Another ex:

continuing after interlude...

$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

- to find $\sigma_1, \vec{v}_1$: maximize $\|A\vec{v}\|$ subject to $\|\vec{v}\| = 1$
- again we consider $\|A\vec{v}\|^2$, $\|\vec{v}\|^2 = 1$ instead.

$$\|A\vec{v}\|^2 = \left\| \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 3x \\ 4x+5y \end{bmatrix} \right\|^2$$

$$= 9x^2 + 16x^2 + 40xy + 25y^2$$

$$= 25x^2 + 25y^2 + 40xy$$

$$= 25(\underbrace{x^2+y^2}_{\|\vec{v}\|^2}) + 40xy$$

$$= F$$

- want to maximize F subject to $x^2+y^2=1$
- rewrite $F = 25 \cdot 1 + 40xy$ ← on constraint curve
$= 25 + 40xy$

this is a problem for... Lagrange multipliers!

let $g = x^2+y^2$

- Lagrange says: F can only be maximized when $\nabla F = \lambda \nabla g$

"Lagrange multiplier" – a constant, though not an eigenvalue i.e. $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = \lambda\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$

$$\left(40y,\ 40x\right) = \lambda\left(2x,\ 2y\right)$$

$$20y = \lambda x \longrightarrow \lambda = \frac{20y}{x}$$

$$20x = \lambda y \longrightarrow 20x = \frac{20y^2}{x} \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$$

since $x^2+y^2=1$, four possibilities

$$\pm\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},\quad \pm\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$$

by inspection $\|A\vec{v}\|$ maximized for

$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ ($-\vec{v}_1$ works too)

$A\vec{v}_1 =$

$$\begin{bmatrix}3 & 0\\4 & 5\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}3\\9\end{bmatrix} = \frac{3}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix}$$

$$= \frac{3\sqrt{10}}{\sqrt{2}}\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}$$

$$= \underbrace{3\sqrt{5}}_{\sigma_1}\underbrace{\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}}_{\vec{u}_1}$$

normalized so $\|\vec{u}_1\|=1$

to find $\vec{v}_2$: maximize $\|A\vec{v}\|$ subject to $\|\vec{v}\| = 1$ and $\vec{v} \perp \vec{v}_1$

only two possibilities:
$\pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

either one (necessarily) works for $\vec{v}_2$

say: $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

then $A\vec{v}_2 = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$= \frac{\sqrt{10}}{\sqrt{2}} \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$$

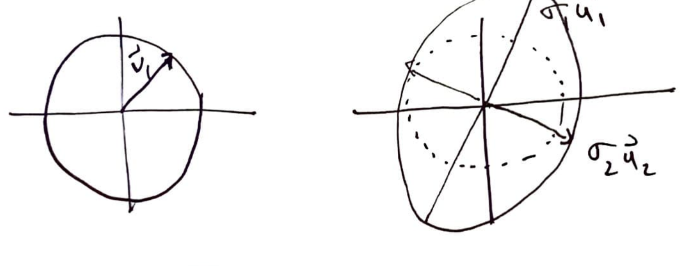
$$= \underbrace{\sqrt{5}}_{\sigma_2} \underbrace{\begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}}_{\vec{u}_2}$$

so $A \overbrace{V}^{[\vec{v}_1 | \vec{v}_2]} = U\Sigma$

where:

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

Hence $A = \underbrace{\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{V^T \ (=V)}$



Symmetric matrices

- Let S be a (real) $n \times n$ symmetric matrix $S = S^T$
- You proved: all eigenvalues of S real, all eigenvectors (can be chosen) real too.

Observe: if $S\vec{x} = \lambda \vec{x}$ and $\vec{y}^T \vec{x} = 0$,
then: $(S\vec{y})^T \vec{x} = \vec{y}^T S^T \vec{x} = \vec{y}^T S \vec{x}$
$= \vec{y}^T \lambda \vec{x} = \lambda \vec{y}^T \vec{x} = 0$

this shows:
↪ if $\vec{x}$ an eigenvector of S, then S maps $\{\vec{x}\}^\perp$ into $\{\vec{x}\}^\perp$!

- we've nearly proved the spectral theorem
- need one more fact:

Claim: S has n independent eigenvectors (i.e. a "full eigenspace")

Sketchy pf.: - S has at least one eigenvalue λ_1 (why?)
- if $\vec{x}_1$ is (real) eigenvector assoc. to λ_1, then we saw: S maps $\{\vec{x}_1\}^\perp$ into $\{\vec{x}_1\}^\perp$

- So we repeat on the $(n-1)$-dim'l vector space $\{\vec{x}\}^\perp$.

↳ S (viewed as lin. trans. of $\{x_1\}^\perp$) has at least one eigenvalue λ_2 (could $= \lambda_1$ or not) which is also eigenvalue of orig. transformation (hence real)

- can find assoc. (real) eigenvector $\vec{x}_2$ ← $\perp$ to $\vec{x}_1$

- repeat on $\{\vec{x}_1, \vec{x}_2\}^\perp$ ... ✓

Now we get

<u>Spectral theorem</u>: if $S = S^T$ is (real) symmetric, then $S = QDQ^T$ for some orthogonal Q and diagonal D.

<u>Pf</u>: - Let $\vec{u}_1, \dots, \vec{u}_n$ be orthonormal set of real eigenvectors for S

(how to get: find indep. orthogonal eigenvectors $\vec{x}_1, \dots, \vec{x}_n$ by above, normalize)

- Let $\lambda_1, \dots, \lambda_n$ be assoc. eigenvalues

- Let $Q = [\vec{u}_1 | \cdots | \vec{u}_n]$
  then $Q^{-1} = Q^T \rightarrow$ i.e. Q is orthogonal.
- Let $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$
- then $S = QDQ^T$ is eigenvalue decomposition of S. ✓