(i)

Recap of last time

Def'n a real symmetric matrix $S = S^T$ is positive definite if all its eigenvalues are > 0

Equiv, S is p.d. iff

(Test 1) $\vec{x}^T S \vec{x} > 0$ for all $\vec{x} \in \mathbb{R}^n$
(proved)

(Test 2) $S = A^T A$ for a matrix A w/ independent columns
(proved)

(Test 3) All pivots of S in Gaussian elimination are > 0
(didn't prove)

Test 3 tells us: We can factor a p.d. S as $S = LU$ w/o row exchange.

We saw in example: U is "L^T up to pivots" (!) i.e. $U = DL^T$ where D is diagonal w/ pivots on its diagonal.

(ii)

Let's prove this happens in general:

- Start from $S = LU$

$\Rightarrow L^{-1} S (L^T)^{-1} = U (L^T)^{-1}$ (L, L^T are invertible — why?)

$\Rightarrow$ hence this is symmetric:

$[L^{-1} S (L^T)^{-1}]^T = [L^{-1} S (L^{-1})^T]^T$ ($(L^{-1})^T = (L^T)^{-1}$ — why?)

$= L^{-1} S^T (L^{-1})^T$

$= L^{-1} S (L^T)^{-1}$

same

$\Rightarrow$ hence $(L^T)^{-1}$ is upper triangular — by HW

Hence so is $U(L^T)^{-1}$

$\Rightarrow$ but this = a symmetric matrix. Only triangular symmetric matrices are diagonal.

$\Rightarrow U(L^T)^{-1} = D$ is diagonal

$\Rightarrow S = LDL^T$

Since LU factorization of S is unique so is $S = LDL^T$

(iii)

Hence if we form D by pulling out strictly positive pivots from U, it must be remaining factor w L^T.

✓

Once we have $S = LDL^T$ we get Cholesky factorization

$$S = L\sqrt{D}\,\sqrt{D}L^T = (L\sqrt{D})(L\sqrt{D})^T$$

which writes $S = A^T A$ w/ A upper triangular $(A = (L\sqrt{D})^T)$ or equiv $S = BB^T$ with B lower triang. $(B = L\sqrt{D})$

Our example was:

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix} = LU$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}}_{L^T}$$

$= L \quad D \quad L^T$

(iv)

$$= L \overset{\sqrt{D}}{\begin{bmatrix} \sqrt{2} & & \\ & \sqrt{3/2} & \\ & & \sqrt{4/3} \end{bmatrix}} \overset{\sqrt{D}}{\begin{bmatrix} \sqrt{2} & & \\ & \sqrt{3/2} & \\ & & \sqrt{4/3} \end{bmatrix}} L^T$$

$$= \underset{\substack{A^T \\ \uparrow \\ \text{lower}}}{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{1/2} & \sqrt{3/2} & 0 \\ 0 & -\sqrt{2/3} & \sqrt{4/3} \end{bmatrix}} \underset{\substack{A \\ \uparrow \\ \text{upper}}}{\begin{bmatrix} \sqrt{2} & -\sqrt{1/2} & 0 \\ 0 & \sqrt{3/2} & -\sqrt{2/3} \\ 0 & 0 & \sqrt{4/3} \end{bmatrix}}$$

(v)

Quadratic forms + Principal Axis Thm
— back to $S = QDQ^T$

Given $A \in \mathbb{R}^{n \times n}$ and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
we have

$$\vec{x}^T A \vec{x} = [x_1 \cdots x_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

check!

$$\begin{aligned} &= a_{11} x_1^2 + a_{12} x_1 x_2 + \cdots + a_{1n} x_1 x_n \\ &+ a_{21} x_2 x_1 + a_{22} x_2^2 + \cdots + a_{2n} x_2 x_n \\ &\vdots \\ &+ a_{n1} x_n x_1 + \cdots \cdots + a_{nn} x_n^2 \end{aligned}$$

<u>Def'n</u> A <u>quadratic form</u> (in n variables $x_1, \dots, x_n$) is a polynomial w/ all terms of degree 2:

$$\begin{aligned} P(x_1, \dots, x_n) = a_{11} x_1^2 + a_{12} x_1 x_2 + \cdots + a_{1n} x_1 x_n \\ + a_{22} x_2^2 + \cdots + a_{2n} x_2 x_n \\ \ddots \\ + a_{nn} x_n^2 \end{aligned}$$

<u>Notice</u>: by splitting cross terms $a_{ij} x_i x_j$ as $\frac{a_{ij}}{2} x_i x_j + \frac{a_{ij}}{2} x_j x_i$ we have:

(vi)

symmetric!

$$P = a_{11} x_1^2 + \frac{a_{12}}{2} x_1 x_2 + \cdots + \frac{a_{1n}}{2} x_1 x_n$$
$$+ \frac{a_{12}}{2} x_2 x_1 + a_{22} x_2^2 + \cdots + \frac{a_{2n}}{2} x_2 x_n$$
$$+$$
$$\vdots$$
$$+ \frac{a_{1n}}{2} x_n x_1 + \frac{a_{2n}}{2} x_n x_2 + \cdots + a_{nn} x_n^2$$

$$= [x_1 \cdots x_n] \, S \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}^T S \vec{x}$$

symmetric

So: every quadratic form P can be represented as $P = \vec{x}^T S \vec{x}$ for a symmetric S.

Def'n a diagonal form is a quadratic form w/ no cross terms:

$$P(x_1, \ldots, x_n) = c_1 x_1^2 + \cdots + c_n x_n^2$$
$$= \vec{x}^T D \vec{x}$$

where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$D = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{bmatrix}$$

(vii)

Given a (not necessarily diagonal) quadratic form $P(x_1, .., x_n) = P(\vec{x})$ we saw

$$P(\vec{x}) = \vec{x}^T S \vec{x} \quad \text{for some symmetric } S$$

By spectral theorem,

$$S = Q D Q^T$$

eigenvalues on diag.

columns are orthonormal eigenvectors of S.

So: $P = \vec{x}^T Q D Q^T \vec{x}$
$= (Q^T\vec{x})^T D (Q^T\vec{x})$
$= \vec{X}^T D \vec{X}$

where $\vec{X}$ is the column vector $Q^T\vec{x}$.
↳ can think of entries of this vector as new variables
↳ if we let X_i = ith coord of $Q^T\vec{x}$ we have, by above:

$$P = \lambda_1 X_1^2 + \cdots + \lambda_n X_n^2$$

Vague interp: spectral theorem allows us to write any quad. form as a diagonal form in an "orthonormal change of variables"