Homework #3

1. i. Spell out the details of the proof of the following fact (that we've used several times in class): if V is a subspace of $\mathbb{R}^n$ of dimension k, there is an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis of V.

   ii. Find a basis for $\mathbb{R}^3$ as above with V the subspace of $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\0\end{bmatrix}$.

2. (Trefethen and Bau 4.1) Determine the SVDs of the following matrices by hand:

$$\begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}, \begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}, \begin{bmatrix}0 & 2\\0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}.$$

3. (Trefethen and Bau 4.2) Suppose that A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper. Do A and B have the same singular values? Either prove the answer is yes or find a counterexample.

4. (Trefethen and Bau 4.4) Two square matrices $A, B \in \mathbb{R}^{n\times n}$ are called *unitarily equivalent* if $A = QBQ^T$ for some orthogonal matrix Q. For each statement below, either prove it or find a counterexample.

   i. If A and B are unitarily equivalent then they have the same singular values.

   ii. If A and B have the same singular values then they are unitarily equivalent.

5. (Trefethen and Bau 5.3) Consider the following matrix:

$$A = \begin{bmatrix}-2 & 11\\-10 & 5\end{bmatrix}.$$

   a. Find the SVD of $A = U\Sigma V^T$. The SVD is not (quite) unique, so find the one that has the minimal number of minus signs in U and V.

   b. Draw a labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A.

   c. Find A^{-1} — not directly, but using the SVD.

   d. Find the eigenvalues λ_1, λ_2 of A.

   e. Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.

   f. What is the area of the "ellipsoid" onto which A maps the unit ball of $\mathbb{R}^2$?

6. (Trefethen and Bau 5.4) Suppose that $A \in \mathbb{R}^{n\times n}$ has SVD $A = U\Sigma V^T$. Find the eigenvalues and associated eigenvectors of the following symmetric matrix:

$$\begin{bmatrix}0 & A^T\\A & 0\end{bmatrix}.$$

   Write down the eigenvalue decomposition (i.e. diagonalize the matrix).

   (*Hint*: By taking the transpose of $A = U\Sigma V^T$ we see: if $A\mathbf{v}_k = \sigma_k\mathbf{u}_k$ then $A^T\mathbf{u}_k = \sigma_k\mathbf{v}_k$. Use these equations to find the sought after eigenvectors.)

7. (Strang I.7.20) From $S = QDQ^T$ compute the positive definite symmetric square root $A = Q\sqrt{D}Q^T$ of the following matrix S.

$$S = \begin{bmatrix}5 & 4\\4 & 5\end{bmatrix}$$

8. (Strang I.7.22) Compute the Cholesky factorization $S = A^T A$ with $A = \sqrt{D}L^T$ for the following matrix:

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}.$$