Assignment 15: Assigned Wed 12/03. Due never

1. (a) If $\int_{\mathbb{R}^d}(1+|x|)|f(x)|\,dx < \infty$, show that $\hat{f}$ is differentiable and $\partial_j\hat{f}(\xi) = -2\pi i(x_j f(x))^\wedge(\xi)$. [NOTE: $(x_j f(x))^\wedge(\xi)$ means $\hat{g}(\xi)$, where $g(x) = x_j f(x)$.]
   (b) If $f \in C_0^1(\mathbb{R}^d)$, and $\nabla f \in L^1$ show that $(\partial_j f)^\wedge(\xi) = +2\pi i\xi_j\hat{f}(\xi)$.
   (c) Show that the mapping $f \mapsto \hat{f}$ is a bijection in the Schwartz space.
2. If μ is a finite Borel measure on $\mathbb{R}^d$ define $\hat{\mu}(\xi) = \int e^{-2\pi i\langle x,\xi\rangle}\,d\mu(x)$. If $\hat{\mu}(\xi) = 0$ for all ξ, show that $\mu = 0$. [HINT: Show that $\int f\,d\mu = 0$ for all $f \in \mathcal{S}$.]
3. For $f \in L^1$, the formula $\hat{f}(\xi) = \int f(x)e^{-2\pi i\langle x,\xi\rangle}$ allows us to prove many identities: E.g. $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$, etc. For $f \in L^2$, the formula $\hat{f}(\xi) = \int f(x)e^{-2\pi i\langle x,\xi\rangle}$ is no longer valid, as the integral is not defined (in the Lebesgue sense). However, most identites remain valid, and can be proved using an approximation argument. I list a couple here.
   (a) For $f \in L^1$ we know $(\tau_x f)^\wedge(\xi) = e^{-2\pi i\langle x,\xi\rangle}\hat{f}(\xi)$. Prove it for $f \in L^2$.
   (b) Similarly, show that $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$ for all $f \in L^2$.
   (c) Let F denote the Fourier transform operator (i.e. $Ff = \hat{f}$), and R denote the reflection operator (i.e. $Rf(x) = f(-x)$). Note that our Fourier inversion formula (for $f \in L^1$, $\hat{f} \in L^1$) is exactly equivalent to saying $F^2 f = Rf$. Prove $F^2 f = Rf$ for all $f \in L^2$.
4. *(Uncertainty principle)* Suppose $f \in \mathcal{S}(\mathbb{R})$. Show that
$$\Big(\int_{\mathbb{R}}|xf(x)|^2\,dx\Big)\Big(\int_{\mathbb{R}}|\xi\hat{f}(\xi)|^2\,d\xi\Big) \geqslant \frac{1}{16\pi^2}\|f\|_{L^2}^2\|\hat{f}\|_{L^2}^2$$
   [This illustrates a nice localisation principle about the Fourier transform. The first integral measures the spread of the function f. The second the spread of the Fourier transform $\hat{f}$. Here you show that this product is bounded below! The proof, once you know enough Physics, reduces to the above inequality.
   Hint: Consider $\int_{\mathbb{R}} xf(x)f'(x)\,dx$.]
5. *(Central limit theorem)* Let $f \in L^1(\mathbb{R})$ be such that $f \geqslant 0$ and $\int x^2 f(x)\,dx < \infty$. Define $g_n = (f * \cdots * f)$ (n-times), and $h_n(x) = \delta_{1/\sqrt{n}}g_n(x) = \sqrt{n}g_n(\sqrt{n}x)$. Show
$$\hat{h}_n(\xi) \xrightarrow{n\to\infty} \exp\big(-2\pi i\mu\xi - 2\pi^2 i\sigma^2\xi^2\big),$$
   where $\mu = \int xf(x)\,dx$ and $\sigma^2 = \int(x-\mu)^2 f(x)\,dx$. [The central limit theorem says that tabulating results of a large number of independent trials of any experiment produces a "bell curve". The key step in the proof, which you will no doubt see next semester, is showing that any function convoved with itself often enough looks like a Gaussian.]
6. *(Sobolev spaces)* For $f \in L^2(\mathbb{R}^d)$ and $s \geqslant 0$ define
$$\|f\|_{H^s}^2 = \int(1+|\xi|^s)^2|\hat{f}(\xi)|^2\,d\xi, \qquad \text{and} \qquad H^s = \{f \in L^2 \mid \|f\|_{H^s} < \infty\}.$$
   Intuitively, we think of H^s as the space of functions with "s" "weak-derivatives" in L^2. (This will be formalized in your functional analysis course.)
   (a) If $f \in C_0^n(\mathbb{R}^d)$ and $D^\alpha f \in L^2$ for all $|\alpha| < n$, then show that $f \in H^n(\mathbb{R}^d)$.
   (b) Let $s \in (0,1)$ and $f \in L^2(\mathbb{R}^d)$. Show that $f \in H^s(\mathbb{R}^d)$ if and only if
$$\int_0^\infty \Big(\frac{\|\tau_h f - f\|_{L^2}}{h^s}\Big)^2\frac{dh}{h} < \infty.$$
7. *(Sobolev embedding)* If $n \in \mathbb{N}$ and $f \in H^s(\mathbb{R}^d)$ for $s > n + \frac{d}{2}$ then show that $f \in C^n$, and further the inclusion map $H^s \to C^n$ is continuous.
8. (a) *(Elliptic regularity)* Let $Lu = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu$, where a_{ij}, b_i, c are constants. Suppose $\exists\lambda > 0$ such that $a_{ij} = a_{ji}$ and $|\sum a_{ij}\xi_i\xi_j| \geqslant \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$ (this assumption is called ellipticity). If $f\mathcal{S}$ and $u, \partial_i u, \partial_i\partial_j u \in L^2 \cap C^0$ are such that $Lu = f$, show that $u \in C^\infty$. [To emphasize why this is surprising, choose for example $L = \Delta$. Then $\Delta u = f$ makes no mention of a mixed derivative $\partial_1\partial_2 u$. Yet, all such mixed derivatives exist and are smooth. Hint: If $f \in H^s$ show that $u \in H^{s+2}$.]
   (b) Show by example that the previous subpart is false without the ellipticity assumption.
9. *(Trace theorems)* Let $p \in \mathbb{R}^m$ be fixed. Given $f : \mathbb{R}^{m+n} \to \mathbb{R}$ define $S_p f : \mathbb{R}^n \to \mathbb{R}$ by $S_p f(y) = f(p, y)$.
   (a) Let $s > m/2$, and $s' = s - m/2$. Show that there exists a constant c such that $\|S_p f\|_{H^{s'}(\mathbb{R}^n)} \leqslant c\|f\|_{H^s(\mathbb{R}^{m+n})}$.
   (b) Show that the section operator S_p extends to a continuous linear operator from $H^s(\mathbb{R}^{m+n})$ to $H^{s'}(\mathbb{R}^n)$. [Given an arbitrary L^2 function on $\mathbb{R}^{m+n}$ it is of course impossible to restrict it to an m-dimensional hyper-plane. However, if your function has more than $n/2$ "Sobolev derivatives", then you can make sense of this restriction, and the restriction still has $s - n/2$ "Sobolev derivatives".]
10. *(Reliech Lemma)* Let $K \subset \mathbb{R}^d$ be compact, $0 \leqslant s_1 < s_2$, and suppose $\{f_n\}$ are a sequence of functions supported in K. If the sequence $\{f_n\}$ is bounded in H^{s_2}, then show that it has a convergent subsequence in H^{s_1}. [This is the generalization of the Arzella-Ascolli theorem in this context.]