,H) = G_3(H,H,T) = G_3(H,T,H) = G_3(T,H,H) = 0$$

$$G_3(H,T,T) = G_3(T,H,T) = G_3(T,T,H) = 3, \quad G_3(T,T,T) = 4.5,$$

so that

$$V_3(H,H,H) = V_3(H,H,T) = V_3(H,T,H) = V_3(T,H,H) = 0,$$

$$V_3(H,T,T) = V_3(T,H,T) = V_3(T,T,H) = 3, \ V_3(T,T,T) = 4.5.$$

We shall compute two values of V_2 in detail and then simply list the other relevant values of V in a table.

1. Suppose that (T,T) has occurred. The current stock price is $S_2(T,T) = 1$, so the intrinsic value of the option is $G_2(T,T) = 4$. Using the algorithm we find that

$$V_2(T,T) = \max\left\{4, \frac{4}{5}\left(\frac{1}{2}(3) + \frac{1}{2}(4.5)\right)\right\} = \max\{4,3\} = 4.$$

2. Suppose that (T,H) has occurred. The current stock price is $S_2(T,H) = 4$, so the intrinsic value of the option is $G_2(T,H) = 1$. Using the algorithm we find that

$$V_2(T,H) = \max\left\{1, \frac{4}{5}\left(\frac{1}{2}(0) + \frac{1}{2}(3)\right)\right\} = \max\{1, 1.20\} = 1.20.$$

We summarize all of the relevant G and V values in a table:

ω	$G_3(\omega)$	$V_3(\omega)$	$G_2(\omega)$	$V_2(\omega)$	$G_1(\omega)$	$V_1(\omega)$	$G_0(\omega)$	$V_0(\omega)$
(H,H,H)	0	0	0	0	0	.48	1	1.392
(H,H,T)	0	0	0	0	0	.48	1	1.392
(H,T,H)	0	0	1	1.20	0	.48	1	1.392
(H,T,T)	3	3	1	1.20	0	.48	1	1.392
(T,H,H)	0	0	1	1.20	3	3	1	1.392
(T,H,T)	3	3	1	1.20	3	3	1	1.392
(T,T,H)	3	3	4	4	3	3	1	1.392
(T,T,T)	4.5	4.5	4	4	3	3	1	1.392

.

We see that an optimal exercise policy is given by

ω	$\tau^*(\omega)$
(H,H,H)	2
(H,H,T)	2
(H,T,H)	3
(H,T,T)	3
(T,H,H)	1
(T,H,T)	1
(T,T,H)	1
(T,T,T)	1

. □

Next, we look at a 2-period example in which there is more than one optimal exercise policy.

Example 9.15: Consider a two-period binomial model with $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, and $S_0 = 4$. For each $n = 0, 1, 2$ and all $\omega \in \Omega$ let

$$M_n(\omega) = \max_{0 \le k \le n} S_k(\omega), \qquad L_n = \min_{0 \le k \le n} S_k(\omega).$$

Consider the American derivative security with intrinsic values given by

$$G_n(\omega) = M_n(\omega) - L_n(\omega).$$

The reader should check that

$$G_0 = 0, \quad G_1(H) = 4, \quad G_1(T) = 2,$$

$$G_2(H,H) = 12, \quad G_2(H,T) = 4, \quad G_2(T,H) = 2, \quad G_2(T,T) = 3.$$

We find that

$$V_1(H) = \max\left\{4, \frac{4}{5}\left[\frac{1}{2}(12) + \frac{1}{2}(4)\right]\right\} = \max\{4, 6.4\} = 6.4,$$

$$V_1(T) = \max\left\{2, \frac{4}{5}\left[\frac{1}{2}(2) + \frac{1}{2}(3)\right]\right\} = \max\{2, 2\} = 2,$$

$$V_0 = \max\left\{0, \frac{4}{5}\left[\frac{1}{2}(6.4) + \frac{1}{2}(2)\right]\right\} = \max\{0, 3.36\} = 3.36.$$

We see that an optimal exercise policy is given by

$$\tau^*(H,H) = 2, \quad \tau^*(H,T) = 2, \quad \tau^*(T,H) = 1, \quad \tau^*(T,T) = 1.$$

Another optimal exercise policy is given by the stopping rule $\tau^{**}(\omega) = 2$ for all $\omega \in \Omega$. This is because if the first coin toss results in tails, the time-1 value of the payoff that one would obtain by waiting until time 2 is the same as the intrinsic value. □

In the following simple, but important, example every stopping rule is an optimal exercise policy.

Example 9.16: In a general N-period binomial model, consider an American option with intrinsic values given by

$$G_n(\omega) = S_n(\omega)$$

for all $\omega \in \Omega$ and all $n = 0, 1, \cdots, n$. Intuitively,we should expect the arbitrage-free price of this option to be S_0 because this option gives us exactly the same financial benefit as owning a share of stock. Also, it seems clear intuitively that there can be no strategy for selling stock that would give a strictly higher value to the corresponding derivative security with random maturity than would any other stopping rule would give, because strategies cannot look into the future. (In particular, the reader should be convinced that a random variable τ that chooses a value of n corresponding to a time at which S_n is maximized is not a stopping rule.) Lemma 9.17 below can be used to verify that for the American derivative security of this example we have

$$\tilde{\mathbb{E}}\left[\frac{G_\tau}{(1+r)^\tau}\right] = S_0$$

for every stopping rule τ. This shows that every exercise policy is, in fact, optimal.
□

Lemma 9.17: Let $k \in \{0, 1, \cdots, N\}$ be given and let τ be a stopping rule with $\tau(\omega) \geq k$ for all $\omega \in \Omega$. Let $(M_n)_{0 \leq n \leq N}$ be an adapted process,

(i) If $(M_n)_{0 \leq n \leq N}$ is a martingale under $\tilde{\mathbb{P}}$ then $\tilde{\mathbb{E}}_k[M_\tau] = M_k$.

(ii) If $(M_n)_{0 \leq n \leq N}$ is a submartingale under $\tilde{\mathbb{P}}$ then $\tilde{\mathbb{E}}_k[M_\tau] \geq M_k$.

(iii) If $(M_n)_{0 \leq n \leq N}$ is a supermartingale under $\tilde{\mathbb{P}}$ then $\tilde{\mathbb{E}}_k[M_\tau] \leq M_k$.

We shall prove a result that is more general than Proposition 9.12. For this more general result, we need a notion of the arbitrage-free value of an American derivative security at each time n, assuming that exercise has not yet occurred. In the general N-period binomial model, we consider an American derivative security with intrinsic values given by the process $(G_n)_{0 \leq n \leq N}$. For each $n \in \{0, 1, \cdots, N\}$, let $\mathcal{S}_n$ denote the collection of all stopping rules τ satisfying $\tau(\omega) \geq n$ for all $\omega \in \Omega$. Notice that $\mathcal{S}_0 = \mathcal{S}$ and that

$$\mathcal{S}_{n+1} \subset \mathcal{S}_n \text{ for all } n = 0, 1, \cdots, N.$$

For each $n = 1, 2, \cdots, N$ we define

$$\mathcal{P}_n^* = \max_{\tau\in\mathcal{S}_n}\tilde{\mathbb{E}}_n\left[\frac{G_\tau}{(1+r)^{\tau-n}}\right]. \tag{19}$$

Notice that if we put $n = 0$ then formula (19) agrees with (18) since $\mathcal{S}_0 = \mathcal{S}$.

We shall now show that

$$\mathcal{P}_n^* = V_n \text{ for all } n = 0, 1, \cdots N,$$

$(V_n)_{0\leq n\leq N}$ is the process determined by the American backward induction algorithm described in Proposition 9.12. We have already observed that the discounted process

$$\left(\frac{V_n}{(1+r)^n}\right)_{0\leq n\leq N}$$

is a supermartingale under $\tilde{\mathbb{P}}$, and that $V_n \geq G_n$ for all $n = 0, 1, \cdots, N$. For future reference we summarize these two properties in a lemma.

Lemma 9.18: The process $(V_n)_{0\leq n\leq N}$ satisfies

(i) $V_n \geq G_n$ for all $n = 0, 1, \cdots, N$.

(ii) The discounted process $(V_n(1+r)^{-n})_{0\leq n\leq N}$ is a supermartingale under $\tilde{\mathbb{P}}$.

The next lemma indicates that the process $(\mathcal{P}_n^*)_{0\leq n\leq N}$ shares these properties.

Lemma 9.19: The process $(\mathcal{P}_n^*)_{0\leq n\leq N}$ satisfies

(i) $\mathcal{P}_n^* \geq G_n$ for all $n = 0, 1, \cdots, N$.

(ii) The discounted process $(\mathcal{P}_n^*(1+r)^{-n})_{0\leq n\leq N}$ is a supermartingale under $\tilde{\mathbb{P}}$.

Proof: To prove (i), we fix a value of $n \in \{0, 1, \cdots, N\}$ and define the stopping rule $\hat{\tau}$ by $\hat{\tau}(\omega) = n$ for all $\omega \in \Omega$. Notice that $\hat{\tau} \in \mathcal{S}_n$. It follows that

$$\mathcal{P}_n^* \geq \tilde{\mathbb{E}}_n\left[\frac{G_{\hat{\tau}}}{(1+r)^{\hat{\tau}-n}}\right] = \tilde{\mathbb{E}}_n[G_n] = G_n.$$

To prove (ii), we fix $n \in \{0, 1, \cdots, N-1\}$ and choose $\tau^* \in \mathcal{S}_{n+1}$ such that

$$\mathcal{P}^*_{n+1} = \tilde{\mathbb{E}}_{n+1}\left[\frac{G_{\tau^*}}{(1+r)^{\tau^*-n-1}}\right]. \tag{20}$$

Observe that $\tau^* \in \mathcal{S}_n$ which implies that

$$\mathcal{P}^*_n \geq \tilde{\mathbb{E}}_n\left[\frac{G_{\tau^*}}{(1+r)^{\tau^*-n}}\right]. \tag{21}$$

Using iterated conditioning and factoring $(1+r)$ out of the denominator, we have

$$\tilde{\mathbb{E}}_n\left[\frac{G_{\tau^*}}{(1+r)^{\tau^*-n}}\right] = \tilde{\mathbb{E}}_n\left[\frac{1}{1+r}\tilde{\mathbb{E}}_{n+1}\left[\frac{G_{\tau*}}{(1+r)^{\tau^*-n-1}}\right]\right]. \tag{22}$$

Combining (20), (21), and (22) we obtain

$$\mathcal{P}^*_n \geq \tilde{\mathbb{E}}_n\left[\frac{\mathcal{P}^*_{n+1}}{1+r}\right]. \tag{23}$$

The desired conclusion is obtained by dividing (23) by $(1+r)^n$. $\square$

Lemma 9.20: Let $(Y_n)_{0\leq n\leq N}$ be an adapted process and assume that (i) and (ii) below hold:

(i) $Y_n \geq G_n$ for all $n = 0, 1, \cdots, N$.

(ii) The discounted process $(Y_n(1+r)^{-n})_{0\leq n\leq N}$ is a supermartingale under $\tilde{\mathbb{P}}$.

Then $Y_n \geq \mathcal{P}^*_n$ for all $n = 0, 1, \cdots, N$.

Proof Let $n \in \{0, 1, \cdots, N\}$ and $\tau \in \mathcal{S}_n$ be given. Using (i), (ii), and Lemma 9.17, we have

$$\tilde{\mathbb{E}}_n\left[\frac{G_\tau}{(1+r)^\tau}\right] \leq \tilde{\mathbb{E}}_n\left[\frac{Y_\tau}{(1+r)^\tau}\right] \leq \frac{Y_n}{(1+r)^n}. \tag{24}$$

Taking the maximum over all $\tau \in \mathcal{S}_n$ and recalling (19), we arrive at

$$\mathcal{P}^*_n \leq Y_n. \quad \square$$

Combining Lemmas 9.18 and 9.20 we obtain

Lemma 9.21: $V_n \geq \mathcal{P}_n^*$ for all $n = 0, 1, \cdots, N$.

The next lemma will allow us to obtain the reverse inequality.

Lemma 9.22: Let $(W_n)_{0 \leq n \leq N}$ be an adapted process and assume that (i) and (ii) below hold:

(i) $W_n \geq G_n$ for all $n = 0, 1, \cdots, N$.

(ii) The discounted process $(W_n(1+r)^{-n})_{0 \leq n \leq N}$ is a supermartingale under $\tilde{\mathbb{P}}$.

Then $V_n \leq W_n$ for all $n = 0, 1, \cdots, N$.

Proof: We proceed by backward induction. Observe first that

$$V_N = G_N \leq W_N.$$

Let $n \in \{0, 1, \cdots, N-1\}$ be given and assume that $V_{n+1} \leq W_{n+1}$. We need to show that $V_n \leq W_n$. Since $V_{n+1} \leq W_{n+1}$ we conclude that

$$\tilde{\mathbb{E}}_n[V_{n+1}] \leq \tilde{\mathbb{E}}_n[W_{n+1}].$$

It follows from (ii) that

$$\frac{1}{1+r}\tilde{\mathbb{E}}_n[W_{n+1}] \leq W_n.$$

Finally, we observe that

$$V_n = \max\left\{G_n, \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}]\right\} \leq \max\left\{G_n, \frac{1}{1+r}\tilde{\mathbb{E}}_n[W_{n+1}]\right\} \leq \max\{G_n, W_n\} = W_n. \ \square$$

Combining Lemmas 9.19 and 9.22 we obtain

Lemma 9.23: $V_n \leq \mathcal{P}_n^*$ for all $n = 0, 1, \cdots, N$.

Combining Lemmas 9.21 and 9.23 we obtain

Theorem 9.24: $\mathcal{P}_n^* = V_n$ for all $n = 0, 1, \cdots$.

We now prove a result on optimal exercise.

Lemma 9.25: The random variable τ^* defined by

$$\tau^*(\omega) = \min\{n = 0, 1, \cdots, N : V_n(\omega) = G_n(\omega)\}$$

is an optimal exercise policy.

Proof: The reader is asked to verify that τ^* is a stopping rule. Define the process $(Y_n)_{0\le n\le N}$ by

$$Y_n(\omega) = \frac{V_{\tau^*(\omega)\wedge n}}{(1+r)^{\tau^*(\omega)\wedge n}},$$

where $k \wedge m = \min\{k, m\}$. Observe that $(Y_n)_{0\le n\le N}$ is an adapted process. We shall show that $(Y_n)_{0\le n\le N}$ is a martingale under $\tilde{\mathbb{P}}$. To this end, let $n \in \{0, 1, \cdots, N-1\}$ and $\hat{\omega}_1, \cdots, \hat{\omega}_n \in \{H, T\}$ be given. Let $\mathcal{A}$ denote the set of all continuations of the string $(\hat{\omega}_1, \cdots, \hat{\omega}_n)$, i.e.

$$\mathcal{A} = \{\omega \in \Omega : \omega_i = \hat{\omega}_i \text{ for all } i = 1, \cdots, n\}.$$

Since τ^* is a stopping rule, either (i) or (ii) below must hold:

(i) $\tau^*(\omega) \le n$ for all $\omega \in \mathcal{A}$.

(ii) $\tau^*(\omega) \ge n+1$ for all $\omega\ \mathcal{A}$.

If (i) holds then

$$Y_{n+1}(\hat{\omega}_1, \cdots, \hat{\omega}_n, H) = Y_{n+1}(\hat{\omega}_1, \cdots, \hat{\omega}_n, T) = Y_n(\hat{\omega}_1, \cdots, \hat{\omega}_n),$$

and clearly we have

$$Y_n(\hat{\omega}_1, \cdots, \hat{\omega}_n) = \tilde{\mathbb{E}}_n[Y_{n+1}](\hat{\omega}_1, \cdots, \hat{\omega}_n).$$

If (ii) holds then

$$Y_{n+1}(\hat{\omega}_1, \cdots, \hat{\omega}_n, H) = \frac{V_{n+1}(\hat{\omega}_1, \cdots, \hat{\omega}_n, H)}{(1+r)^{n+1}},$$

$$Y_{n+1}(\hat{\omega}_1, \cdots, \hat{\omega}_n, T) = \frac{V_{n+1}(\hat{\omega}_1, \cdots, \hat{\omega}_n, T)}{(1+r)^{n+1}},$$

and consequently we have

$$\tilde{\mathbb{E}}_n[Y_{n+1}](\hat{\omega}_1, \cdots, \hat{\omega_n}) = \frac{V_n(\hat{\omega}_1, \cdots, \hat{\omega}_n)}{(1+r)^n} = Y_n(\hat{\omega}_1, \cdots, \hat{\omega}_n),$$

which completes the proof that $(Y_n)_{0 \leq N}$ is a martingale under $\tilde{\mathbb{P}}$.

Notice that

$$Y_0 = V_0 = \mathcal{P}_0^*.$$

Furthermore, since $(Y_n)_{0 \leq n \leq N}$ is a martingale under $\tilde{\mathbb{P}}$, we have

$$Y_0 = \tilde{\mathbb{E}}[Y_N].$$

Using the definitions of τ^* and Y_N we see that

$$Y_N = \frac{V_{\tau^*}}{(1+r)^{\tau^*}} = \frac{G_{\tau^*}}{(1+r)^{\tau^*}}.$$

We conclude that

$$\mathcal{P}_0^* = Y_0 = \tilde{\mathbb{E}}[Y_N] = \tilde{\mathbb{E}}\left[\frac{G_{\tau^*}}{(1+r)^{\tau^*}}\right],$$

which shows that τ^* is optimal. $\square$

Remark 9.26: Consider an American option in which the intrinsic value at time n depends only on the stock price at time n, i.e.

$$G_n(\omega) = g_n(S_n(\omega))$$

for some functions $g_0, g_1, \ldots, g_N : \mathbb{R} \to \mathbb{R}$. It can be shown that the process $(V_n)_{0 \leq n \leq N}$ described in Proposition 9.12 can be determined by the following algorithm.

(i) $v_N(s) = g_N(s)$

(ii) For $n < N$, $v_n(s) = \max\ \left\{g_n(s),\ \frac{1}{1+r}\left[\tilde{p}v_{n+1}(us) + \tilde{q}v_{n+1}(ds)\right]\right\}$. The process $(V_n)_{0 \leq n \leq N}$ is given by

$$V_n(\omega) = v_n(S_n(\omega)).$$

This observation can streamline the computations very significantly when N is large.

American Call Options

In this section we prove a theorem which shows that there is no early exercise premium for an American call when $r \geq 0$. (As usual, we assume that the stock does not pay dividends.) In order to get a feel for why it is sometimes optimal to exercise American puts early, but not American calls, we look at an example involving options that are always in the money.

Example 9.27: Consider a general N-period binomial model with up factor u, down factor d, interest rate r and initial stock price S_0. Let K^p and K^c be given real numbers satisfying

$$K^p > S_0 u^N, \quad \text{and} \quad 0 < K^c < S_0 d^N.$$

Let us consider an American put with strike K^p and expiration N and an American call with strike K^c and expiration N. The inequalities we assumed regarding the strike prices ensure that both options will always be "in the money". The intrinsic values of the put and call are therefore given by

$$G_n^p = K^p - S_n, \qquad G_n^c = S_n - K^c.$$

When the put is exercised the holder receives K^p and pays the stock price. When the call is exercised, the holder receives the stock price and pays K^c. The value at time 0 of a security that pays $S_{\tau(\omega)}$ at time $\tau(\omega)$ is S_0 for every stopping rule τ. Consequently, as far as the put and call are concerned optimal exercise is determined by when it is optimal to receive K^p and when it is optimal to pay K^c. Notice that these amounts are independent of the exercise time. Assuming that $r > 0$, it is better to receive K^p as early as possible and to pay K^c as late as possible.

To cast this idea into formulas, let τ be a stopping rule. Then we have

$$\tilde{\mathbb{E}}\left[\frac{G_\tau^p}{(1+r)^\tau}\right] = K^p \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^\tau}\right] - S_0, \tag{25}$$

$$\tilde{\mathbb{E}}\left[\frac{G_\tau^c}{(1+r)^\tau}\right] = S_0 - K^c \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^\tau}\right]. \tag{26}$$

If $r \geq 0$ then clearly the quantity in (25) is maximized by taking $\tau(\omega) = 0$ and the quantity in (26) is maximized by taking $\tau(\omega) = N$ for all $\omega \in \Omega$. The reader is invited to think about what happens if $r < 0$ and if the strike prices depend on time. $\square$

Before giving the main result of this section, we state a lemma that will be used in the proof.

Lemma 9.28: Let τ be a stopping rule and assume that $(M_n)_{0\leq n\leq N}$ is a submartingale under $\tilde{\mathbb{P}}$. Then we have

$$\tilde{\mathbb{E}}[M_\tau] \leq \tilde{\mathbb{E}}[M_N].$$

Theorem 9.29: Assume that $r \geq 0$ and let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function with $g(0) = 0$. Let V_0^A denote the price at time 0 of the American derivative security having intrinsic values given by $G_n(\omega) = g(S_n(\omega))$ and let V_0^E denote the price at time 0 of the European derivative security that pays the amount $V_N^E(\omega) = g(S_N(\omega))$ at time N. Then we have

$$V_0^A = V_0^E.$$

Proof: Since g is convex we have

$$g(\lambda s) = g(\lambda s + (1-\lambda)0) \leq \lambda g(s) + (1-\lambda)g(0) \quad \text{for all } \lambda \in [0,1],\ s \in \mathbb{R}.$$

Since $g(0) = 0$, we conclude that

$$g(\lambda s) \leq \lambda g(s) \quad \text{for all } \lambda \in [0,1],\ s \in \mathbb{R}. \tag{27}$$

The idea will be to show that the process

$$\left(\frac{g(S_n)}{(1+r)^n}\right)_{0\leq n\leq N}$$

is a submartingale under $\tilde{\mathbb{P}}$ and apply Lemma 9.28.

Since

$$S_n = \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{1+r}\right],$$

we may write

$$g(S_n) = g\left(\tilde{\mathbb{E}}_n\left(\frac{S_{n+1}}{1+r}\right)\right). \tag{28}$$

Using Jensen's inequality and (27) (with $\lambda = (1+r)^{-1}$) in (28) we find that

$$g(S_n) \leq \tilde{\mathbb{E}}_n \left[\frac{g(S_{n+1})}{1+r} \right].$$

We conclude that

$$\left(\frac{g(S_n)}{(1+r)^n} \right)_{0 \leq n \leq N}$$

is a submartingale under $\tilde{\mathbb{P}}$.

Applying Lemma 9.28 we find that

$$\tilde{\mathbb{E}} \left[\frac{g(S_\tau)}{(1+r)^\tau} \right] \leq \tilde{\mathbb{E}} \left[\frac{g(S_N)}{(1+r)^N} \right],$$

for all stopping rules τ. Taking the maximum over all stopping rules we obtain the inequality

$$V_0^A \leq V_0^E.$$

Since we already know that $V_0^E \leq V_0^A$, the proof is complete. $\square$