Coercivity and stability results for an extended Navier-Stokes system

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In this paper, we study a system of equations that is known to *extend* Navier-Stokes dynamics in a well-posed manner to velocity fields that are not necessarily divergence-free. Our aim is to contribute to an understanding of the role of divergence and pressure in developing energy estimates capable of both controlling the nonlinear terms, and being useful at the time-discrete level. We address questions of global existence and stability in bounded domains with no-slip boundary conditions. Through use of new H^1 coercivity estimates for the linear equations, we establish a number of global existence and stability results, including results for small divergence and a time-discrete scheme. We also prove global existence in 2D for any initial data, provided sufficient divergence damping is included. © *2012 American Institute of Physics.* [http://dx.doi.org/10.1063/1.4738637]

Dedicated to Peter Constantin, on the occasion of his 60th birthday.

I. INTRODUCTION

The zero-divergence constraint and the associated pressure field are the source of both difficulties and benefits in the study of the Navier-Stokes equations for the flow of viscous incompressible fluids. On one hand, the divergence constraint complicates analysis and approximation in a number of ways. For example, it produces a well-known inf-sup compatibility condition for mixed approximations that makes it difficult to achieve high accuracy with simple kinds of discretization. On the other hand, the incompressibility constraint is responsible for the energy inequality, an estimate which is fundamental to global existence theory.

In this paper, we study global existence and stability questions for a non-degenerate parabolic system that is known to *extend* Navier-Stokes dynamics in a well-posed manner to velocity fields that are not necessarily divergence-free. This system appeared recently in Ref. 10, and begins to explain the good performance of certain numerical schemes where the pressure is computed by solving boundary-value problems.[12] The idea to determine pressure by solving boundary-value problems was also a feature of an earlier analytical study by Grubb and Solonnikov,[5,6] and the system we consider is equivalent to one of their several "reduced" models.

Explicitly, we study the initial-boundary value problem

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\nabla p = (I - P)(\Delta u - \nabla\nabla \cdot u - u \cdot \nabla u), \tag{1.2}$$

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$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.3}$$

$$u = u_0 \qquad \text{in } \Omega, \text{ when } t = 0. \tag{1.4}$$

Here, $u = u(x, t)$ is the velocity field, $p = p(x, t)$ the pressure, and P is the standard Leray projection of $L^2(\Omega, \mathbb{R}^d)$ onto the subspace of divergence-free vector fields which are tangential at the boundary. For simplicity, we have taken the kinematic viscosity to be unity and omitted body forces.

For the system (1.1)–(1.4), neither the initial data nor the solution are required to be divergence free. Equation (1.2) *defines* the pressure gradient, and replaces the incompressibility constraint

$$\nabla \cdot u = 0 \qquad \text{in } \Omega \tag{1.5}$$

that appears in the standard incompressible Navier-Stokes system. However, if initially $\nabla \cdot u_0 = 0$, then the incompressibility constraint (1.5) holds for all time. This follows because (1.1)–(1.3) show that $\nabla \cdot u$ satisfies the heat equation with no-flux boundary conditions

$$\begin{cases} \partial_t \nabla \cdot u = \Delta \nabla \cdot u \text{ in } \Omega, \\ \dfrac{\partial}{\partial \nu} \nabla \cdot u = 0 \qquad \text{for } x \in \partial\Omega, t > 0, \end{cases} \tag{1.6}$$

where $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outward unit normal to $\partial\Omega$.[15] Thus, if $\nabla \cdot u_0 = 0$, then the system (1.1)–(1.4) reduces to the standard incompressible Navier-Stokes equations, and in this sense we say that the system (1.1)–(1.4) extends the dynamics of the standard incompressible Navier-Stokes equations.

Of course, the dynamics of the standard incompressible Navier-Stokes equations could alternately be extended by completely omitting the $\nabla\nabla \cdot u$ term from (1.2). However, the presence of this term is crucial to the theory for two reasons. First, the diffusion term and Neumann boundary condition for $\nabla \cdot u$ in (1.6) are a direct result of the $\nabla\nabla \cdot u$ term in (1.2), providing exponential stability of the divergence-free subspace. Thus, from a numerical perspective, errors in the divergence should be exponentially damped.

The second and perhaps deeper reason is the essential role played by $\nabla\nabla \cdot u$ in the well-posedness results of Refs. 10, 5, and 6. Define the *Stokes pressure gradient*, $\nabla p_s(u)$, by

$$\nabla p_s(u) = (I - P)(\Delta u - \nabla\nabla \cdot u). \tag{1.7}$$

In context we often use p_s to denote $p_s(u)$. Given any $u \in H^2(\Omega, \mathbb{R}^d)$, the function $p_s(u)$ is determined as the unique mean-zero solution to the boundary-value problem

$$\Delta p_s = 0 \quad \text{in } \Omega, \qquad \nu \cdot \nabla p_s = \nu \cdot (\Delta - \nabla\nabla\cdot)u \quad \text{on } \partial\Omega. \tag{1.8}$$

Without the $\nabla\nabla \cdot u$ term, the boundary condition in (1.8) would not make sense for all $u \in H^2(\Omega)$. With the $\nabla\nabla \cdot u$ term, however, $\Delta u - \nabla\nabla \cdot u$ is L^2 and divergence-free; hence a standard trace theorem [Proposition 1.4 of Ref. 3.] makes sense of the boundary condition in $H^{-1/2}(\partial\Omega)$. The Grubb-Solonnikov[5,6] approach is based on using the boundary-value problem (1.8) to determine the contribution of $\nabla p_s(u)$ to ∇p, and proves well-posedness of (1.1)–(1.4) using a theory of parabolic pseudo-differential initial-boundary value problems in L^p-based Sobolev spaces.

The convergence arguments of Liu *et al.*,[10] on the other hand, result in a comparatively simple local well-posedness proof for (1.1)–(1.4) for initial velocity in $H^1_0(\Omega)$. This proof is based instead on the expression of the Stokes pressure gradient as a Laplace-Leray commutator

$$\nabla p_s = (\Delta P - P\Delta)u. \tag{1.9}$$

This follows directly from (1.7) using the fact that

$$\nabla\nabla \cdot u = \Delta(I - P)u. \tag{1.10}$$

Even in this approach, the "extra" $\nabla\nabla \cdot u$ term in (1.7) is directly responsible for the commutator representation (1.9). The key idea used in Ref. 10 is to treat the Stokes pressure gradient as the Laplace-Leray commutator (1.9), and show (Theorem 3.1, below) that it is dominated by the Laplacian (cf. (3.1)) to leading order.[16]

While the methods of Refs. 5, 6, and 10 effectively address local well-posedness of (1.1)–(1.4), they do not address global existence or stability. For the standard incompressible Navier-Stokes equations in three-space dimensions, global existence of strong solutions is a well-known fundamental open problem.[2,4] However, classical results establish global existence and regularity if the flow is two-dimensional,[9] or the initial data are suitably small.[3,9,13]

In this paper, we establish a few such global existence results for the system (1.1)–(1.4). The main difficulty in proving a small-data global existence result for (1.1)–(1.4) is not the nonlinearity. The root of the problem is that the *linear* terms are not coercive under the standard L^2 inner product. We remedy this difficulty by using the commutator estimate in Ref. 10 to construct an adjusted inner product under which the linear terms are coercive. This allows us to establish global existence for small initial data in two or three dimensions, and unconditional global stability of a time discrete scheme for the linear equations. This leads to an improved understanding of how the divergence and pressure can be handled to obtain energy estimates capable of controlling the nonlinear terms.

In 2D, we can extend our small-data global existence results to initial data with small divergence. For arbitrary initial data, we can add a sufficiently large divergence damping term to (1.1)–(1.4) to obtain global existence. However, presently we are not able to prove global existence for (1.1)–(1.4) for arbitrary initial data. The difficulty is that for the energy balance using the standard L^2 inner-product, the nonlinear term is skew-symmetric, and does not contribute; however, the linear terms are not coercive. On the other hand, for the energy balance using the adjusted inner products we consider, the linear terms are coercive; however, nonlinearity is no longer skew symmetric, and contributes non-trivially.

Coercivity of the linear terms (albeit under a non-standard inner product) allows one to treat (1.1)–(1.4) as a non-degenerate parabolic system. While this has helped simplify existence theory and the analysis of certain numerical approximation schemes, some other questions apparently become more difficult. In particular, while global existence of the standard incompressible Navier-Stokes equations is well known in 2D, the techniques here do not easily show 2D global existence for (1.1)–(1.4) for general initial data.

II. MAIN RESULTS

A. Coercivity of the extended Stokes operator

In the study of parabolic problems, an extremely useful (and often crucial) property is coercivity of the underlying linear operator. For (1.1), the linear operator in question is the *extended Stokes operator*, A, defined by

$$Au \stackrel{\text{def}}{=} -\Delta u + \nabla p_s(u) = -P\Delta u - \nabla\nabla\cdot u. \tag{2.1}$$

Note that the last equality follows from the identity (1.10). Under periodic boundary conditions, the extended Stokes operator A is coercive. Indeed, under periodic boundary conditions, $P\Delta = \Delta P$, and so

$$\langle u, Au\rangle = \|\nabla u\|_{L^2}^2, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard L^2 inner product on the torus.

Under no-slip (0-Dirichlet) boundary conditions, the situation is surprisingly more complicated. The extended Stokes operator *fails* to be positive, let alone coercive, under the standard L^2 inner product. To briefly explain why, observe that for $u \in H^2 \cap H_0^1(\Omega, \mathbb{R}^d)$,

$$\langle u, Au\rangle = \int_\Omega u\cdot Au = \int_\Omega |\nabla u|^2 + \int_\Omega u\cdot\nabla p_s. \tag{2.3}$$

Now if $\nabla\cdot u \neq 0$, the second term on the right need not vanish. In view of the commutator relation (1.9), one might expect $\|\nabla p_s\|_{L^2}$ to be dominated by $\|\nabla u\|_{L^2}$. This, however, is known to be false, and control of the Stokes pressure p_s requires *more* than one derivative on u. Consequently, if $\nabla\cdot u \neq 0$, then the second term on the right of (2.3) can dominate the first, and destroy positivity of A.

Since our primary interest in the extended Stokes operator is to study (1.1)–(1.4), and the divergence of solutions to (1.1)–(1.4) is well controlled, one may hope to rectify non-positivity of A by a coercivity estimate of the form

$$\langle u, Au\rangle \geqslant \varepsilon\|\nabla u\|_{L^2}^2 - C\|\nabla\cdot u\|_{L^2}^2. \tag{2.4}$$

But again, this turns out to be false.

Proposition 2.1 (Failure of Coercivity): Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected C^3 domain. For any ε, $C \geq 0$, there exists a function $u \in C^2(\bar\Omega)$ such that

$$u = 0 \text{ on } \partial\Omega, \qquad \text{and} \qquad \langle u, Au\rangle \leqslant \varepsilon\|\nabla u\|_{L^2}^2 - C\|\nabla\cdot u\|_{L^2}^2. \tag{2.5}$$

The key idea in the proof is to identify the *harmonic conjugate of the Stokes pressure as the harmonic extension of the vorticity*. Since this is independent of our main focus, we present the proof of Proposition 2.1 in the Appendix, towards the end of this paper. We remark, however, that if $u \in H^2 \cap H$, where

$$H \stackrel{\text{def}}{=} \{v \in L^2(\Omega) | v = Pv\} = \{v \in L^2(\Omega) | \nabla\cdot v = 0 \text{ in } \Omega, \text{ and } v\cdot\nu = 0 \text{ on } \partial\Omega\},$$

then the second equality in (2.1) shows that the extended Stokes operator A reduces to the standard Stokes operator $-P\Delta$. In the space $H^2 \cap H_0^1 \cap H$ coercivity of the standard Stokes operator is well known. Namely, (2.2) holds for all $u \in H^2 \cap H_0^1 \cap H$ (see for, instance, [Chap. 4 of Ref. 3]). Unfortunately, when we consider vector fields for which $u \notin H$, Proposition 2.1 shows that coercivity fails for the extended Stokes operator.

The key to global existence results for the nonlinear system (1.1)–(1.4) is to remedy the negative results in Proposition 2.1 in a manner that interacts well with the nonlinear term. This can be done by introducing a stabilizing higher order term, and a compensating gradient projection term, as we now describe.

For any $u \in H^1(\Omega)$ define $Q(u)$, the primitive of the gradient projection, to be the unique mean zero H^1 function such that

$$\nabla Q(u) = (I - P)u.$$

Given constants ε, $C > 0$, we define a H^1-equivalent inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\varepsilon,C}$ by

$$\langle\!\langle u, v\rangle\!\rangle_{\varepsilon,C} = \langle u, v\rangle + \varepsilon\langle\nabla u, \nabla v\rangle + C\langle Q(u), Q(v)\rangle, \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $L^2(\Omega)$. Our main result shows that for all ε sufficiently small, we can find C large enough to ensure coercivity of A under the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\varepsilon,C}$.

Proposition 2.2 (H^1-equivalent coercivity). Let $\Omega \subset \mathbb{R}^d$ be a C^3 domain. There exists positive constants $\varepsilon_0 = \varepsilon_0(\Omega)$ and $c = c(\Omega)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a constant $C_\varepsilon = C_\varepsilon(\Omega) > 0$, such that for the inner product $\langle\cdot,\cdot\rangle_\varepsilon$ defined by

$$\langle\cdot,\cdot\rangle_\varepsilon \stackrel{\text{def}}{=} \langle\!\langle\cdot,\cdot\rangle\!\rangle_{\varepsilon,C_\varepsilon},$$

we have

$$\langle u, Au\rangle_\varepsilon \geqslant \frac{1}{c}\left(\|\nabla u\|_{L^2}^2 + \varepsilon\|\Delta u\|_{L^2}^2 + C_\varepsilon\|\nabla q\|_{L^2}^2\right) \tag{2.7}$$

for all $u \in H^2 \cap H_0^1$. Consequently, there exists a constant $C'_\varepsilon = C'_\varepsilon(\varepsilon, \Omega)$ such that

$$\langle u, Au\rangle_\varepsilon \geqslant \frac{1}{c}\langle u, u\rangle_\varepsilon, \quad \text{and} \quad \langle u, Au\rangle_\varepsilon \geqslant \frac{1}{C'_\varepsilon}\langle\nabla u, \nabla u\rangle_\varepsilon, \tag{2.8}$$

for all $u \in H^2 \cap H_0^1$.

We prove this proposition in Sec. III. The main ingredient in the proof is an estimate for the Laplace-Leray commutator (1.9) that is proved in Ref. 10 and stated in Theorem 3.1 below. A couple of further consequences of this Theorem are worth mentioning here. First, A is invertible on L^2 with

compact resolvent (Lemma 3.3). And, due to Theorem 3.1 and the self-adjointness of the Laplacian, an elementary result about sectorial operators [Theorem 1.3.2 of Ref. 7] directly implies that A is a sectorial operator on L^2 with domain $D(A) = D(-\Delta) = H^2 \cap H_0^1$.

The result of Proposition 2.2 raises the question of whether coercivity of A can be obtained in a space with less regularity than H^1 by using an equivalent inner product. In this regard, we have two remarks. First, in Proposition 3.5, we will describe an inner product $\langle \cdot, \cdot \rangle'_\varepsilon$ for which A is coercive that is equivalent to the usual inner product on the space

$$H_{\text{div}} = \left\{ v \in L^2(\Omega) \mid \nabla \cdot v \in L^2(\Omega) \text{ and } v \cdot \nu = 0 \text{ on } \partial\Omega \right\}.$$

Second, we expect that a bilinear form defined by

$$\langle u, v \rangle''_\varepsilon \stackrel{\text{def}}{=} \langle A^{-1/2} u, A^{-1/2} v \rangle_\varepsilon \tag{2.9}$$

determines an L^2-equivalent inner product under which A is coercive. Coercivity for $u \in D(A)$ would follow from Proposition 2.2, and L^2 continuity by well-known interpolation estimates. However, an L^2-coercivity bound $\langle u, u \rangle''_\varepsilon \geqslant c \|u\|^2_{L^2}$ appears not to be easy to prove — it may involve proving A has bounded imaginary powers (see Ref. 1) in order to establish the expected characterization $D(A^{1/2}) = H_0^1$.

In any case, unfortunately the inner products $\langle \cdot, \cdot \rangle'_\varepsilon$ and $\langle \cdot, \cdot \rangle''_\varepsilon$ do not seem to interact well with the nonlinearity in (1.1). Thus, for questions of global existence and stability for the nonlinear extended Navier-Stokes equations and their discretizations, it is more convenient to use the inner product in Proposition 2.2. The rest of the paper can be read independently of Proposition 3.5 or its proof.

B. Energy decay for the extended Stokes equations

A first step to global existence results for (1.1)–(1.4) is the study of long time behaviour for the underlying linear equations. These are the extended Stokes equations

$$\begin{cases} \partial_t u - \Delta u + \nabla p_s(u) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{2.10}$$

A direct consequence of Proposition 2.1 is that the energy of solutions to (2.10) can increase, at least initially.

Corollary 2.3: There exists $u_0 \in C^2(\Omega)$ with $u_0 = 0$ on $\partial\Omega$, and $t_0 > 0$ such that the solution u to (2.10) with initial data u_0 satisfies

$$\|u(t_0)\|_{L^2} > \|u_0\|_{L^2}.$$

The proof of Corollary 2.3 can be found at the end of the Appendix, following the proof of Proposition 2.2.

In contrast to the extended Stokes equations, solutions to the standard Stokes equations (with initial data in H) always have monotonically decaying L^2 norm. This follows because if $u(t) \in H$, then multiplication by u and integration by parts produces the standard energy inequality

$$\frac{1}{2} \partial_t \|u(t)\|^2_{L^2} + \|\nabla u\|^2_{L^2} = 0. \tag{2.11}$$

The Poincaré inequality now yields strict exponential decay

$$\|u(t)\|^2_{L^2} \leqslant e^{-ct} \|u_0\|^2_{L^2} \tag{2.12}$$

for all solutions to the standard Stokes equations with initial data in H.

Despite the counter-intuitive initial energy increase, the extended Stokes system is a well-posed, *non-degenerate* parabolic system. This was proved in Refs. 5 and 10, and is a direct consequence of Theorem 3.1. Indeed, since A is sectorial it generates an analytic semigroup e^{-At}, showing well-posedness of the initial boundary-value problem (2.10). Because no eigenvalue of A has non-positive

real part by Proposition 2.2, one can quickly show that while the L^2 energy of solutions to (2.10) can increase initially, it must eventually decay exponentially. Explicitly, this means that solutions to (2.10) must satisfy

$$\|u(t)\|^2_{L^2} \leqslant C e^{-ct} \|u_0\|^2_{L^2} \tag{2.13}$$

for some constants $C, c > 0$.

To digress briefly, we remark that with a little work, one can explicitly characterize the spectrum of A. Indeed, if A_S denotes the (standard) Stokes operator with no-slip boundary conditions, and Δ_N denotes the Laplace operator with homogeneous Neumann boundary conditions, then

$$\sigma(A) = \sigma(A_S) \cup \sigma(-\Delta_N) - \{0\}.$$

Seeing $\sigma(A)$ is contained in the right hand side above is immediate. The reverse inclusion requires a little work, and was communicated to us by Kelliher.[11]

Unfortunately, an abstract spectral-theoretic proof of (2.13) is not of direct help for studying the stability of time-discrete schemes, which was a primary motivation for introducing these equations. Further, (2.13) does not recover (2.11) for solutions with initial data in H. For this reason, we search for a direct energy-method proof of (2.13), and for an idea which also allows the study of time discrete schemes.

Observe first that if we multiply (2.10) by u, integrate, use the commutator estimate (3.1) and Gronwall's lemma, we obtain exponential *growth*, not decay, of $\|u\|^2_{L^2}$. If we involve a higher derivative, coercivity of A in Proposition 2.2 (or Proposition 3.5) and Gronwall's lemma guarantee eventual exponential decay of $\|u\|_{H^1}$ (or $\|u\|_{H_{\text{div}}}$). However, for (2.10), we can obtain a more satisfactory decay estimate by considering non-quadratic form energies.

Proposition 2.4. Let u be a solution to (2.10) with $u_0 \in H^1(\Omega)$. Then for any $\varepsilon > 0$, there exists constants $c_1 = c_1(\Omega)$ and $c_2 = c_2(\Omega, \varepsilon)$, such that $c_1, c_2 > 0$ and

$$\partial_t \mathcal{E}_{c_1,c_2}(u) + \mathcal{E}'_\varepsilon(u) \leqslant 0, \tag{2.14}$$

where $\mathcal{E}_{c_1,c_2}$ and $\mathcal{E}'_\varepsilon$ are defined by

$$\mathcal{E}_{c_1,c_2}(u) \stackrel{\text{def}}{=} \|u\|^2_{L^2} + c_1 \|\nabla u\|_{L^2} \|\nabla Q(u)\|_{L^2} + c_2 \|\nabla Q(u)\|^2_{L^2}, \tag{2.15}$$

$$\mathcal{E}'_\varepsilon(u) \stackrel{\text{def}}{=} (2-\varepsilon)\|\nabla u\|^2_{L^2} + \|\Delta u\|_{L^2} \|\nabla Q(u)\|_{L^2} + \|\Delta Q(u)\|^2_{L^2}. \tag{2.16}$$

The proof of Proposition 2.4 is in Sec. IV. While (2.14) does not imply eventual exponential decay controlled only by the L^2 norm as in (2.13), it does provide an estimate that reduces to the energy inequality for extended Stokes equations (2.11) when the initial data is in H. To see this, note that if $u_0 \in H$, then $(I - P)u(t) = 0$ for all $t > 0$ because $\nabla \cdot u$ satisfies the heat equation (1.6). Consequently $Q(u) \equiv 0$, and Eq. (2.14) reduces to

$$\partial_t \|u\|^2_{L^2} + (2-\varepsilon)\|\nabla u\|^2_{L^2} \leqslant 0.$$

Thus, in the limit $\varepsilon \to 0$, we naturally recover the energy decay for the Stokes equation (2.11) for initial data in H.

We also notice that the "energy" $\mathcal{E}_{c_1,c_2}$ of solutions must in fact decrease exponentially. This is because $\int_\Omega \nabla u = 0 = \int_\Omega Q(u)$, and so the Poincaré inequality can be applied to both the terms $\|\nabla u\|_{L^2}$ and $\|Q(u)\|_{L^2}$. Thus, Eq. (2.14) immediately implies

$$\mathcal{E}_{c_1,c_2}(u(t)) \leqslant e^{-ct} \mathcal{E}_{c_1,c_2}(u_0),$$

for some small constant $c = c(c_1, c_2, \varepsilon, \Omega)$. Unfortunately, however, for the extended Navier-Stokes equations, the "energy" $\mathcal{E}_{c_1,c_2}$ does not interact well with the nonlinearity.

C. Uniform stability for a time-discrete scheme

Before moving on to the non-linear system (1.1)–(1.4), we study stability of a time-discrete scheme for (2.10), of the type treated in Ref. 10. One main motivation for studying the system (1.1)–(1.4), or the linear system (2.10), is that this kind of time-discrete scheme is naturally implicit only in the viscosity term, and explicit in the pressure. We will show that the ideas used in the proof of Proposition 2.2 give *globally uniform* stability estimates for such time-discrete schemes.

Given an approximation u^n to the velocity at time $n\,\delta t$, we determine ∇p^n from the weak-form Poisson equation

$$\langle \nabla p^n, \nabla\varphi\rangle = \langle \Delta u^n - \nabla\nabla\cdot u^n + f^n, \nabla\varphi\rangle \qquad \forall \varphi \in H^1(\Omega). \tag{2.17}$$

Now, we determine u^{n+1} by solving the elliptic boundary value problem

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\delta t} - \Delta u^{n+1} + \nabla p^n = f^n & \text{in } \Omega, \\ u^{n+1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.18}$$

where $f^n = \frac{1}{\delta t}\int_{n\delta t}^{(n+1)\delta t} f(s)\,ds$ is a time-discretized forcing term.

Proposition 2.5: Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with C^3 boundary. Then there exist positive constants κ_0, ε, C_ε, C, C', depending only on Ω, such that whenever $0 < \delta t < \kappa_0$, then for all $N > 0$ we have

$$\begin{aligned} &\|u^N\|_{L^2}^2 + \varepsilon\|\nabla u^N\|_{L^2}^2 + C_\varepsilon\|\nabla q^N\|_{L^2}^2 \\ &\quad + \frac{1}{C}\sum_{k=0}^{N-1}\left(\|\nabla u^k\|_{L^2}^2 + \varepsilon\|\Delta u^k\|_{L^2}^2 + C_\varepsilon\|\Delta q^{k+1}\|_{L^2}^2\right)\delta t \\ &\qquad \leqslant \|u^0\|_{L^2}^2 + \varepsilon\|\nabla u^0\|_{L^2}^2 + C_\varepsilon\|\nabla q^0\|_{L^2}^2 \\ &\qquad\quad + C\,\delta t\left(\|\nabla u^0\|_{L^2}^2 + \varepsilon\|\Delta u^0\|_{L^2}^2 + \sum_{k=0}^{N-1}\|f^k\|_{L^2}^2\right) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\|u^N\|_{L^2}^2 + \varepsilon\|\nabla u^N\|_{L^2}^2 + C_\varepsilon\|\nabla q^N\|_{L^2}^2 \\ &\quad \leqslant (1-C\delta t)^N\left(\|u^0\|_{L^2}^2 + \varepsilon\|\nabla u^0\|_{L^2}^2 + C_\varepsilon\|\nabla q^0\|_{L^2}^2\right) \\ &\qquad + C'\sum_{k=0}^{N-1}\|f_k\|_{L^2}^2(1-C\delta t)^{N-1-k}\delta t. \end{aligned} \tag{2.20}$$

The proof of this proposition is in Sec. V.

D. Global existence results for the extended Navier-Stokes equations

When one seeks an L^2 energy estimate for (1.1)–(1.4), multiplying (1.1) by u, the nonlinearity produces the term

$$\int_\Omega u\cdot(u\cdot\nabla)u = -\frac{1}{2}\int_\Omega(\nabla\cdot u)|u|^2. \tag{2.21}$$

In general this is non-zero, but is morally harmless since $\nabla\cdot u$ is a solution of (1.6) and is well controlled. This is indeed the case in two dimensions, but under *periodic* boundary conditions (see Proposition 2.9, and the remark following it). The key ingredient for proving global existence for periodic boundary conditions is the coercivity (2.2) of the linear terms. Consequently, despite the extra nonlinear term arising from (2.21), the L^2 energy balance closes and the well-known existence results for the standard incompressible Navier-Stokes equations continue to hold with minor modifications.

The situation is more complicated under no-slip boundary conditions, however, since now coercivity (2.2) *fails*. To get any mileage from the linear terms, we need to use an inner-product under which the linear terms are coercive. Using the inner product in Proposition 2.2, and a "brutal" estimate on the nonlinearity, we can obtain a two- or three-dimensional small-data global existence result.

Theorem 2.6 (Small data global existence): *Let $d = 2$ or 3, $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^3 boundary. There exists a small constant $V_0 = V_0(\Omega) > 0$ such that if $u_0 \in H^1_0(\Omega)$ with*

$$\|u_0\|_{H^1} < V_0,$$

then there exists a global strong solution to (1.1)–(1.4) with

$$u \in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega)), \tag{2.22}$$

for any $T > 0$. Consequently, $u \in C([0,\infty); H^1_0)$ and $\nabla\cdot u \in C^\infty((0,\infty)\times\Omega)$.

The proof of this theorem is in Sec. VI. Two-dimensional global existence, however, poses a different problem. A key ingredient in 2D global existence for the standard incompressible Navier-Stokes equations is the L^2 energy balance: the nonlinearity cancels, and does not contribute! Unfortunately, for (1.1)–(1.4), the L^2-energy balance does not close because of the higher order contribution from the Stokes pressure gradient.

In the absence of an L^2 energy inequality, we are only able to prove a perturbative result. If the initial data are divergence free, then (1.1)–(1.4) reduces to the standard incompressible Navier-Stokes equations, for which 2D global existence is well known. Thus, for initial data with small divergence, we can prove 2D global existence for (1.1)–(1.4).

Theorem 2.7 (Small divergence global existence in 2D): *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^3 domain, $v_0 \in H^1_0(\Omega)$ with $\nabla\cdot v_0 = 0$ be arbitrary. There exists a small constant $U_0 = U_0(\Omega, \|v_0\|_{H^1_0(\Omega)}) > 0$ such that if*

$$u_0 \in H^1_0(\Omega), \quad P_0u_0 = v_0 \quad \text{and} \quad \|\nabla\cdot u_0\|_{L^2(\Omega)} < U_0, \tag{2.23}$$

then there exists a global strong solution to (1.1)–(1.4) with initial data u_0 such that (2.22) holds for all $T > 0$.

The operator P_0 above is the H^1_0-orthogonal projection of $H^1_0(\Omega)$ onto the subspace of divergence free vector fields, and is described in Sec. VII along with the proof of Theorem 2.7. One strategy to avoid the small divergence assumption is to further damp the divergence. Namely, for arbitrary initial data (in 2D), if we add a strong enough divergence-damping term to (1.1)–(1.2), we can guarantee global existence.

Corollary 2.8 (Divergence-damped global existence in 2D): Let $\Omega \subset \mathbb{R}^2$ be a C^3, bounded domain and $u_0 \in H^1_0(\Omega)$ be arbitrary. There exists a constant $\alpha_0 = \alpha_0(\Omega, \|\nabla\cdot u_0\|_{L^2(\Omega)}) > 0$ such that if $\alpha \geq \alpha_0$, then the system

$$\begin{cases} \partial_t u + P((u\cdot\nabla)u) + Au + \alpha(I-P)u = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.24}$$

has a global strong solution u such that (2.22) holds for all $T > 0$.

The main idea in proving Corollary 2.8 is to verify that the divergence-damped extended Stokes operator B_α defined by

$$B_\alpha \overset{\text{def}}{=} A + \alpha(I-P) \tag{2.25}$$

is coercive, with coercivity constant *independent* of α. Consequently, the proofs of Theorems 2.6 and 2.7 work verbatim for the system (2.24), with constants independent of α. Combining these existence theorems, and using the added divergence damping gives Corollary 2.8, a better existence result as an easy corollary. We devote Sec. VIII to the coercivity of B_α (Proposition 8.1), and the proof of Corollary 2.8.

So far, our two-dimensional global existence results under no-slip boundary conditions required either a small initial divergence assumption, or an additional strong divergence damping term. Such requirements are not needed under periodic boundary conditions, primarily because of (2.2). We observe, then, that the identity (2.2) will still hold in domains with boundary, provided we consider functions u with boundary conditions

$$Pu \cdot \tau = 0 \text{ on } \partial\Omega \qquad \text{and} \qquad u \cdot \nu = 0 \text{ on } \partial\Omega, \tag{2.26}$$

where ν and τ are the unit normal and tangential vectors, respectively. These boundary conditions (2.26) reduce to the usual no-slip conditions in the physically relevant situation where $u = Pu$.

Armed with (2.2), we obtain a 2D global existence result without a smallness assumption, or any additional divergence damping.

Proposition 2.9. Let $\Omega \subset \mathbb{R}^2$ be locally Lipschitz and bounded, and let $u_0 \in H^1(\Omega)$. There exists a time-global strong solution to (1.1)–(1.2) with initial data u_0 and boundary conditions (2.26).

We prove the identity (2.2) and Proposition 2.9 in Sec. IX. The proof of Proposition 2.9 emphasizes another (analytical) advantage of the boundary conditions (2.26). Under all the boundary conditions, we consider (no-slip, periodic, and (2.26)) the evolution equation for the gradient projection is always linear, self-contained, and decays at an explicitly known rate. The evolution equation for the Leray projection (Eq. (9.2)), is coupled to the gradient projection; however, the coupling terms are harmless. What causes trouble under the no-slip boundary conditions is that the evolution of the Leray projection is also coupled to the gradient projection through boundary conditions! This proves problematic in the case of 2D global existence. On the other hand, periodic boundary conditions, or the boundary conditions (2.26) provide an explicit *de-coupled* boundary condition for the Leray projection, which simplifies the analysis greatly. Unfortunately, the price paid is that the boundary conditions (2.26) are much harder to implement numerically.

III. COERCIVITY OF THE EXTENDED STOKES OPERATOR.

As mentioned earlier, the extended Stokes operator is *not* coercive under the standard L^2 inner product. However, it *is* coercive under a non-standard, but H^1-equivalent, inner product. This is the main tool we use in studying the extended Navier-Stokes. The aim of this section is to prove Proposition 2.2 (coercivity under the adjusted H^1 inner product). The main ingredient in the proof is the following estimate on the Laplace-Leray commutator.

Theorem 3.1 (Liu, Liu, and Pego[10]). *Let Ω be a connected, bounded domain with C^3 boundary. For any $\delta > 0$, there exists $C_\delta \geq 0$ such that*

$$\|\nabla p_s(u)\|_{L^2}^2 \leqslant \left(\frac{1}{2} + \delta\right) \|\Delta u\|_{L^2}^2 + C_\delta \|\nabla u\|_{L^2}^2 \tag{3.1}$$

for all $u \in H^2 \cap H_0^1(\Omega)$.

We refer the reader to Ref. 10 for the proof of Theorem 3.1.

A. H^1 equivalent coercivity on $D(A^2)$

The idea behind the proof of Proposition 2.2 is to use Theorem 3.1 and prove coercivity assuming the "extra" boundary condition $Au \in H_0^1$. We will later use an approximation argument to prove the proposition for all $H^2 \cap H_0^1$ functions.

Lemma 3.2: For any $\varepsilon > 0$ sufficiently small, there exists a constant $c = c(\Omega)$, independent of ε, and a constant $C_\varepsilon = C_\varepsilon(\Omega) > 0$, depending on ε and Ω, such that (2.7) holds for all $u \in H^2 \cap H_0^1$, such that $Au \in H_0^1$.

Proof. Observe first that there exists a constant $C = C(\Omega)$ such that for all $u \in H^2 \cap H_0^1$, we have

$$\|\nabla p_s(u)\|_{L^2} \leqslant C \|\Delta u\|_{L^2}. \tag{3.2}$$

While this immediately follows from Theorem 3.1 and the Poincaré inequality, we can see it directly from (1.7) because, due to elliptic regularity,

$$\|\nabla p_s(u)\|_{L^2}^2 \leqslant C \left(\|\Delta u\|_{L^2}^2 + \|\nabla \nabla \cdot u\|_{L^2}^2\right) \leqslant C \|\Delta u\|_{L^2}^2. \tag{3.3}$$

Now let $u \in H^2 \cap H_0^1$ be such that $Au \in H_0^1$, and $q = Q(u)$ be the unique mean zero function such that $\nabla q = (I - P)u$. Then,

$$\begin{aligned} \langle u, Au \rangle &= \langle u, -\Delta u \rangle + \langle u, \nabla p_s \rangle = \|\nabla u\|_{L^2}^2 + \langle \nabla q, \nabla p_s \rangle \\ &\geqslant \|\nabla u\|_{L^2}^2 - \|\nabla q\|_{L^2} \|\nabla p_s\|_{L^2} \geqslant \|\nabla u\|_{L^2}^2 - C \|\nabla q\|_{L^2} \|\Delta u\|_{L^2} \\ &\geqslant \|\nabla u\|_{L^2}^2 - \frac{\varepsilon}{16} \|\Delta u\|_{L^2}^2 - C_\varepsilon \|\nabla q\|_{L^2}^2, \end{aligned} \tag{3.4}$$

where the second last inequality followed from (3.2), and C_ε is some constant depending only on Ω and ε.

Since $Au = 0$ on $\partial\Omega$ by assumption, we can integrate the H^1-term by parts. This gives

$$\langle \nabla u, \nabla Au \rangle = -\langle \Delta u, -\Delta u + \nabla p_s \rangle \geqslant \frac{1}{8} \|\Delta u\|_{L^2}^2 - C_1 \|\nabla u\|^2, \tag{3.5}$$

where C_1 is the constant that arises from Theorem 3.1.

Thus, if $\varepsilon < \frac{1}{2C_1}$, Eqs. (3.4) and (3.5) give

$$\langle u, Au \rangle + \varepsilon \langle \nabla u, \nabla Au \rangle \geqslant \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{\varepsilon}{16} \|\Delta u\|_{L^2}^2 - C_\varepsilon \|\nabla q\|_{L^2}^2. \tag{3.6}$$

Now let r be the unique mean zero function such that $\nabla r = (I - P)Au$. Observe that

$$(I - P)Au = (I - P)(-P\Delta u - \nabla \nabla \cdot u) = -\nabla \nabla \cdot u.$$

Since $\int_\Omega \nabla \cdot u = 0$, we must have $r = -\nabla \cdot u = -\Delta q$. Thus,

$$\langle q, r \rangle = \langle q, -\Delta q \rangle = \|\nabla q\|_{L^2}^2.$$

Combining this with (3.6), we get (2.7) as desired. □

B. Properties of the extended Stokes operator

Consider the extended Stokes operator A as an operator from $L^2(\Omega)$ into $L^2(\Omega)$ with domain $D(A) = H^2 \cap H_0^1$. In this context, we recall that Proposition 2.2 asserts (2.7) for all $u \in D(A)$; however, Lemma 3.2 only proves (2.7) for all $u \in D(A^2)$. To address this gap, and finish the proof of Proposition 2.2, we need a few basic properties of the extended Stokes operator.

Lemma 3.3 (Regularity and invertibility): The extended Stokes operator A has a compact inverse. Furthermore, there exists a constant $c = c(\Omega) > 0$ such that

$$\frac{1}{c}\|u\|_{H^2} \leqslant \|Au\|_{L^2} \leqslant c\|u\|_{H^2}, \quad \text{for all } u \in H^2 \cap H_0^1. \tag{3.7}$$

Proof. Our first step is to obtain estimates for the operator $A + \lambda I$ with λ large enough. For an arbitrary $u \in H^2 \cap H_0^1$, let $f = (A + \lambda I)u$. Multiplying by $-\Delta u$ and integrating gives

$$\begin{aligned}\lambda\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 &= \int_\Omega \nabla p_s \cdot \Delta u \, dx - \int_\Omega f \Delta u \, dx \\ &\leqslant \frac{1}{2}\|\Delta u\|_{L^2}^2 + \frac{1}{2}\|\nabla p_s\|_{L^2}^2 + \frac{1}{16}\|\Delta u\|_{L^2}^2 + 4\|f\|_{L^2}^2 \\ &\leqslant \left(\frac{1}{2} + \frac{3}{8} + \frac{1}{16}\right)\|\Delta u\|_{L^2}^2 + c\|\nabla u\|_{L^2}^2 + 4\|f\|_{L^2}^2,\end{aligned}$$

where the last inequality followed from Theorem 3.1, and $c = c(\Omega)$ is a constant. This gives

$$(\lambda - c)\|\nabla u\|_{L^2}^2 + \frac{1}{16}\|\Delta u\|_{L^2}^2 \leqslant 4\|f\|_{L^2}^2.$$

Thus, when $\lambda > c$, we immediately see

$$\|u\|_{H^2} \leqslant C\|f\|_{L^2} = C\|(A + \lambda I)u\|_{L^2}, \quad \text{when } u \in H^2 \cap H_0^1. \tag{3.8}$$

One can use the last relation to check that A is closed. We claim further that $A + \lambda I$ is surjective for some large enough λ. This can be proved by a Neumann-series perturbation argument based on the identity

$$A + \lambda I = (I + B)(\lambda I - \Delta), \tag{3.9}$$

where $B = \nabla p_s \circ (\lambda I - \Delta)^{-1}$. That is,

$$Bu = \nabla p_s(v), \quad v = (\lambda I - \Delta)^{-1}u.$$

It suffices to prove that the operator norm of B on L^2 is strictly less than one, if λ is positive and large enough. By easy energy estimates, we have that $\lambda\|v\|_{L^2} \leq \|u\|_{L^2}$ and $\|\Delta v\|_{L^2} \leq \|u\|_{L^2}$. Then due to Theorem 3.1 and interpolation, we have

$$\|Bu\|_{L^2}^2 = \int_\Omega |\nabla p_s(v)|^2 \leqslant \beta\|\Delta v\|_{L^2}^2 + C_\beta\|v\|_{L^2}^2 \leq \left(\beta + C_\beta\lambda^{-2}\right)\|u\|_{L^2}^2,$$

and the coefficient on the right is less than 1 for λ large enough. Thus, $I + B$ is an isomorphism on L^2, hence $A + \lambda I$ is surjective.

Further, the Rellich-Kondrachov compact embedding theorem and the bound (3.8) imply that $A + \lambda I$ has compact inverse. Since we have shown that the resolvent of A contains at least one element with a compact inverse, the spectrum of A consists only of (isolated) eigenvalues, of finite multiplicity (see, for instance, [Theorem III.6.29 of Ref. 8]. Thus, to prove invertibility of A, it suffices to show that 0 is not an eigenvalue of A.

To see this, suppose $u \in D(A)$ is such that $Au = 0$. Then $-P\Delta u = \nabla\nabla \cdot u \in L^2(\Omega)$. Since the range of the Leray projection (by definition) is orthogonal to gradients, we must have $\nabla\nabla \cdot u = P\Delta u = 0$, and hence $\nabla \cdot u$ must be constant. Since $\int_\Omega \nabla \cdot u = \int_{\partial\Omega} u \cdot \nu = 0$, this forces $\nabla \cdot u = 0$. Thus, $u = Pu$, and is orthogonal to gradients. Since $Au = 0$, we have

$$0 = \int_\Omega u \cdot Au \, dx = -\int_\Omega u \cdot \Delta u \, dx + \int_\Omega Pu \cdot \nabla p_s(u) \, dx = \|\nabla u\|_{L^2}^2 + 0,$$

forcing $u = 0$. Hence, 0 is not an eigenvalue of A, and we conclude that A is invertible.

It remains to establish (3.7). The upper bound follows immediately from (2.1) and (3.2). To prove the lower bound, observe first that boundedness of A^{-1} implies

$$\|A^{-1}u\|_{L^2} \leqslant C\|u\|_{L^2}, \tag{3.10}$$

for some constant $C = C(\Omega)$, which we subsequently allow to change from line to line. Thus, using the operator identity

$$A^{-1} = (A + \lambda I)^{-1}\left(I + \lambda A^{-1}\right)$$

and the inequalities (3.8), (3.10), we see

$$\begin{aligned}\|A^{-1}u\|_{H^2} &= \|(A + \lambda I)^{-1}\left(I + \lambda A^{-1}\right)u\|_{H^2} \\ &\leqslant C\|\left(I + \lambda A^{-1}\right)u\|_{L^2} \leqslant C(1 + C\lambda)\|u\|_{L^2}\end{aligned}$$

proving the lower bound in (3.7). □

Lemma 3.4: For the extended Stokes operator, $D(A^2)$ is dense in $D(A)$.

Proof: Let $u \in D(A)$, and $v = Au$. Since $v \in L^2(\Omega)$, we can find $v_n \in H^2 \cap H_0^1$ such that $(v_n) \to v$ in L^2. Since $D(A) = H^2 \cap H_0^1$ by letting $u_n \stackrel{\text{def}}{=} A^{-1}v_n$, we have $u_n \in D(A^2)$. Finally, by Lemma 3.3 we see

$$\|u_n - u\|_{H^2} \leqslant c\|Au_n - Au\|_{L^2} = c\|v_n - v\|_{L^2} \to 0,$$

concluding the proof. □

C. H^1-equivalent coercivity on $D(A)$

Lemmas 3.2 and 3.4 quickly imply Proposition 2.2.

Proof of Proposition 2.2: Let $u \in D(A)$. By Lemma 3.4, there exists a sequence $u_n \in D(A^2)$ such that $(u_n) \to u$ in H^2. By Lemma 3.2, there exist constants $c(\Omega)$, $C_\varepsilon(\Omega) > 0$ such that

$$\langle u_n, Au_n\rangle_\varepsilon \geqslant \frac{1}{c}\left(\|\nabla u_n\|_{L^2}^2 + \varepsilon\|\Delta u_n\|_{L^2}^2 + C_\varepsilon\|\nabla q_n\|_{L^2}^2\right),$$

where q_n is the unique, mean-zero function such that $\nabla q_n = (I - P)u_n$. Since $(u_n) \to u$ in H^2, taking limits as $n \to \infty$ yields (2.7). Now using the Poincaré inequality, (2.8) follows. □

D. H_{div} equivalent coercivity

We conclude this section by proving coercivity under an H_{div}-equivalent inner product. The rest of this paper is independent of this result and its proof.

Proposition 3.5 (H_{div}-coercivity): Let $\Omega \subset \mathbb{R}^3$ be a C^3 domain. There exist positive constants $\varepsilon_0 > 0$ and $c = c(\Omega)$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a constant $C_\varepsilon = C_\varepsilon(\Omega) > 0$ such that the following hold.

(1) Let $\langle\cdot,\cdot\rangle_\varepsilon'$ be defined by

$$\begin{aligned}\langle u, v\rangle_\varepsilon' \stackrel{\text{def}}{=} \langle u, v\rangle + \varepsilon\langle\nabla \cdot u, \nabla \cdot v\rangle + C_\varepsilon\langle A^{-1}u, A^{-1}v\rangle_\varepsilon - \\ -\langle u, \nabla p_s(A^{-1}v)\rangle - \langle\nabla p_s(A^{-1}u), v\rangle,\end{aligned} \tag{3.11}$$

where $\langle u, v\rangle$ is the standard inner product on $L^2(\Omega)$, and $\langle\cdot,\cdot\rangle_\varepsilon$ denotes the inner product from Proposition 2.2. Then

$$\frac{1}{c}\|u\|_{L^2} + \varepsilon\|\nabla \cdot u\|_{L^2}^2 \leqslant \langle u, u\rangle_\varepsilon' \leqslant c(1 + C_\varepsilon)\|u\|_{L^2}^2 + \varepsilon\|\nabla \cdot u\|_{L^2}^2 \tag{3.12}$$

for any $u \in L^2(\Omega)$ with $\nabla \cdot u \in L^2(\Omega)$ and $u \cdot \nu = 0$ on $\partial\Omega$.

(2) *For any* $u \in H^2 \cap H_0^1$ *with* $\nabla \cdot u \in H^l$, *we have*

$$\langle u, Au\rangle'_\varepsilon \geqslant \|\nabla u\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla\nabla\cdot u\|_{L^2}^2 \tag{3.13}$$

$$\text{and} \quad \langle u, Au\rangle'_\varepsilon \geqslant \frac{1}{c}\langle u, u\rangle'_\varepsilon. \tag{3.14}$$

Proof: We begin by proving (3.12) for all $u \in D(A)$. Density of $D(A)$ in L^2 and a standard approximation argument will now establish (3.12) for all $u \in H_{\text{div}}$. We will assume C_ε and c are constants that can change from line to line, provided their dependence on parameters is as required in the proposition. Let $u \in D(A)$, and $v = A^{-1}u$. Then from (3.11) we have

$$\begin{aligned}\langle u, u\rangle'_\varepsilon &= \langle Av, Av\rangle - 2\langle Av, \nabla p_s(v)\rangle + \varepsilon\|\nabla\cdot u\|_{L^2}^2 + C_\varepsilon\langle v, v\rangle_\varepsilon\\ &= \langle -\Delta v + \nabla p_s(v), -\Delta v - \nabla p_s(v)\rangle + \varepsilon\|\nabla\cdot u\|_{L^2}^2 + C_\varepsilon\langle v, v\rangle_\varepsilon\\ &= \|\Delta v\|_{L^2}^2 - \|\nabla p_s(v)\|_{L^2}^2 + \varepsilon\|\nabla\cdot u\|_{L^2}^2 + C_\varepsilon\langle v, v\rangle_\varepsilon\\ &\geqslant \frac{1}{4}\|\Delta v\|_{L^2}^2 - c\|v\|_{L^2}^2 + \varepsilon\|\nabla\cdot u\|_{L^2}^2 + C_\varepsilon\langle v, v\rangle_\varepsilon,\end{aligned} \tag{3.15}$$

where the last inequality followed from Theorem 3.1 and interpolation. Now since $Av = u$, we immediately see

$$\|u\|_{L^2}^2 = \|-P\Delta v - \nabla\nabla\cdot v\|_{L^2}^2 \leqslant c\|\Delta v\|_{L^2}^2.$$

Finally, by definition of $\langle v, v\rangle_\varepsilon$, we have $\langle v, v\rangle_\varepsilon \geqslant \|v\|_{L^2}^2$. Thus, if $C_\varepsilon > c$, the lower bound in Eq. (3.12) will hold for all $\varepsilon > 0$.

For the upper bound in (3.12), observe that by definition of ∇p_s, and Lemma 3.3 we have

$$\|\nabla p_s(A^{-1}u)\|_{L^2} \leqslant c\|A^{-1}u\|_{H^2} \leqslant c\|u\|_{L^2}.$$

Combined with the estimate $\|A^{-1}u\|_{L^2} \leqslant c\|u\|_{L^2}$, which is also a consequence of Lemma 3.3, we immediately obtain the upper bound in (3.12).

Finally, it remains to prove the inequality (3.13). We will prove (3.13) for $u \in D(A^2)$; since $D(A^2)$ is dense in $D(A)$, the same approximation argument from the proof of Proposition 2.2 will show that (3.13) holds on $D(A)$. In keeping with the above notation, we again set $v = A^{-1}u$. This gives

$$\langle u, Au\rangle'_\varepsilon = \langle u, Au\rangle - \langle u, \nabla p_s(u)\rangle - \langle \nabla p_s(v), Au\rangle + \varepsilon\langle\nabla\cdot u, \nabla\cdot Au\rangle + C_\varepsilon\langle v, Av\rangle_\varepsilon.$$

We deal with the terms on the right individually. Combining the first two terms, the dangerous term involving the Stokes pressure cancels. This gives

$$\langle u, Au\rangle - \langle u, \nabla p_s(u)\rangle = \langle u, -\Delta u\rangle = \|\nabla u\|_{L^2}^2.$$

For the third term,

$$-\langle\nabla p_s(v), Au\rangle = -\langle\nabla p_s(v), (I-P)Au\rangle = \langle\nabla p_s(v), \nabla\nabla\cdot u\rangle.$$

For the fourth term, observe that if $u \in D(A^2)$, then $Au = 0$ on $\partial\Omega$. Thus, integrating by parts gives

$$\langle\nabla\cdot u, \nabla\cdot Au\rangle = -\langle\nabla\nabla\cdot u, Au\rangle = \|\nabla\nabla\cdot u\|_{L^2}^2.$$

Combining these identities, we have

$$\begin{aligned}\langle u, Au\rangle'_\varepsilon &= \|\nabla u\|_{L^2}^2 + \langle\nabla p_s(v), \nabla\nabla\cdot u\rangle + \varepsilon\|\nabla\nabla\cdot u\|_{L^2}^2 + C_\varepsilon\langle v, Av\rangle_\varepsilon\\ &\geqslant \|\nabla u\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla\nabla\cdot u\|_{L^2}^2 - \frac{1}{2\varepsilon}\|\nabla p_s(v)\|_{L^2}^2 + C_\varepsilon\langle v, Av\rangle_\varepsilon\\ &\geqslant \|\nabla u\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla\nabla\cdot u\|_{L^2}^2 - \frac{c}{\varepsilon}\|\Delta v\|_{L^2}^2 + C_\varepsilon\langle v, Av\rangle_\varepsilon\\ &\geqslant \|\nabla u\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla\nabla\cdot u\|_{L^2}^2 + \left(\frac{\varepsilon C_\varepsilon}{2c_1} - \frac{c}{\varepsilon}\right)\|\Delta v\|_{L^2}^2 + \frac{C_\varepsilon}{2}\langle v, Av\rangle_\varepsilon,\end{aligned}$$

where the last inequality followed from Proposition 2.2, and c_1 is the constant in (2.7). We note that Proposition 2.2 guarantees that c_1 is independent of ε. Now we choose C_ε large enough so that $\frac{\varepsilon C_\varepsilon}{2c_1} - \frac{c}{\varepsilon} > 1$, giving

$$\langle u, Au\rangle'_\varepsilon \geqslant \|\nabla u\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla\nabla\cdot u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \frac{C_\varepsilon}{2}\langle v, Av\rangle_\varepsilon, \tag{3.16}$$

from which inequality (3.13) follows.

Finally for (3.14), observe that from (3.15) we have

$$\begin{aligned}\langle u, u\rangle'_\varepsilon &= \|\Delta v\|_{L^2}^2 - \|\nabla p_s(v)\|_{L^2}^2 + \varepsilon\|\nabla\cdot u\|_{L^2}^2 + C_\varepsilon\langle v, v\rangle_\varepsilon\\ &\leqslant c_2\|\Delta v\|_{L^2}^2 + \varepsilon\|\nabla\cdot u\|_{L^2}^2 + C_\varepsilon\langle v, v\rangle_\varepsilon,\end{aligned}$$

for some constant $c_2 = c_2(\Omega)$. Now using Proposition 2.2 and (3.16), the inequality (3.14) follows. □

IV. DECAY OF A NON-QUADRATIC FORM ENERGY.

This section comprises the proof of Proposition 2.4, addressing the long time behaviour of solutions to the extended Stokes equations (2.10). The result and proof are independent of the rest of this paper.

Proof of Proposition 2.4: In this proof, we use C to denote an intermediate constant that depends only on Ω whose value can change from line to line. We use $C_1, C_2, \ldots$ to denote fixed positive constants that depend only on Ω, whose values *do not* change from line to line.

As usual, let $q = Q(u)$ be the unique mean zero function such that $\nabla q = (I - P)u$. We begin by establishing the energy inequalities

$$\partial_t\|u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2 \leqslant C_1\|\nabla q\|_{L^2}\|\Delta u\|_{L^2}, \tag{4.1}$$

$$\partial_t\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\Delta u\|_{L^2}^2 \leqslant C\|\nabla u\|_{L^2}^2, \tag{4.2}$$

$$\partial_t\|\nabla q\|_{L^2}^2 + 2\|\nabla\cdot u\|_{L^2}^2 \leqslant 0. \tag{4.3}$$

Before proving the above inequalities, we remark that the L^2 balance (4.1) does not close by itself. On the other hand, the H^1 balance (4.2) closes, but does not give decay. A combination of the norms, however, gives us the desired exponential decay.

For the proof of (4.1), multiply (2.10) by u and integrate over Ω to obtain

$$\frac{1}{2}\partial_t\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = -\int_\Omega u\cdot\nabla p_s = -\int_\Omega \nabla q\cdot\nabla p_s \leqslant C_1\|\nabla q\|_{L^2}\|\Delta u\|_{L^2},$$

where we used (3.3). This establishes (4.1).

Turning to (4.2), we multiply (2.10) by $-\Delta u$ and integrate over Ω to obtain

$$\frac{1}{2}\partial_t\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int_\Omega \nabla p_s\cdot\Delta u. \tag{4.4}$$

Using Theorem 3.1, we know that for any $\delta > 0$ there exists a constant $C_\delta = C_\delta(\Omega)$ such that

$$\|\nabla p_s\|_{L^2}^2 \leqslant \left(\frac{1+\delta}{2}\right)\|\Delta u\|_{L^2}^2 + C_\delta\|\nabla u\|_{L^2}^2.$$

Hence,

$$\left|\int_\Omega \nabla p_s\cdot\Delta u\right| \leqslant \frac{1}{2}\|\nabla p_s\|_{L^2}^2 + \frac{1}{2}\|\Delta u\|_{L^2}^2 \leqslant \left(\frac{3+\delta}{4}\right)\|\Delta u\|_{L^2}^2 + \frac{1}{2}C_\delta\|\nabla u\|_{L^2}^2. \tag{4.5}$$

Choosing $\delta = \frac{1}{2}$, Eq. (4.4) reduces to (4.2) as desired.

Finally for (4.3), we apply $(I - P)$ to (2.10) to get

$$\partial_t \nabla q + (I - P)(-\Delta u + \nabla p_s(u)) = 0.$$

Since $\nabla \cdot u = \Delta q$ and

$$(I - P)(-\Delta u + \nabla p_s(u)) = (I - P)(P\Delta u - \nabla\nabla \cdot u) = \nabla\nabla \cdot u = \nabla \Delta q,$$

we see

$$\partial_t \nabla q - \Delta \nabla q = 0,$$

and hence $\partial_t q - \Delta q = C(t)$, where C is constant in space. Now, since $u = 0$ on $\partial\Omega$, we must have $\frac{\partial q}{\partial \nu} = 0$ on $\partial\Omega$. This means $\int_\Omega \Delta q = 0$; since $\int_\Omega q = 0$ by our choice of q, we must have $C(t) = 0$. Thus, we obtain

$$\partial_t q - \Delta q = 0 \quad \text{in } \Omega, \qquad \text{with } \frac{\partial}{\partial \nu} q = 0 \text{ for } x \in \partial\Omega. \tag{4.6}$$

Multiplying by $-\Delta q$ and integrating over Ω gives (4.3) as desired.

Now we combine (4.1)–(4.3) to obtain the desired exponential decay. First from (4.2), (4.3) and the Poincaré inequality, we have

$$\partial_t \|\nabla q\|_{L^2} + \frac{1}{C_2}\|\nabla q\|_{L^2} \leqslant 0,$$

$$\partial_t \|\nabla u\|_{L^2} + \frac{1}{C_2}\|\Delta u\|_{L^2} \leqslant C_3 \|\nabla u\|_{L^2}.$$

Thus,

$$\partial_t \left(\|\nabla u\|_{L^2}\|\nabla q\|_{L^2}\right) + \frac{1}{C_2}\|\Delta u\|_{L^2}\|\nabla q\|_{L^2} \leqslant C_4 \|\nabla u\|_{L^2}\|\nabla q\|_{L^2},$$

where $C_4 = C_3 - \frac{1}{C_2}$. Using this in (4.1), we see

$$\partial_t \left(\|u\|_{L^2}^2 + C_2(C_1 + 1)\|\nabla u\|_{L^2}\|\nabla q\|_{L^2}\right) + 2\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}\|\nabla q\|_{L^2} \leqslant C_5 \|\nabla u\|_{L^2}\|\nabla q\|_{L^2},$$

where $C_5 = C_2(C_1 + 1)C_4$. Now letting $c_1 = C_2(C_1 + 1)$, and $c_2 = c_2(\Omega, \varepsilon)$ to be chosen later, we see that

$$\partial_t \left(\|u\|_{L^2}^2 + c_1\|\nabla u\|_{L^2}\|\nabla q\|_{L^2} + c_2\|\nabla q\|_{L^2}^2\right) + 2\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}\|\nabla q\|_{L^2} + 2c_2\|\nabla \cdot u\|_{L^2}^2 \leqslant \varepsilon\|\nabla u\|_{L^2}^2 + \frac{C_5^2}{2\lambda_1 \varepsilon}\|\nabla \cdot u\|_{L^2}^2,$$

where λ_1 is the best constant in the Poincaré inequality

$$\lambda_1 \|\nabla q\|_{L^2}^2 \leqslant \|\Delta q\|_{L^2}^2 = \|\nabla \cdot u\|_{L^2}^2.$$

Thus, choosing $c_2 = \frac{C_5^2}{4\lambda_1 \varepsilon} + \frac{1}{2}$, we obtain (2.14). □

V. GLOBAL STABILITY OF TIME DISCRETIZATION FOR THE EXTENDED STOKES EQUATIONS

We devote this section to proving Proposition 2.5. The main idea again is similar: to introduce a stabilizing, high order term in the definition of the energies.

Proof of Proposition 2.5. In the following, we use $C_1, C_2, \ldots$ to denote fixed positive constants that depend only on Ω, whose values *do not* change from line to line and a generic constant C whose value might change from one line to the next, depending only on Ω.

Let p_s^n be the Stokes pressure for u^n hence $\nabla p_s^n = (\Delta P - P\Delta)u^n$ and thus

$$\int_\Omega \nabla p_s^n \cdot \nabla \varphi = \int_\Omega (\Delta u^n - \nabla\nabla \cdot u^n) \cdot \nabla \varphi, \quad \forall \varphi \in H^1(\Omega).$$

Using (2.17) with $\varphi = p^n$ and combining it with the last relation, we obtain

$$\|\nabla p^n\|_{L^2} \leqslant \|\nabla p_s^n\|_{L^2} + \|f^n\|_{L^2}. \tag{5.1}$$

We derive first the discrete H^1 estimate just as in Ref. 10. Taking the L^2 inner product of (2.18) with $-\Delta u^{n+1}$ gives

$$\begin{aligned}
&\tfrac{1}{2\delta t}\left(\|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2 + \|\nabla u^{n+1} - \nabla u^n\|_{L^2}^2\right) + \|\Delta u^{n+1}\|_{L^2}^2 \\
&\quad \leqslant \|\Delta u^{n+1}\|_{L^2}\left(2\|f^n\|_{L^2} + \|\nabla p_s^n\|_{L^2}\right) \\
&\quad \leqslant \frac{\varepsilon_1}{2}\|\Delta u^{n+1}\|_{L^2}^2 + \frac{2}{\varepsilon_1}\|f^n\|_{L^2}^2 + \frac{1}{2}(\|\Delta u^{n+1}\|_{L^2}^2 + \|\nabla p_s^n\|_{L^2}^2)
\end{aligned}$$

for all $\varepsilon_1 > 0$. This implies

$$\frac{1}{\delta t}\left(\|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2\right) + \frac{1}{\delta t}\|\nabla u^{n+1} - \nabla u^n\|_{L^2}^2 + (1 - \varepsilon_1)\|\Delta u^{n+1}\|_{L^2}^2 \leqslant \frac{4}{\varepsilon_1}\|f^n\|_{L^2}^2 + \|\nabla p_s^n\|_{L^2}^2. \tag{5.2}$$

Fix any $\beta \in (\frac{1}{2}, \frac{2}{3})$. By Theorem 3.1, we have

$$\|\nabla p_s^n\|_{L^2}^2 \leqslant \frac{3}{2}\beta\|\Delta u^n\|_{L^2}^2 + \frac{2C_\beta}{3}\|\nabla u^n\|_{L^2}^2.$$

Using this in (5.2) and dividing by $2C_\beta$, we get

$$\begin{aligned}
&\frac{1}{2C_\beta \delta t}\left(\|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2 + \|\nabla u^{n+1} - \nabla u^n\|_{L^2}^2\right) \\
&\quad + \frac{(1 - \varepsilon_1)}{2C_\beta}\left(\|\Delta u^{n+1}\|_{L^2}^2 - \|\Delta u^n\|_{L^2}^2\right) + \frac{(2 - 2\varepsilon_1 - 3\beta)}{4C_\beta}\|\Delta u^n\|_{L^2}^2 \\
&\quad \leqslant \frac{2}{\varepsilon_1 C_\beta}\|f^n\|_{L^2}^2 + \frac{1}{3}\|\nabla u^n\|_{L^2}^2,
\end{aligned} \tag{5.3}$$

and we may assume that $\varepsilon_1 > 0$ is small enough so that $1 - \varepsilon_1 - \frac{3}{2}\beta > 0$.

We continue by obtaining the discrete L^2 estimate. We dot Eq. (2.18) by u^{n+1} in L^2 and obtain

$$\begin{aligned}
&\frac{1}{2\delta t}\left(\|u^{n+1}\|_{L^2}^2 - \|u^n\|_{L^2}^2 + \|u^{n+1} - u^n\|_{L^2}^2\right) \\
&\quad + \frac{1}{3}\|\nabla u^{n+1}\|_{L^2}^2 + \frac{2}{3}\left(\|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2\right) + \frac{2}{3}\|\nabla u^n\|_{L^2}^2 \\
&\quad = \int_\Omega (f^n - \nabla p^n) \cdot u^{n+1} = \int_\Omega (Pf^n) \cdot u^{n+1} - \int_\Omega \nabla p_s(u^n) \cdot \nabla q^{n+1} \\
&\quad \leqslant \|f^n\|_{L^2}\|u^{n+1}\|_{L^2} + \|\nabla p_s(u^n)\|_{L^2}\|\nabla q^{n+1}\|_{L^2} \\
&\quad \leqslant \frac{\lambda_0}{3}\|u^{n+1}\|_{L^2}^2 + \frac{2 - 2\varepsilon_1 - 3\beta}{8C_\beta}\|\Delta u^n\|_{L^2}^2 + C_1(\|f^n\|_{L^2}^2 + \|\nabla q^{n+1}\|_{L^2}^2),
\end{aligned} \tag{5.4}$$

where $q^{n+1} = Q(u^{n+1})$, and λ_0 is the principal eigenvalue of the Laplacian on Ω with zero Dirichlet boundary conditions.

Since $p^n = p_s^n + Q(f^n)$, by applying $I - P$ to (2.18) we find that q^n satisfies the time-discrete inhomogeneous heat equation

$$\frac{1}{\delta t}\left(q^{n+1} - q^n\right) - \Delta q^{n+1} = p_s^{n+1} - p_s^n. \tag{5.5}$$

Then we find after testing with $-\Delta q^{n+1}$ that, as above (and as in p. 1477 of Ref. 10),

$$\frac{1}{\delta t}\left(\|\nabla q^{n+1}\|_{L^2}^2 - \|\nabla q^n\|_{L^2}^2\right) + \|\Delta q^{n+1}\|_{L^2}^2 \leqslant \|p_s^{n+1} - p_s^n\|_{L^2}^2. \tag{5.6}$$

Also, since p^n satisfies a Neumann boundary value problem, we have the estimate

$$\|p_s^{n+1} - p_s^n\|_{L^2}^2 \leqslant C\|u^{n+1} - u^n\|_{L^2}^{\frac{1}{2}}\|u^{n+1} - u^n\|_{H^2}^{\frac{3}{2}}. \tag{5.7}$$

Now choose C_2 large enough to ensure

$$C_1\|\nabla q^{n+1}\|_{L^2}^2 \leqslant \frac{C_2}{2}\|\Delta q^{n+1}\|_{L^2}^2,$$

and ε_2 small enough so that $4\varepsilon_2 < 1 - \varepsilon_1 - \beta$. Combining (5.6) and (5.7), we obtain

$$\begin{aligned}
&\frac{1}{\delta t}\left(\|\nabla q^{n+1}\|_{L^2}^2 - \|\nabla q^n\|_{L^2}^2\right) + \|\Delta q^{n+1}\|_{L^2}^2 \\
&\quad\leqslant \frac{\varepsilon_2}{4C_\beta C_2}\|\Delta u^{n+1} - \Delta u^n\|_{L^2}^2 + C_3\|u^{n+1} - u^n\|_{L^2}^2 \\
&\quad\leqslant \frac{\varepsilon_2}{4C_\beta C_2}\|\Delta u^{n+1} - \Delta u^n\|_{L^2}^2 + C_4\|\nabla u^{n+1} - \nabla u^n\|_{L^2}^2
\end{aligned} \tag{5.8}$$

for large enough constants C_3 and C_4.

Assume that δt is small enough so that $(2C_\beta C_2 C_4)\delta t < 1$. Multiplying (5.8) by C_2, and adding it to (5.3) and (5.4) then gives

$$\begin{aligned}
&\frac{C_2}{\delta t}\left(\|\nabla q^{n+1}\|_{L^2}^2 - \|\nabla q^n\|_{L^2}^2\right) + \frac{C_2}{2}\|\Delta q^{n+1}\|_{L^2}^2 + \frac{1}{2\delta t}\left(\|u^{n+1}\|_{L^2}^2 - \|u^n\|_{L^2}^2\right) \\
&\quad + \left(\frac{1}{2C_\beta\delta t} + \frac{2}{3}\right)\left(\|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2\right) + \frac{1}{3}\|\nabla u^n\|_{L^2}^2 \\
&\quad + \frac{1-\varepsilon_1}{2C_\beta}\left(\|\Delta u^{n+1}\|_{L^2}^2 - \|\Delta u^n\|_{L^2}^2\right) + \frac{2 - 2\varepsilon_1 - 3\beta}{8C_\beta}\|\Delta u^n\|_{L^2}^2 \\
&\quad\leqslant \frac{\varepsilon_2}{2C_\beta}\left(\|\Delta u^{n+1}\|_{L^2}^2 + \|\Delta u^n\|_{L^2}^2\right) + \left(\frac{2}{\varepsilon_1 C_\beta} + C_1\right)\|f^n\|_{L^2}^2.
\end{aligned} \tag{5.9}$$

Now summing from $n = 0$ to $N - 1$ the last inequality gives (for small enough δt, and for a suitable constant $C > 0$) the claimed inequality (2.19).

We rearrange (5.9) and obtain

$$\begin{aligned}
&\frac{1}{\delta t}\left(\|\nabla q^{n+1}\|_{L^2}^2 - (1 - \hat{C}\delta t)\|\nabla q^n\|_{L^2}^2\right) \\
&+\frac{1}{\delta t}\left(\|u^{n+1}\|_{L^2}^2 - (1 - \hat{C}\delta t)\|u^n\|_{L^2}^2\right) + \frac{1}{\delta t}\left(\|\nabla u^{n+1}\|_{L^2}^2 - (1 - \hat{C}\delta t)\|\nabla u^n\|_{L^2}^2\right) \\
&\quad + \left(\|\nabla u^{n+1}\|_{L^2}^2 - (1 - \hat{C}\delta t)\|\nabla u^n\|_{L^2}^2\right) \\
&\quad +(1+\delta t)\left(\|\Delta u^{n+1}\|_{L^2}^2 - (1 - \hat{C}\delta t)\|\Delta u^n\|_{L^2}^2\right) \\
&+ \left(\|\Delta q^{n+1}\|_{L^2}^2 - (1 - \hat{C}\delta t)\|\Delta q^n\|_{L^2}^2\right) + (1 - \hat{C}\delta t)\|\Delta q^n\|_{L^2}^2 \leqslant C\|f^n\|_{L^2}^2
\end{aligned} \tag{5.10}$$

provided that δt is small enough, for suitable constants C and $\hat{C}$. Defining

$$\begin{aligned}
a_n \overset{\text{def}}{=} &\frac{1}{\delta t}\|\nabla q^n\|_{L^2}^2 + \frac{1}{\delta t}\|u^n\|_{L^2}^2 + \frac{1}{\delta t}\|\nabla u^n\|_{L^2}^2 \\
&+\|\nabla u^n\|_{L^2}^2 + (1+\delta t)\|\Delta u^n\|_{L^2}^2 + \|\Delta q^n\|_{L^2}^2,
\end{aligned}$$

(5.10) becomes

$$a_{n+1} - (1 - \hat{C}\delta t)a_n \leqslant C\|f_n\|_{L^2}^2. \tag{5.11}$$

Solving this recurrence relation yields (2.20). □

VI. SMALL DATA GLOBAL EXISTENCE FOR THE EXTENDED NAVIER-STOKES EQUATIONS

This section is devoted to the proof of a long time, small data existence result (Theorem 2.6) for the system (1.1)–(1.4). As bounds for the linear terms have already been established (Proposition 2.2), we begin with a bound on the nonlinear term. When obtaining energy estimates for solutions to (1.1), the explicit, exponential decay of $\nabla \cdot u$ allows sharper estimates for many terms. However, in order to exploit coercivity of the linear terms, we are forced to use a H^1-equivalent inner product. In this case, the "worst" term that arises from the nonlinearity is not aided by decay of $\nabla \cdot u$, and must be estimated brutally. Consequently, estimating the remaining terms similarly does not weaken the final result. Thus, we begin with a lemma that provides a "brutal" estimate on the nonlinearity.

Lemma 6.1: Let $f, g, h \in H^2 \cap H_0^1(\Omega)$ with $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ a bounded domain with C^3 boundary. Then there exists a constant $C = C(\Omega) > 0$ such that[17]

$$|\langle P((f\cdot\nabla)g), h\rangle_\varepsilon| = C\|f\|_{H^1}\|\nabla g\|_{H^{\frac{1}{2}}}\|\Delta h\|_{L^2}.$$

Proof: Observe first that

$$\langle P((f\cdot\nabla)g), h\rangle_\varepsilon = \langle P((f\cdot\nabla)g), h\rangle + \varepsilon\langle\nabla P((f\cdot\nabla)g), \nabla h\rangle,$$

since $(I - P)P((f\cdot\nabla)g) = 0$. Thus, to prove the lemma, it suffices to show the estimates

$$|\langle P((f\cdot\nabla)g), h\rangle| \leqslant C\|f\|_{H^1}\|\nabla g\|_{H^{\frac{1}{2}}}\|h\|_{L^2}, \tag{6.1}$$

and

$$|\langle\nabla P((f\cdot\nabla)g), \nabla h\rangle| \leqslant C\|f\|_{H^1}\|\nabla g\|_{H^{\frac{1}{2}}}\|\Delta h\|_{L^2}, \tag{6.2}$$

for some constant $C = C(\Omega)$.

The inequality (6.1) follows directly from the Sobolev embedding theorem. Indeed, for any three functions f_1, f_2, f_3, we know[18]

$$\left|\int_\Omega f_1 f_2 f_3\right| \leqslant C\|f_1\|_{H^{s_1}}\|f_2\|_{H^{s_2}}\|f_3\|_{H^{s_3}}, \tag{6.3}$$

provided $0 \leq s_i \leq 3$, $s_1 + s_2 + s_3 \geqslant \frac{d}{2}$ and at least two of $s_1, \ldots, s_3$ are non-zero (see, for instance, the proof of Proposition 6.1 in Ref. 3). Choosing $s_1 = 1$, $s_2 = 1/2$, and $s_3 = 0$, we have

$$|\langle P((f\cdot\nabla)g), h\rangle| = |\langle(f\cdot\nabla)g, Ph\rangle| \leqslant C\|f\|_{H^1}\|\nabla g\|_{H^{\frac{1}{2}}}\|h\|_{L^2},$$

proving (6.1).

For (6.2), we first integrate by parts and observe ΔP is a regular differential operator (identity (1.10)). Now we can integrate by parts again to obtain the desired estimate. Explicitly,

$$\begin{aligned}
\langle\nabla P((f\cdot\nabla)g), \nabla h\rangle &= -\langle\Delta P((f\cdot\nabla)g), h\rangle \\
&= -\langle(\Delta - \nabla\nabla\cdot)((f\cdot\nabla)g), h\rangle = -\langle(f\cdot\nabla)g, (\Delta - \nabla\nabla\cdot)h\rangle,
\end{aligned}$$

where all boundary integrals vanish because $f, g, h \in H^1_0$. Now using (6.3) with $s_1 = 1$, $s_2 = 1/2$, $s_3 = 0$, and elliptic regularity we have

$$|\langle \nabla P((f\cdot\nabla)g), \nabla h\rangle| = |\langle (f\cdot\nabla)g, (\Delta - \nabla\nabla\cdot)h\rangle| \leqslant C\|f\|_{H^1}\|\nabla g\|_{H^{\frac{1}{2}}}\|\Delta h\|_{L^2}.$$

This concludes the proof. □

We now return to the proof of Theorem 2.6.

Proof: We assume there exists a smooth solution u of (1.1)–(1.4) on the time interval $[0, T]$ for some $T > 0$. We will find appropriate *a priori* estimates for the norm of u (see relation (2.22), below) in terms of the initial data and T. Now a standard approximating scheme (e.g., the one constructed in Ref. 10) will prove global existence of solutions.

Fix $\varepsilon > 0$ to be small enough so that Proposition 2.2 holds, and $\langle\cdot,\cdot\rangle_\varepsilon$ denote the H^1 equivalent inner product from Proposition 2.2. Then

$$\frac{1}{2}\partial_t\|u\|^2_{H^1_\varepsilon} + \langle P((u\cdot\nabla)u), u\rangle_\varepsilon + \langle u, Au\rangle_\varepsilon = 0. \tag{6.4}$$

where

$$\|v\|_{H^1_\varepsilon} \overset{\text{def}}{=} \sqrt{\langle v, v\rangle_\varepsilon}.$$

By Lemma 6.1, for any $c_0 > 0$, we can find a constant $C = C(\varepsilon, c_0, \Omega) > 0$ such that

$$\begin{aligned}|\langle P((u\cdot\nabla)u), u\rangle_\varepsilon| &\leqslant \|u\|_{H^1}\|\nabla u\|_{H^{1/2}}\|\Delta u\|_{L^2}\\ &\leqslant C\|\nabla u\|^{3/2}_{L^2}\|\Delta u\|^{3/2}_{L^2} \leqslant C\|\nabla u\|^6_{L^2} + \frac{\varepsilon}{8c_0}\|\Delta u\|^2_{L^2}.\end{aligned} \tag{6.5}$$

We will subsequently fix c_0 to be the constant c that appears on the right of (2.7).

Using Proposition 2.2 and Eqs. (6.4) and (6.5), we obtain

$$\partial_t\|u\|^2_{H^1_\varepsilon} + \frac{2}{c_0}\left(\|\nabla u\|^2_{L^2} + \frac{\varepsilon}{2}\|\Delta u\|^2_{L^2} + C_\varepsilon\|\nabla q(u)\|^2_{L^2}\right) \leqslant C\|\nabla u\|^6_{L^2},$$

where C_ε is the constant in (2.7). Allowing the constant $C = C(\varepsilon, c_0, \Omega)$ to change from line to line, and using the Poincaré inequality, we obtain

$$\partial_t\|u\|^2_{H^1_\varepsilon} + \frac{1}{c_0}\left(\|\nabla u\|^2_{L^2} + \frac{\varepsilon}{2}\|\Delta u\|^2_{L^2} + C_\varepsilon\|\nabla q(u)\|^2_{L^2}\right) \leqslant C\|u\|^6_{H^1_\varepsilon} - \frac{1}{c_1}\|u\|^2_{H^1_\varepsilon},$$

for some constant $c_1 = c_1(\varepsilon, \Omega)$. Thus, if at time $t = 0$ we have

$$\|u_0\|_{H^1_\varepsilon} \leqslant \frac{1}{(Cc_1)^{1/4}},$$

then for all $t > 0$,

$$\|u(t)\|^2_{H^1_\varepsilon} + \frac{1}{c_0}\int_0^t \left(\|\nabla u\|^2_{L^2} + \frac{\varepsilon}{2}\|\Delta u\|^2_{L^2} + C_\varepsilon\|\nabla q(u)\|^2_{L^2}\right) ds \leqslant \|u_0\|^2_{H^1_\varepsilon}.$$

Now using the local existence result in Ref. 10, and the fact that $\|\cdot\|_{H^1_\varepsilon}$ is equivalent to the usual H^1 norm, we conclude the proof of Theorem 2.6. □

VII. TWO-DIMENSIONAL SMALL DIVERGENCE GLOBAL EXISTENCE FOR THE EXTENDED NAVIER-STOKES EQUATIONS

The aim of this section is to prove Theorem 2.7. We recall first the H^1_0-orthogonal projection onto divergence free vector fields. For $u_0 \in H^1_0(\Omega)$, we define $v_0, w_0 \in H^1_0(\Omega)$ to be solutions of the PDEs

$$\begin{cases} -\Delta v_0 + \nabla\phi = -\Delta u_0 & \text{in } \Omega,\\ \nabla\cdot v_0 = 0 & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega,\end{cases} \quad \text{and} \quad \begin{cases} -\Delta w_0 + \nabla\psi = 0 & \text{in } \Omega,\\ \nabla\cdot w_0 = \nabla\cdot u_0 & \text{in } \Omega,\\ w_0 = 0 & \text{on } \partial\Omega,\end{cases} \tag{7.1}$$

respectively. Note that $u_0 \in H^1_0(\Omega)$ guarantees the required compatibility condition $\int_\Omega \nabla\cdot u_0 = 0$, and so the existence of v_0, w_0 satisfying (7.1) is well known (see, for instance, [Sec. 2 of Ref. 14]. Clearly, $u_0 = v_0 + w_0$, and orthogonality of v_0 and w_0 in H^1_0 follows from the identity:

$$\langle v_0, w_0\rangle_{H^1_0(\Omega)} \overset{\text{def}}{=} \langle \nabla v_0, \nabla w_0\rangle = \langle v_0, -\Delta w_0\rangle = \langle v_0, -\nabla\psi\rangle = 0.$$

Let v be the solution to Eqs. (1.1)–(1.4) with initial data v_0. As shown earlier, $\nabla\cdot v_0 = 0$ implies that $\nabla\cdot v = 0$ for all time, and consequently v is a solution of the 2*D* Navier-Stokes with initial data v_0. Let $w = u - v$, and observe

$$\begin{cases} \partial_t w + P((w\cdot\nabla)w) & \\ \quad + P((v\cdot\nabla)w) + P((w\cdot\nabla)v) = \Delta w - \nabla p_s(w) & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega.\end{cases} \tag{7.2}$$

The strategy to prove Theorem 2.7 is as follows. First, standard existence theory for the 2*D* Navier-Stokes equations implies that for any initial data $v_0 \in H^1_0$, with $\nabla\cdot v_0 = 0$, we have global existence of a strong solution v. Further, after a long time T_0, the solution v becomes small. Now making $u_0 - v_0$ is sufficiently small, we can guarantee that w, a solution to (7.2) with initial data $u_0 - v_0$, both exists on the time interval $[0, T_0]$, and is small at time T_0. Thus, $u = v + w$ is a solution to (1.1)–(1.4) defined, which is small at time T_0. Now a small data global existence result (Theorem 2.6) will allow us to continue this solution for all time.

We begin with a Lemma concerning the existence and smallness of solutions to (7.2).

Lemma 7.1: Let $\Omega \subset \mathbb{R}^2$ be a bounded C^3 domain, and $v_0 \in H^1_0(\Omega)$ with $\nabla\cdot v_0 = 0$. Let $u_0 \in H^1_0(\Omega)$ be such that $P_0 u_0 = v_0$. Then, for any T_0, $\delta_0 > 0$ there exists a (small) constant $W_0 = W_0(\Omega, \|v_0\|_{H^1}, T_0, \delta_0)$ such that if

$$\|w_0\|_{H^1_\varepsilon} \leqslant W_0,$$

then there exists a solution of (7.2) on the interval [0, T_0] and

$$\|w(T_0)\|_{H^1} \leqslant \delta_0.$$

Momentarily postponing the proof of the lemma, we prove Theorem 2.7.

Proof of Theorem 2.7: We let V_0 be as in Theorem 2.6, and let v be the solution to the 2D Navier-Stokes equations with initial data $v_0 = P_0 u_0 \in H^1_0$. It is well known (see, for instance, Refs. 3 and 14) that there exists T_0 large enough, so that $\|v(T_0)\|_{H^1_0} \leqslant \frac{1}{2}V_0$. Indeed, from the standard L^2 energy identity we can choose T_0 to satisfy $T_0(\frac{1}{2}V_0)^2 \leq \|v_0\|^2_{L^2}$.

By Lemma 7.1, there exists $W_0 > 0$ small enough so that if initially

$$\|w_0\|_{H^1_\varepsilon} \leqslant W_0, \tag{7.3}$$

then the solution w to (7.2) exists up to time T_0, and further $\|w(T_0)\|_{H^1} \leqslant \frac{1}{2}V_0$. From (7.1), we know $\|w_0\|_{H^1} \leqslant c\|\nabla\cdot u_0\|_{L^2}$ (see [Sec. 2 of Ref. 14]). Since the norms $\|\cdot\|_{H^1_\varepsilon}$ and $\|\cdot\|_{H^1}$ are equivalent, making U_0 small enough will guarantee (7.3), thus allowing to apply Lemma 7.1 and obtain the existence of w on the interval $[0, T_0]$. Then we obtain that (1.1)–(1.4) has a solution $u = w + v$ on $[0, T_0]$ and moreover $\|u(T_0)\|_{H^1} \leqslant \|w(T_0)\|_{H^1} + \|v(T_0)\|_{H^1} \leqslant V_0$. Applying Theorem 2.6, we can continue the solution u on the interval $[T_0, \infty)$.

It remains to prove the lemma.

Proof of Lemma 7.1. As with the proof of Theorem 2.6, it suffices to obtain an *a priori* estimate for $\|w\|_{H^1}$. Fix $\varepsilon > 0$ to be small enough so that Proposition 2.2 holds. Then

$$\begin{aligned}\frac{1}{2}\partial_t\|w\|^2_{H^1_\varepsilon} + \overbrace{\langle P((w\cdot\nabla)w), w\rangle_\varepsilon + \langle w, Aw\rangle_\varepsilon}^{\mathcal{J}_1}&\\ = -\underbrace{\langle P((v\cdot\nabla)w), w\rangle_\varepsilon}_{\mathcal{J}_2} - \underbrace{\langle P((w\cdot\nabla)v), w\rangle_\varepsilon}_{\mathcal{J}_3},&\end{aligned} \tag{7.4}$$

where $\langle\cdot,\cdot\rangle_\varepsilon$ denotes the inner product defined in Proposition 2.2, and $\|\cdot\|_{H^1_\varepsilon}$ the induced norm.

We estimate each term individually. The term $\mathcal{J}_1$ is identical to the term that appears in the proof of Theorem 2.6, and thus

$$\mathcal{J}_1 \geqslant \frac{1}{c_0}\left(\|\nabla w\|_{L^2}^2 + \frac{\varepsilon}{2}\|\Delta w\|_{L^2}^2 + C_\varepsilon\|\nabla q\|_{L^2}^2\right) - \left(c_2\|w\|_{H^1_\varepsilon}^6 - \frac{1}{c_1}\|w\|_{H^1_\varepsilon}^2\right),$$

where q is the unique, mean-zero function such that $\nabla q = (I-P)w$. As before, c_0 is the constant that appears in (2.7), and $c_1 = c_1(\varepsilon,\Omega)$, $c_2 = c_2(\varepsilon,\Omega)$ are positive constants.

Using $C = C(\varepsilon,\Omega) > 0$ to denote an intermediate constant that can change from line to line, Lemma 6.1 bounds $\mathcal{J}_2$ and $\mathcal{J}_3$ by

$$\begin{aligned}|\mathcal{J}_2| + |\mathcal{J}_3| &\leqslant C\left(\|v\|_{H^1}\|\nabla w\|_{H^{1/2}}\|\Delta w\|_{L^2} + \|w\|_{H^1}\|\nabla v\|_{H^{1/2}}\|\Delta w\|_{L^2}\right)\\ &\leqslant \frac{\varepsilon}{4c_0}\|\Delta w\|_{L^2}^2 + c_3\left(\|v\|_{H^1}^4 + \|\nabla v\|_{H^{1/2}}^2\right)\|w\|_{H^1_\varepsilon}^2\end{aligned}$$

for some constant $c_3 = c_3(\varepsilon,\Omega)$.

Combining our estimates,

$$\begin{aligned}&\frac{1}{2}\partial_t\|w\|_{H^1_\varepsilon}^2 + \frac{1}{c_0}\left(\|\nabla w\|_{L^2}^2 + \frac{\varepsilon}{4}\|\Delta w\|_{L^2}^2 + C_\varepsilon\|\nabla q\|_{L^2}^2\right)\\ &\qquad\leqslant c_2\|w\|_{H^1_\varepsilon}^6 - \left(\frac{1}{c_1} - c_3\left(\|v\|_{H^1}^4 + \|\nabla v\|_{H^{1/2}}^2\right)\right)\|w\|_{H^1_\varepsilon}^2. \qquad (7.5)\end{aligned}$$

Since v is a strong solution to the 2D incompressible Navier-Stokes equations with initial data $v_0 \in H^1_0$, we have (see, for instance, [p. 78 of Ref. 3]) that

$$\sup_{t\geqslant 0}\|v(t)\|_{H^1}^2 + \int_0^\infty \|v(s)\|_{H^1}^2\,ds < C$$

for some constant C depending only on Ω and $\|v_0\|_{H^1}$. Using this in (7.5) will prove local well-posedness of (7.2). Further, for any T_0, $\delta_0 > 0$, Eq. (7.5) will also show that the solution to (7.2) exists up to time T_0, and $\|w(T_0)\|_{H^1} < \delta_0$, provided $\|w_0\|_{H^1}$ is small enough.

VIII. DIVERGENCE-DAMPED EQUATIONS

The aim of this section is to prove coercivity of B_α (defined in (2.25)), with constants independent of α, and 2D global existence with strong enough divergence damping (Corollary 2.8).

Proposition 8.1. For any $\alpha \geq 0$ and $u \in D(B_\alpha)$, we have

$$\langle u, B_\alpha u\rangle_\varepsilon = \langle u, Au\rangle_\varepsilon + \alpha\left(\|\nabla Q(u)\|_{L^2}^2 + C_\varepsilon\|Q(u)\|_{L^2}^2 + \varepsilon\|\Delta Q(u)\|_{L^2}^2\right). \qquad (8.1)$$

Proof: By linearity,

$$\langle u, B_\alpha u\rangle_\varepsilon = \langle u, Au\rangle_\varepsilon + \alpha\langle u, (I-P)u\rangle_\varepsilon. \qquad (8.2)$$

For the second term on the right,

$$\langle u,(I-P)u\rangle_\varepsilon = \langle u,(I-P)u\rangle + C_\varepsilon\langle Q(u), Q((I-P)u)\rangle + \varepsilon\langle\nabla u, \nabla(I-P)u\rangle.$$

The first two terms on the right are equal to $\|(I-P)u\|_{L^2}^2$ and $C_\varepsilon\|Q(u)\|_{L^2}^2$, respectively. For the last term,

$$\begin{aligned}\langle\nabla u, \nabla(I-P)u\rangle &= -\langle u, \Delta(I-P)u\rangle + \int_{\partial\Omega} u_i\frac{\partial}{\partial\nu}((I-P)u)_i\\ &= -\langle u, \nabla\nabla\cdot u\rangle + 0\\ &= \|\nabla\cdot u\|_{L^2}^2 - \int_{\partial\Omega}(\nabla\cdot u)u\cdot\nu = \|\nabla\cdot u\|_{L^2}^2.\end{aligned}$$

Consequently,

$$\langle u, (I-P)u\rangle_\varepsilon = \|(I-P)u\|_{L^2}^2 + C_\varepsilon\|Q(u)\|_{L^2}^2 + \varepsilon\|\nabla\cdot u\|_{L^2}^2, \qquad (8.3)$$

and using (8.2), we obtain (8.1). □

Before moving to the proof of Corollary 2.8, we digress briefly to remark that we can also consider higher order divergence damped operators of the form

$$B'_\alpha \stackrel{\text{def}}{=} A - \alpha\nabla\nabla\cdot$$

The results we obtain for (1.1)–(1.2) with a zeroth order damping term will also apply when we add the second order damping term above. However, while the operator B'_α has a stronger (second order) damping term, it is not as easy to deal with numerically. The zeroth order damping terms in B_α, on the other hand, can easily be implemented numerically, and has a strong enough damping effect to give a better existence result (Corollary 2.8). We now return to prove Corollary 2.8.

Proof of Corollary 2.8: Since Theorems 2.6 and 2.7 work verbatim for (2.24), there exists a time $T_0 = T_0(\|u_0\|_{H^1}, \Omega)$, independent of α, such that there exists a solution u of (2.24) on the interval $[0, T_0]$, with $\|u(T_0)\|_{H^1}$ bounded, independent of α. Let U_0 be the constant from Theorem 2.7. Observe that (2.24) implies that $\nabla\cdot u$ satisfies

$$\begin{cases}\partial_t\nabla\cdot u + \alpha\nabla\cdot u = \Delta\nabla\cdot u & \text{in } \Omega,\\ \dfrac{\partial}{\partial\nu}\nabla\cdot u = 0 & \text{for } x\in\partial\Omega,\ t>0.\end{cases}$$

Consequently,

$$\|\nabla\cdot u(t)\|_{L^2}^2 \leqslant e^{-(\lambda_1+\alpha)t}\|\nabla\cdot u_0\|_{L^2}^2,$$

where $\lambda_1 > 0$ is the smallest non-zero eigenvalue of the Laplacian with Neumann boundary conditions. Thus, there exists $\alpha_0 > 0$, such that

$$\|\nabla\cdot u(T_0)\|_{L^2} < U_0,$$

for all $\alpha > \alpha_0$. Now, by Theorem 2.7 the solution to (2.24) also exists and is regular on the time interval $[T_0, \infty)$. □

IX. EXISTENCE RESULTS UNDER COERCIVE BOUNDARY CONDITIONS.

The aim of this section is to show that the extended Stokes operator is coercive under the boundary conditions (2.26), and prove Proposition 2.9. We begin with coercivity.

Proposition 9.1: If either u and v are in $H^1(\mathbb{T}^d)$ and periodic, or if u and v are in $H^2(\Omega)$ and satisfy the boundary conditions (2.26), then

$$\langle Au, v\rangle = \langle u, Av\rangle \quad\text{and}\quad \langle u, Au\rangle = \int_\Omega |\nabla u|^2. \qquad (9.1)$$

Proof: In the periodic case, $P\Delta = \Delta P$. Thus, $\nabla p_s = 0$, $A = -\Delta$, and both equalities in (9.1) follow easily.

Suppose now u, v satisfy (2.26). In view of (2.1), we have

$$\begin{aligned}\langle Au, v\rangle &= -\langle P\Delta u, v\rangle - \langle\nabla\nabla\cdot u, v\rangle = -\langle P\Delta Pu, v\rangle - \langle\nabla\nabla\cdot u, v\rangle\\ &= \langle\nabla Pu, \nabla Pv\rangle + \langle\nabla\cdot u, \nabla\cdot v\rangle - \int_{\partial\Omega}\left[(Pv)_i\frac{\partial(Pu)_i}{\partial\nu} - (\nabla\cdot u)v\cdot\nu\right].\end{aligned}$$

Observe that $Pv = 0$ on $\partial\Omega$, because $Pv\cdot\nu = 0$ by definition of P, and $Pv\cdot\tau = 0$ by (2.26). Thus, both the above boundary integrals vanish, giving

$$\langle Au, v\rangle = \langle\nabla Pu, \nabla Pv\rangle + \langle\nabla\cdot u, \nabla\cdot v\rangle.$$

A similar calculation shows

$$\langle Av, u\rangle = \langle \nabla Pu, \nabla Pv\rangle + \langle \nabla\cdot u, \nabla\cdot v\rangle$$

proving that A is self-adjoint.

Now because $Pu = Pv = 0$ on $\partial\Omega$, a direct calculation shows that

$$\langle \nabla Pu, \nabla Pv\rangle = \langle \nabla\times Pu, \nabla\times Pv\rangle = \langle \nabla\times u, \nabla\times v\rangle.$$

Consequently, we see

$$\langle Au, v\rangle = \langle \nabla\times u, \nabla\times v\rangle + \langle \nabla\cdot u, \nabla\cdot v\rangle = \langle \nabla u, \nabla v\rangle.$$

Setting $u = v$, the second assertion in (9.1) follows. □

Finally, we turn to Proposition 2.9. Before presenting the proof, we remark that if we instead impose periodic boundary conditions, Proposition 2.9 and its proof (below) go through almost unchanged. The only modification required is the justification of the Poincaré inequality that will be (implicitly) used in many estimates. For this justification, observe that with periodic boundary conditions, the mean of solutions to (1.1)–(1.2) is conserved. Thus, by switching to a moving frame, we can assume that the initial data, and hence the solution for all time, are mean zero. This will justify the use of the Poincaré inequality in the proof. With this, we prove Proposition 2.9.

Proof of Proposition 2.9: Let $v = Pu$, and $q = Q(u)$. Since

$$P((\nabla q\cdot\nabla)\nabla q) = P(\nabla|\nabla q|^2/2) = 0,$$

applying P to (1.1) gives

$$\begin{cases} \partial_t v - P\Delta v + P((v\cdot\nabla)(v+\nabla q) + (\nabla q\cdot\nabla)v) = 0 & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.2}$$

where the boundary condition on v comes from (2.26). The point is that energy estimates can be used directly to estimate v, since it satisfies explicit boundary conditions.

Since $Pv = v$, multiplying (9.2) by v and integrating yields,

$$\begin{aligned} \frac{1}{2}\partial_t\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 &= \frac{1}{2}\int_\Omega |v|^2\Delta q - \int_\Omega v\cdot((v\cdot\nabla)\nabla q)\\ &\leqslant C\|v\|_{L^4}^2\|\nabla^2 q\|_{L^2} \leqslant C\|v\|_{L^2}\|\nabla v\|_{L^2}\|\Delta q\|_{L^2}\\ &\leqslant \frac{1}{2}\|\nabla v\|_{L^2}^2 + C\|v\|_{L^2}^2\|\nabla\cdot u\|_{L^2}^2. \end{aligned} \tag{9.3}$$

Here, we used elliptic regularity to control $\|\nabla^2 q\|_{L^2}$ by $\|\Delta q\|_{L^2}$, which is valid since $\frac{\partial q}{\partial \nu} = 0$ on $\partial\Omega$. We also used the (2D) Ladyzhenskaya inequality $\|v\|_{L^4}^2 \leqslant C\|v\|_{L^2}\|\nabla v\|_{L^2}$, which is valid since $v = 0$ on $\partial\Omega$.

Since $\nabla\cdot u$ is a mean-zero solution of (1.6), we know that

$$\int_0^\infty \|\nabla\cdot u(t)\|_{L^2}^2\,dt \leqslant \frac{1}{2\lambda_1}\|\nabla\cdot u_0\|_{L^2}^2,$$

where λ_1 is the smallest non-zero eigenvalue of the Laplacian with Neumann boundary conditions. Thus, Gronwall's lemma and (9.3) give the closed estimate

$$\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(s)\|_{L^2}^2\,ds \leqslant \exp\left(C\|\nabla\cdot u_0\|_{L^2}^2\right)\|v_0\|_{L^2}^2. \tag{9.4}$$

Since $Pv = v$, regularity of the (standard) Stokes operator tells us that the norms $\|-P\Delta v\|_{L^2}$ and $\|v\|_{H^2}$ are equivalent (see, for instance, [Chap. 4 of Ref. 3]). Multiplying (9.2) by $-P\Delta v$, integrating by parts, and using (6.3) to bound the nonlinear term in the usual way gives

$$\partial_t\|\nabla v\|_{L^2}^2 + \frac{1}{c}\|\Delta v\|_{L^2}^2 \leqslant C\left(\|v\|_{L^2}^2\|\nabla v\|_{L^2}^2 + \|\nabla\nabla\cdot u\|_{L^2}^2 + \|\nabla\cdot u\|_{L^2}^4\right)\|\nabla v\|_{L^2}^2.$$

Using Gronwall's lemma, Eqs. (9.4) and (1.6), we obtain

$$\|\nabla v(t)\|_{L^2}^2 + \frac{1}{c}\int_0^t \|\Delta v(s)\|_{L^2}^2\,ds \leqslant K\|\nabla v_0\|_{L^2}^2, \tag{9.5}$$

for some constant $K = K(\Omega, \|\nabla\cdot u_0\|_{L^2}, \|v_0\|_{L^2})$. In fact, one can bound K above by

$$K \leqslant C\exp\left(C\left(\exp\left(C\|\nabla\cdot u_0\|_{L^2}^2\right)\|v_0\|_{L^2}^4 + \|\nabla\cdot u_0\|_{L^2}^2 + \|\nabla\cdot u_0\|_{L^2}^4\right)\right)$$

for some constant $C = C(\Omega)$.

Finally, we consider a Galerkian scheme for (1.1)–(1.2) using eigenfunctions of the Stokes operator (with no-slip boundary conditions), and gradients of eigenfunctions of the Laplacian (with no-flux boundary conditions). It is easy to check that these Galerkian approximations satisfy the same energy estimates (9.4) and (9.5). A bound for $\partial_t u$ will then follow from (1.1), and standard techniques will prove global existence. □

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APPENDIX: FAILURE OF COERCIVITY UNDER THE STANDARD INNER PRODUCT

Most of this section is devoted to the proof that Stokes operator is not positive under the standard L^2 inner product (Proposition 2.1).

Proof of Proposition 2.1: As mentioned earlier, the key idea in the proof is to identify the harmonic conjugate of the Stokes pressure as the harmonic extension of the vorticity. We begin by working up to this. Since

$$\Delta p_s = \nabla\cdot\nabla p_s = \nabla\cdot(\Delta P - P\Delta)u = 0,$$

the Poincaré lemma guarantees the existence of q_s such that

$$\nabla p_s = \nabla^\perp q_s \overset{\text{def}}{=} \begin{pmatrix} -\partial_2 q_s\\ \partial_1 q_s\end{pmatrix}. \tag{A1}$$

Observe that both p_s and q_s are harmonic. Indeed,

$$\Delta q_s = \nabla\times\nabla^\perp q_s = \nabla\times\nabla p_s = 0. \tag{A2}$$

We remark that Eqs. (A1) and (A2) above show that $-q_s$ is the harmonic conjugate of p_s.

To obtain boundary conditions for q_s, let $\tau = -\nu^\perp$ be the unit tangent vector on $\partial\Omega$. To clarify our sign convention, if $\nu = \binom{\nu_1}{\nu_2}$, then $\tau \overset{\text{def}}{=} \binom{\nu_2}{-\nu_1}$. Now observe

$$\begin{aligned} \frac{\partial q_s}{\partial\tau} = \nabla q_s\cdot\tau = \nabla^\perp q_s\cdot\nu = \frac{\partial p_s}{\partial\nu} &= \nu\cdot(\Delta P - P\Delta)u\\ &= \nu\cdot(\Delta u - \nabla\nabla\cdot u) = \nu\cdot\nabla^\perp\nabla\times u = \tau\cdot\nabla\omega = \frac{\partial\omega}{\partial\tau}, \end{aligned}$$

where, as before, $\nabla \times u = \partial_1 u_2 - \partial_2 u_1$ is the two-dimensional curl, and $\omega = \nabla \times u$. Thus, adding a constant to q_s, we may, without loss of generality assume

$$q_s = \omega \quad \text{on } \partial\Omega. \tag{A3}$$

A direct calculation shows

$$\int_\Omega u \cdot Au = \int_\Omega |\nabla u|^2 + \int_\Omega \nabla p_s \cdot u = \int_\Omega \left(\omega^2 + |\nabla \cdot u|^2\right) - \int_\Omega q_s \cdot \omega,$$

where we used the boundary condition $u = 0$ on $\partial\Omega$ to integrate by parts. Thus, to prove Proposition 2.1, it is enough to produce a function u, satisfying the required boundary conditions, such that

$$\int_\Omega q_s \omega \geqslant \|\omega\|_{L^2}^2 + (C+1)\|\nabla \cdot u\|_{L^2}^2. \tag{A4}$$

We prove the existence of such functions separately.

Lemma A.1: For any $C > 0$, there exists $u \in H_0^1(\Omega)$ such that (A4) holds. As usual, $\omega = \nabla \times u$, and q_s is the solution of the Dirichlet problem

$$\begin{cases} \Delta q_s = 0 & \text{in } \Omega, \\ q_s = \omega & \text{on } \partial\Omega. \end{cases} \tag{A5}$$

The lemma immediately finishes the proof of Proposition 2.1. □

Proof of Lemma A.1: We look for u of the form $u = v + \nabla p$ with $\nabla \cdot v = 0$ in Ω and $v \cdot \nu = 0$ on $\partial\Omega$, where ν denotes the outward pointing normal vector on the boundary. Then there exists ψ on Ω so that $v = \nabla^\perp \psi$. The boundary condition $v \cdot \nu = 0$ becomes $\frac{\partial \psi}{\partial \tau} = 0$ where τ denotes the tangential direction on $\partial\Omega$.

We note that $u = 0$ and $v \cdot \nu = 0$ on $\partial\Omega$ imply

$$\frac{\partial p}{\partial \tau} = -v \cdot \tau = -\frac{\partial \psi}{\partial \nu}, \quad \frac{\partial p}{\partial \nu} = 0, \quad v \cdot \nu = \frac{\partial \psi}{\partial \tau} = 0.$$

As $\frac{\partial \psi}{\partial \tau} = 0$ and ψ is determined up to a constant we can assume without loss of generality that $\psi = 0$ on $\partial\Omega$ and then for a given u the stream function ψ is uniquely determined as the solution of the Dirichlet problem

$$\begin{aligned} \Delta \psi &= \omega \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then, we have

$$\begin{aligned} \int_\Omega q_s \omega \, dx = \int_\Omega q_s \Delta \psi \, dx &= \int_\Omega \Delta q_s \psi \, dx + \int_{\partial\Omega} \frac{\partial \psi}{\partial \nu} q_s \, d\sigma - \int_{\partial\Omega} \frac{\partial q_s}{\partial \nu} \psi \, d\sigma \\ &= \int_{\partial\Omega} \frac{\partial \psi}{\partial \nu} q_s \, d\sigma = \int_{\partial\Omega} \frac{\partial \psi}{\partial \nu} \Delta \psi \, d\sigma, \end{aligned}$$

and (A4) becomes

$$\int_{\partial\Omega} \frac{\partial \psi}{\partial \nu} \Delta \psi \, d\sigma \geqslant \|\Delta \psi\|_{L^2}^2 + (C+1)\|\Delta p\|_{L^2}^2. \tag{A6}$$

Summarizing it suffices to find ψ, p such that (A6) holds together with the boundary conditions

$$\psi = 0, \quad \frac{\partial p}{\partial \nu} = 0, \quad \text{and} \quad \frac{\partial p}{\partial \tau} = -\frac{\partial \psi}{\partial \nu} \quad \text{on } \partial\Omega. \tag{A7}$$

Fix some point $x_0 \in \partial\Omega$ and let $s \mapsto \hat{x}(s)$ be an arclength parametrization of the (C^3) boundary $\partial\Omega$, such that $x_0 = \hat{x}(0)$ and oriented so that the outward unit normal $\hat{\nu}(s)$ at $\hat{x}(s)$ satisfies $\hat{\nu}(s)^\perp = \hat{x}'(s)$.

Then the map $(s, r) \mapsto x = \hat{x}(s) - r\hat{\nu}(s)$ is C^2 and is locally invertible near x_0, providing orthogonal coordinates $x \mapsto (s, r) \in (-\varepsilon, \varepsilon) \times (0, \varepsilon)$ in some neighborhood of x_0 in Ω.

We fix p to be of the form $p(x) = \alpha(s)\beta(r)$ where α and β are in $C_c^\infty((-\varepsilon, \varepsilon))$ and $\beta(0) = 1$, $\beta'(0) = 0$. We will then choose ψ of the form $\psi(x) = \alpha'(s)\gamma(r)$ where $\gamma \in C_c^\infty((-\varepsilon, \varepsilon))$ with $\gamma(0) = 0$ and $\gamma'(0) = 1$. Then (A7) will hold, and direct calculation shows

$$\int_{\partial\Omega} \frac{\partial \psi}{\partial \nu} \Delta \psi \, d\sigma = \int_{-\varepsilon}^{\varepsilon} \alpha'(s)^2 (\gamma''(0) + \kappa(s)) \, ds,$$

where $\kappa(s) = \Delta r(\hat{x}(s))$ is the curvature of the boundary. The right-hand side of (A6) on the other hand, is easily computed to be bounded by $C + C\|\gamma\|_{H^2}^2$, with a constant C independent of the choice of γ. It is clear that γ can be chosen to make $\gamma''(0)$ arbitrarily large while $\|\gamma\|_{H^2}^2$ remains bounded. Thus, (A6) holds for some ψ and p. □

Finally, to conclude this section we turn to the proof of Corollary 2.3. Of course the proof is immediate from Proposition 2.1, and we only present it here for completeness.

Proof of Corollary 2.3: Choose $u_0 \in C^2(\bar{\Omega})$ to be such that (2.5) holds, and let u be the solution to (2.10) with initial data u_0. By continuity in time, we must have

$$\int_\Omega u(t) \cdot Au(t) < 0$$

for all t in some small interval $[0, t_0]$. Thus, $\partial_t \|u\|_{L^2}^2 = -\int_\Omega u \cdot Au > 0$ on the interval $(0, t_0]$ which immediately completes the proof. □

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[15] Note that $\int_\Omega \nabla \cdot u_0 = \int_{\partial\Omega} u_0 \cdot \nu = 0$, and so the compatibility condition for (1.6) is satisfied.
[16] When Ω is the whole space or the torus, the commutator $\Delta P - P\Delta = 0$ and the Stokes pressure vanishes. In this case, one can alternatively determine the pressure and obtain an integro-differential equation for u by solving the Poisson equation $-\Delta p = \mathrm{tr}(\nabla u)^2$. This in fact leads to the classical formulation by Leray, as explained in [Sec. 1.8 of Ref. 13]. for instance, where it is shown that $\nabla \cdot u$ then satisfies an advection-diffusion equation.
[17] Our estimates on the nonlinear term are not optimal. Using "optimal" estimates here would be at the expense of simplicity, and obfuscate the main idea. Further, the "optimal" estimates are still insufficient to prove global existence without a smallness assumption on the initial data.
[18] For non-integer values of s, we define the fractional Sobolev norms by interpolation. See, for instance, [p. 50 of Ref. 3].