Homework #2

1. Prove the following facts about matrix ranks:

   i. $\texttt{rank}(AB) \leq \texttt{rank}(A)$ and $\texttt{rank}(AB) \leq \texttt{rank}(B)$

   ii. $\texttt{rank}(A + B) \leq \texttt{rank}(A) + \texttt{rank}(B)$

   iii. $\texttt{rank}(A^T A) = \texttt{rank}(AA^T) = \texttt{rank}(A) = \texttt{rank}(A^T)$ (*Hint*: recall HW1 #3)

2. (Strang I.3.4) If $S = S^T$ is symmetric then of course $C(S) = C(S^T)$ (i.e. the column and row spaces of S coincide) and also $N(S) = N(S^T)$. Does the converse hold? That is, if A is square with $C(A) = C(A^T)$ and $N(A) = N(A^T)$, is A necessarily symmetric? Either prove the answer is yes or find a counterexample.

3. (Trefethen and Bau 2.1) Show that if a square matrix A is both triangular and orthogonal, then it is diagonal.

4. (Strang I.5.4) Suppose Q is $n \times n$ and orthogonal. Check that $||Q\mathbf{x}||^2 = ||\mathbf{x}||^2$ for any $\mathbf{x} \in \mathbb{R}^n$ (so orthogonal matrices don't change lengths of vectors). Check that in fact $(Q\mathbf{x})^T(Q\mathbf{y}) = \mathbf{x}^T\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ (so orthogonal matrices don't change angles between vectors).

5. (Trefethen and Bau 2.4) What are the possible eigenvalues of an orthogonal matrix Q? Remember, though its entries are real, Q may have complex eigenvalues.

6. Prove that eigenvectors associated to distinct eigenvalues of an orthogonal matrix Q are orthogonal. Remember, vectors $\mathbf{x}, \mathbf{y}$ with possibly complex entries are orthogonal when $\mathbf{x}^*\mathbf{y} = 0$, where $\mathbf{x}^* = \overline{\mathbf{x}}^T$ is the complex conjugate of the transpose of $\mathbf{x}$.

7. (Trefethen and Bau 2.3) Suppose that $S = S^T$ is a symmetric matrix (entries from $\mathbb{R}$).

   i. Prove that all eigenvalues of S are real. Conclude that for every eigenvalue we can find an associated eigenvector which is real.
   (*Hint*: use the identity $(AB)^* = B^*A^*$ and the symmetry of S to prove $\overline{\lambda}\mathbf{x}^*\mathbf{x} = \lambda\mathbf{x}^*\mathbf{x}$ for a given eigenvalue λ and an associated eigenvector $\mathbf{x}$. For the conclusion about real eigenvectors, it may help to use that $S\mathbf{x} = \lambda\mathbf{x}$ iff $(S - \lambda I)\mathbf{x} = \mathbf{0}$.)

   ii. Prove that eigenvectors associated to distinct eigenvalues of S are orthogonal.
   (*One way*: say why this holds when the first eigenvalue $\lambda_1 = 0$. In the general case consider the shifted matrix $S - \lambda_1 I$.)

8. (Strang I.6.22) Consider the following matrix:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

   Diagonalize A (i.e. factor $A = XDX^{-1}$ where D is diagonal) and use this diagonalization to find a formula for A^k.

9. (Strang I.6.12) The matrix A below is singular of rank one. Find three eigenvalues and three corresponding eigenvectors for A:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$