Finding eigenvectors / eigenvalues:

- One way: compute characteristic polynomial.
- $A\vec{x} = \lambda\vec{x}$ equiv. to $(A-\lambda I)\vec{x} = \vec{0}$

$\vec{x}$ eigenvector iff in nullspace of $A-\lambda I$

must have $\det(A-\lambda I) = 0$

nth degree poly in λ.

- Can use $\det(A-\lambda I) = 0$ to solve for λ (at most n distinct eigenvalues — at most n sol'ns)

Ex: if $A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{pmatrix}$$
$$= (8-\lambda)(7-\lambda) - 6$$
$$= 56 - 15\lambda + \lambda^2 -$$
$$= \lambda^2 - 15 + 50$$
$$= (\lambda-5)(\lambda-10)$$
$$= 0 \text{ if } \lambda = 5, 10$$

lucky: both real

if $\lambda = 5$

$$(A-5I)\vec{x} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$
$$\Rightarrow x_1 + x_2 = 0$$
$$\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

eigenvector

if $\lambda = 10$:

$$(A-10I)\vec{x} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$
$$\Rightarrow 2x_1 - 3x_2 = 0 \Rightarrow \begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix}$$
$$= x_1 \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ assoc. to $\lambda_1 = 5$ check: $\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (20)

$= \begin{pmatrix} 5 \\ -5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$ assoc. to $\lambda_2 = 10$

$\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 10 \\ \frac{20}{3} \end{pmatrix}$

$= 10\begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}$

✓

- Given a scalar s, $A + sI$ has same eigenvectors, shifted eigenvalues as A:

if $A\vec{x} = \lambda\vec{x}$ then $(A+sI)\vec{x} = A\vec{x} + s\vec{x}$

$= \lambda\vec{x} + s\vec{x}$

$= (\lambda + s)\vec{x}$

so $\vec{x}$ still eigenvector now assoc. to $\lambda + s$.

- if C is <u>similar</u> to A, i.e. $C = BAB^{-1}$ for some B, then C has same eigenvalues, transformed eigenvectors as A:

if $A\vec{x} = \lambda\vec{x}$

then for $\vec{x}' = B\vec{x}$

we have: $C\vec{x}' = BAB^{-1}(B\vec{x})$

$= BA\vec{x}$

$= B\lambda\vec{x} = \lambda(B\vec{x}) = \lambda\vec{x}'$

Useful Facts :

1) Sum of eigenvalues = trace of A
2) Product of eigenvalues = det(A)
3) triangular matrices have eigenvalues on diagonal.

SVD

~~useful def'n: Given a subspace $V \subseteq \mathbb{R}^n$ define the orthogonal complement~~

Notation $\vec{u} \perp \vec{v}$ means $\vec{u}$ is orthog to $\vec{v}$ i.e. $\vec{u}^T \vec{v} = 0$

Def'n Given a subspace $V \subseteq \mathbb{R}^n$, define the orthogonal complement of V:

$$V^\perp = \{\vec{u} \in \mathbb{R}^n \mid \vec{u} \perp \vec{v} \text{ for every } \vec{v} \in V\}$$

ez to check: $V^\perp$ is a subspace

and: if $\dim V = k$

then $\dim V^\perp = n - k$

why: form orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ so that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for V.

Then: $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is a basis for $V^\perp$.

Sps $A \in \mathbb{R}^{m \times n}$ (think of as linear map $\mathbb{R}^n \to \mathbb{R}^m$)

<u>Key fact 1</u>: it's possible to maximize $\|A\vec{v}\|$ over the n-sphere,

i.e. let $\sigma_1 = \sup_{\|\vec{v}\|=1} \|A\vec{v}\|$

<u>then</u>: σ_1 is realized, i.e. $\exists \vec{v}_1 \in \mathbb{R}^n$ s.t. $\|A\vec{v}_1\| = \sigma_1$ with $\|\vec{v}_1\| = 1$

first "singular value" of A

<u>why</u>: the function

$\vec{v} \to \|A\vec{v}\|$

is continuous and the n-sphere $\{\vec{v} \in \mathbb{R}^n : \|\vec{v}\| = 1\}$ is <u>compact</u>

$\hookrightarrow$ cts functions realize their extrema on compact domains

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since $\|A\vec{v}_1\| = \sigma_1$ we can find $\vec{u}_1 \in \mathbb{R}^m$ with $\|\vec{u}_1\| = 1$ s.t. $A\vec{v}_1 = \sigma_1 \vec{u}_1$