Positive Definite Matrices

* Standing assumption: $S = S^T$ is symmetric and real *

__Def'n__ S is __positive definite__ (p.d.) if all of its eigenvalues λ are strictly positive (__positive semidefinite__ (p.s.d.) iff all $\lambda \geq 0$)

↪ know $S = QDQ^T$ so given such a factorization can see if p.d. by checking if D's diagonal entries > 0.

__e.g.__: $\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$ is p.d.

$Q \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} Q^T$ is p.d. for any orthog Q.

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is p.s.d.

$\begin{bmatrix} 2 & 0 \\ 0 & -6 \end{bmatrix}$ is __not__ p.d.

Vague geometric intuition:
p.d. matrices only "stretch" vectors
do not reflect vectors

<u>Test 1</u>: S is p.d. iff $\vec{x}^T S \vec{x} > 0$ for all $\vec{x} \in \mathbb{R}^n$.

<u>Pf</u>: ($\Rightarrow$) if S is p.d. and $\vec{x} \in \mathbb{R}^n$:
can write $\vec{x}$ as linear combo of eigenvectors $\vec{x}_1, \dots, \vec{x}_n$ of S (poss. by spectral theorem — S is symmetric!) orthonormal!

$$\vec{x} = c_1 \vec{x}_1 + \dots + c_n \vec{x}_n$$

Then $\vec{x}^T S \vec{x} = (c_1 \vec{x}_1 + \dots + c_n \vec{x}_n)^T (\lambda_1 c_1 \vec{x}_1 + \dots + \lambda_n c_n \vec{x}_n)$

$$= \lambda_1 c_1^2 \|\vec{x}_1\|^2 + \dots + \lambda_n c_n^2 \|\vec{x}_n\|^2$$

all cross terms $c_i c_j \lambda_i \lambda_j \vec{x}_i^T \vec{x}_j = 0$

> 0 since $\lambda_1, \dots, \lambda_n > 0$ by p.d.-ness of S. ✓

($\Leftarrow$) if $\vec{x}^T S \vec{x} > 0$ for all $\vec{x}$;
Sps $\vec{x}_i$ is an eigenvector:

$$S\vec{x}_i = \lambda_i \vec{x}_i$$

then $\vec{x}_i^T S \vec{x}_i = \lambda_i \|x_i\|^2 > 0$ by hypothesis

$$\Rightarrow \lambda_i > 0 \checkmark$$

This test gives us an easy proof of the following:

<u>Fact</u>: if S_1, S_2 are p.d. then so is $S_1 + S_2$.

<u>Pf</u>: given $\vec{x} \in \mathbb{R}^n$:

$$\vec{x}^T (S_1 + S_2) \vec{x} = \underbrace{\vec{x}^T S_1 \vec{x}}_{>0} + \underbrace{\vec{x}^T S_2 \vec{x}}_{>0} \text{ by p.d.-ness of } S_1, S_2$$

$$> 0. \checkmark$$

<u>Test 2</u>: S is p.d. iff $S = A^T A$ for some A with independent columns.

<u>Pf</u>: ($\Rightarrow$) Sps S is p.d. Then (by spectral theorem) $S = QDQ^T$ where diagonal

entries of D are positive.

Let $A = Q\sqrt{D}\,Q^T$, where $\sqrt{D}$ is matrix obtained by taking $\sqrt{\ }$'s of all entries of D.

Then $A^T = A = Q\sqrt{D}\,Q^T$

and $A^T A = Q\sqrt{D}\,\underbrace{Q^T Q}_{I}\sqrt{D}\,Q^T$

$= QDQ^T = S$

of course: A has independent columns (why?)

($\Leftarrow$) Sps $A^T A = S$ and rank $A = n$

Let λ be an eigenvalue, $\vec{x}$ an associated eigenvector: $A^T A\vec{x} = \lambda\vec{x}$

Then: $\vec{x}^T(A^T A\vec{x}) = \vec{x}^T\lambda\vec{x} = \lambda\|\vec{x}\|^2$

$\| $

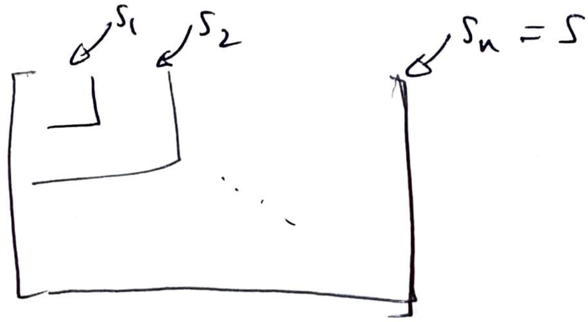
$(A\vec{x})^T(A\vec{x}) = \|A\vec{x}\|^2 > 0$

since $\vec{x} \neq 0$ and A has trivial nullspace

So all eigenvalues positive

$\lambda > 0$ ✓

- Given S, consider the "upper left square" matrices $S_1, \dots, S_n = S$

$S =$ 

- call the determinants of these matrices $D_1, \dots, D_n$

<u>Test 3</u>: S is p.d. iff $D_1, \dots, D_n > 0$.
↳ won't prove.

<u>One upshot</u>: the D_i's are connected to the pivots in Gaussian elimination:

<u>Fact</u>: kth pivot $= \dfrac{D_k}{D_{k-1}}$

So: if S is p.d. then in particular <u>all pivots</u> <u>are nonzero</u> (in fact >0)
↳ elimination succeeds w/o row exchange

and we can factor $S = LU$.

Moreover: if we factor as usual (1's on diagonal of L, pivots on diagonal of U) then L and U must be closely related — U is essentially L^T up to multiplication by the pivots which we can pull out as a diagonal matrix.

Proof by example:

Given $S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

we don't know S is p.d. — yet.

Let's try to factor: $S = LU$.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & \frac{2}{3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{2} & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

pull out pivots

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

L^T as if by magic

$$= L D L^T$$

↳ since pivots (diag entries in D) are > 0 can take $\sqrt{D} = \sqrt{D}^T$ and absorb left and right:

$$= (L\sqrt{D})\sqrt{D}^T L^T = (L\sqrt{D})(L\sqrt{D})^T$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{4/3} \end{bmatrix} \right) \left(\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{4/3} \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{1/2} & \sqrt{3/2} & 0 \\ 0 & -\sqrt{2/3} & \sqrt{4/3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{1/2} & 0 \\ 0 & \sqrt{3/2} & -\sqrt{2/3} \\ 0 & 0 & \sqrt{4/3} \end{bmatrix}$$

This writes S as $A^T A$ (with $A = (L\sqrt{D})^T$) where A has indep columns — so S is p.d.

But this factorization has a special form: A is upper triangular!

~~[illegible]~~ This procedure always works when S is p.d.