- does <u>not</u> mean $R(A) = C(A)$, usually in fact $R(A) \neq C(A)$.

to future self: pp 15–20 first then 12–15 column space

<u>Null spaces $N(A)$ and $N(A^T)$</u>

- We just saw: given A $m \times n$: we have $C(A)$ (subspace of $\mathbb{R}^m$) and $C(A^T)$ (row space) (subspace of $\mathbb{R}^n$); dimensions agree (hence both $\leq \min\{m,n\}$)
- $C(A)$ sometimes called <u>range</u> of A:
  - for every $\vec{v} \in \mathbb{R}^n$, $A\vec{v}$ is linear combo of columns, i.e. $A\vec{v} \in C(A)$
  - conversely every combo of columns can be written $A\vec{v}$ for some $\vec{v} \in \mathbb{R}^n$,
  - hence $C(A) =$ outputs of map $\vec{v} \to A\vec{v}$

- We now consider the <u>null spaces</u> $N(A)$ of A and $N(A^T)$ of A^T.

- $N(A) = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \vec{0}\}$ (nonempty since $\vec{0} \in N(A)$)

  $N(A^T) = \{\vec{u} \in \mathbb{R}^m : A^T\vec{u} = \vec{0}\}$

- vectors $\vec{u}, \vec{v}$ over $\mathbb{R}$ are called <u>orthogonal</u> if $\vec{u}^T\vec{v} = 0$

  (if over $\mathbb{C}$ change to: $\vec{u}^*\vec{v} = 0$)

Notice: - if $\vec{v} \in N(A)$ then $A\vec{v} = \vec{0}$

$\Rightarrow \vec{r}_i^T \vec{v} = 0$ for every row $\vec{r}_i^T$ of A

- conversely if $r_i^T \vec{v} = 0$ for every row of A then $\vec{v} \in N(A)$ since $A\vec{v} = \vec{0}$.

- so: $\vec{v} \in N(A)$ iff $\vec{v}$ orthogonal to every row of A
- but in this case: $\vec{v}$ orthog to <u>every</u> $\vec{u}$ in row space of A (why?) (distributivity of mult.)
- we say: $\vec{v}$ is orthogonal to rowspace $C(A^T)$

- $\vec{v} \in N(A)$ was arbitrary; we say: $N(A)$ orthogonal to $C(A^T)$

- similarly $N(A^T)$ orthog. to $C(A)$ in some sense.

- we know $\dim(C(A)) = \dim(C(A^T))$

can also find dims of nullspace:

<u>Theorem</u>: (rank + nullity thm) if A is ~~[illegible]~~ $m \times n$ and $\underset{\text{"rank of A"}}{\dim(C(A)) = \dim(C(A^T))} = r$ then

$\underset{\text{"nullity of A"}}{\dim(N(A))} = n - r$

i.e. $\text{rank}(A) + \text{nullity}(A) = n$

Proof idea: - iteratively form orthogonal basis (i.e. vectors in basis pairwise orthogonal) $\{\vec{v}_1, \ldots, \vec{v}_r, \vec{v}_{r+1}, \ldots, \vec{v}_{r+k}\}$ of $\mathbb{R}^n$ (so $r+k=n$)
So that $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is a basis for rowspace $C(A^T)$
(we'll see how to form orthog basis later)
- then $\{\vec{v}_{r+1}, \ldots, \vec{v}_{r+k}\}$ is a basis for nullspace $N(A)$
(Why? See if you can prove) ✓

Summary: if A is $m \times n$ and $\dim C(A) = r$ and likewise $= \dim C(A^T)$ then $\dim N(A) = n-r$
$\dim N(A^T) = m - r$.

Ex if $A = 3\left\{\underbrace{\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 7 \end{bmatrix}}_{4}\right. = \overbrace{\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}^{C} \overbrace{\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}}^{R}$

$\operatorname{rank}(A) = 2$ so nullity $= \dim(N(A)) = 2$ (Why?)

notice $A\vec{u} = \vec{0}$ iff $R\vec{u} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 2x_2 - x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

$\Rightarrow x_4 = x_1 + 2x_2$
$x_3 = -2x_4 = -2x_1 - 4x_2$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -2x_1 - 4x_2 \\ x_1 + 2x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \\ 2 \end{bmatrix}$ form basis for $N(A)$

Important rank facts

1. $\text{rank}(AB) \leq \text{rank}(A), \text{rank}(B)$

   (Why? can you prove?)

2. $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

   (Why?)

3. $\text{rank}(A^T A) = \text{rank}(A A^T) = \text{rank}(A) = \text{rank}(A^T)$

   (HW).

AB as sum of rank 1 matrices:

- if A is $m \times n = \left[\vec{c}_1 \middle| \cdots \middle| \vec{c}_n \right]$

  B is $n \times k = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_n^{\,T} \end{bmatrix}$

we can take yet another "left to right" view of AB: now "column times row"

- $AB = \vec{c}_1 \vec{r}_1^{\,T} + \cdots + \vec{c}_n \vec{r}_n^{\,T}$
- each $\vec{c}_i \vec{r}_i^{\,T}$ is $(m \times 1) \times (1 \times k) = m \times k$
- observe: $\vec{c}_i \vec{r}_i^{\,T}$ is always rank 1 (if $\vec{c}_i, \vec{r}_i^{\,T} \neq \vec{0}$)

e.g. $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix} = \begin{bmatrix}1 & 1 & 2 & 0\\2 & 2 & 4 & 0\\3 & 3 & 6 & 0\\4 & 4 & 8 & 0\end{bmatrix}$

- So: $AB = \vec{c}_1\vec{r}_1^{\,T} + \cdots + \vec{c}_n\vec{r}_n^{\,T}$ writes AB as sum of rank 1 matrices.

- e.g. $\begin{bmatrix}1 & 1\\2 & 0\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}4 & 6\\2 & 4\\2 & 2\end{bmatrix}$

$$= \begin{bmatrix}1\\2\\-1\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2\\2 & 4\\-1 & -2\end{bmatrix} + \begin{bmatrix}3 & 4\\0 & 0\\3 & 4\end{bmatrix}$$

<u>exercise</u>: convince yourself this way of taking product AB always same as usual.

- we'll see decompositions into rank 1 sums frequently (e.g. with $A = LU$ and later when factoring symmetric matrices).

$A = LU$ for square matrices!

- Recall: system of n linear equations in n unknowns can be represented by matrix eq'n $A\vec{x} = \vec{b}$ where A is $n \times n$

- e.g.

$$\begin{aligned} x + y + z &= 1 \\ 2x + y - 3z &= 0 \\ -x + 2y - 2z &= 1 \end{aligned}$$

leads to

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{b}}$$

- one way to solve: Gaussian elimination on A (in this case w/o row exchanges)
  - ↳ let's forget about $\vec{b}$ for now, focus on A

- first step:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & -2 \end{bmatrix}}_{A} \xrightarrow[R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 3 & -1 \end{bmatrix}}_{\text{row equivalent } A'}$$

- in matrix terms, can write this as:

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}}_{\text{rank 1 matrix}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -5 \\ 0 & 3 & -1 \end{bmatrix}}_{\text{"2x2" matrix}}$$

- notice: needed first pivot nonzero for this to work $\begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix} = A$ nonzero

(or left column of zeroes)

- <u>now</u>: can pivot again and write:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -5 \\ 0 & 3 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -5 \\ 0 & 3 & 15 \end{bmatrix}}_{\text{rank 1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -16 \end{bmatrix}}_{\text{"1x1"}}$$

(again: needed nonzero pivot)

- so: overall:

already rank 1 can stop algorithm

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -5 \\ 0 & 3 & 15 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -16 \end{bmatrix}$$

= sum of rank 1 matrices

- but rank 1 matrices can be factored! column × row

i.e. $A = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \begin{bmatrix} 0 & -1 & -5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -16 \end{bmatrix}$

which, by our previous notes ~~factor~~ as:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -16 \end{bmatrix} = LU$$

"upper triangular"

"lower triangular"

- Note: factoring $A = LU$ only works when we have nonzero pivots at every stage of elimination (or get lucky w/ column of 0s when we have a 0 pivot)

- when it works L has 1's along diagonal
  U has the pivots along diag.

- there are variations on $A = LU$, but heart of process always same:
  elimination along a column of A can be viewed as writing
  $A =$ (rank 1 matrix) + (matrix w/ a row and column of 0's)

- after iterating we can factor rank 1 matrices as column × row then reassemble into a big matrix factorization for A.

Ex: if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 7 \\ 2 & 4 & 8 \end{bmatrix}$ then $A = LU$ fails since first pivot 0

Consider $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = "permutation matrix" flips R1 and R2

then $PA = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix}$ elimination on this matrix leads to no zero pivots so we can factor $PA = LU$

2e

Check:

$$PA = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 0 \\ 2 & 6 & 14 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 0 \\ 2 & 6 & 14 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} = LU \checkmark$$